Frictionless Trains and Palindromes

Puzzle Corner
by
Allan J. Gottlieb

As many readers noticed, there were no Kings on the board of our chess problem in December (DEC 1); somehow they were printed as Queens. So change them back to Kings, try again, and I will print the solution in the July/August issue.

Problems

M/A 1 We begin this month's selection with a bridge problem from Russell A. Nahigian: South is declarer at a contract of six spades. How can he make the contract after West leads ♥K?

	♠K Q 3	
	♥ –	
	♦Q 8 3	
	♣A K 9 8 6 4 2	
♠J 10 8		♠6
♥A K 6 4		♥9 8 7 5 3 2
♦K J 7 4		♦10 9 2
♣7 3		♣Q
	♠A 9 7 5 4 2	
	♥Q J 10	
	♦A 6 5	
	♣5	

M/A 2 John Prussing wants you to recall the Fibbonacci numbers defined by: $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$. This sequence begins 1,1,2,3,5,8,. . . . The problem is to prove that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

M/A 3 Jack Parsons has a problem derived from Lewis Carroll: A frictionless train runs by gravity in a *straight* tunnel between two points on the earth's surface. Find the maximum velocity and the time for a round trip. Show that the latter is independent of the length of the tunnel.

M/A 4 The following entertaining problem is from Dr. Ben Whang: A palindrome is, of course, a word or a sentence which spells the same thing when spelled backwards. Among the notable are:

Raddar
Rotator
Madam, I'm Adam.
Able was I ere I saw Elba.
Lewd did I live & evil I did dwel'.
A man, a plan, a canal — PANAMA.

Extending the concept, Dr. Whang became curious about the word to describe a word (or a sentence) which spelled another word (or a sentence) when spelled backwards. For example,
deer, stop
devil, sung,
reviled, repaid,
reward, etc. (There are hundreds more.)

After doing a bit of research at the Library of Congress, and running into words like apocope, epenthesis, and metathesis, Dr. Whang found that the word he is looking for does not exist. Therefore, he has decided to coin it! The word is "drow." Its preferred pronunciation rhymes with "brow." A palindrome then would be a special case of a drow. The beauty of "drow" is that it is a drow itself, while the word palindrome is not a palindrome. The longest drow Dr. Whang knows has eight letters. He wants *Review* readers to find the word. And, he writes, "I wish my last name was Wang, which of course is a drow."

M/A 5 This problem, from Bill Saidell, is reminiscent of Perm 1: Construct the integers from 1 to 30 using four 4s. For example $4 = \sqrt{4} + 4 + 4 + 4$. Note that the "greatest integer function" is not allowed.

Speed Department

M/A SD1 A timely problem from Eric Jamin:
Z.P.G.? It is well known that all families want a son. The method to satisfy people and limit population growth is thus obvious: You may have children until you have a son, but then you must stop having children! What will the reproduction rate of the population be? How will its male-female ratio, starting at one, evolve? (Assume monogamy, a maximum of marriages with no remarriage, immortality for married people before a son is obtained, no twin births, no other limits on family size than the one stated.)

M/A SD2 While "flying from N.O. to N.Y." Joseph Horton wondered how far away the horizon was, for a given altitude. Presumably Mr. Horton was in an airplane at the time.

Solutions

DEC 1 As mentioned in the introduction, the Kings were misprinted as Queens. Solutions will appear in the July/August issue.

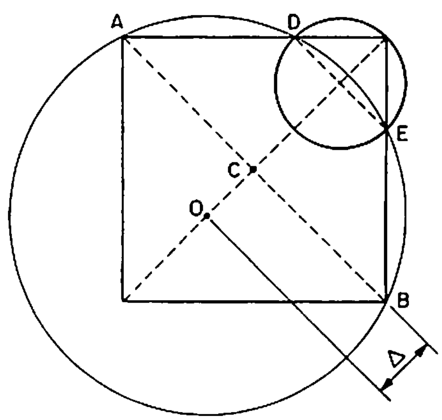
DEC 2 What is the minimum total area of two circles which cover a unit square? Three circles? Four circles?

Most people felt that the best one can do is to use one circumscribing circle and the rest null circles (radius zero). If null circles are technically excluded, most proposed small circles (radius 1/N, N large). Thus the minimum area would be

$$\tfrac{1}{2}\pi + (K - 1)\pi/N^2 \approx \tfrac{1}{2}\pi$$

where K = 2,3,4 is the required number of circles.

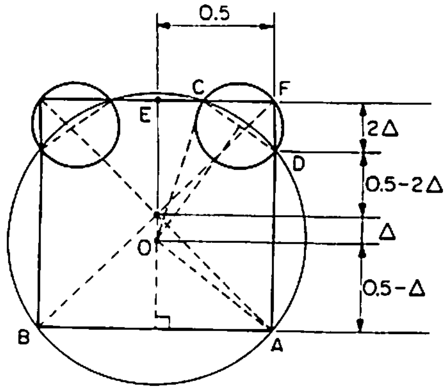
Harry Zaremba, however, achieves a smaller area for K = 3. His solution follows:



In the two-circle case, the large circle is drawn through A and B with its center at O, a distance Δ from the geometric center C of the square. The small circle is drawn with its radius equal to one half of the intercepted chord DE. If Δ is decreased, the area of the small circle can be made as small as it is desired to imagine. Since AB $= \sqrt{2}$, the limit of the area of the two circles will be the following minimum:

$$A = \pi/4(\sqrt{2})^2 = \pi/2 = 1.570796.$$

In the three-circle case, the large circle is drawn through A and B with its center O a distance Δ below the center of the square. The centers of the two small circles are at



the midpoints of the chords intercepted between the sides of the square. From the figure, the radius

$$OA = OC = [(0.5)^2 + (0.5 - \Delta)^2]^{1/2}$$

$$EC = (OC^2 - OE^2)^{1/2} = [(0.5)^2 - (0.5 - \Delta)^2 - (0.5 + \Delta)^2]^{1/2} = (0.25 - 2\Delta)^{1/2}$$

$$CF = EF - EC = 0.5 - (0.25 - 2\Delta)^{1/2}$$

$$CD = (CF^2 - FD^2)^{1/2} = \{[0.5 - (0.25 - 2\Delta)^{1/2}]^2 + (2\Delta)^2\}^{1/2}.$$

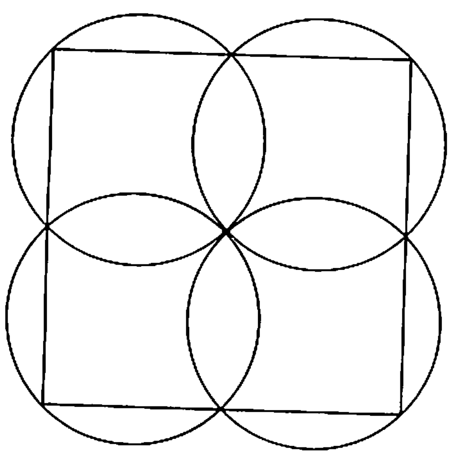
The area of the circles,

$$A = \pi[OA^2 + 2(CD/2)^2] = \pi[0.75 - 2\Delta + 3\Delta^2 - 0.5(0.25 - 2\Delta)^{1/2}].$$

Setting the derivative of A with respect to Δ to zero, and simplifying:

$$288\Delta^3 - 228\Delta^2 + 56\Delta - 3 = 0.$$

Solving for Δ gives $\Delta = 0.07355$. Substituting in the expression for A yields the minimum area A = 1.44117.



In the four-circle case, the radius of each circle $R = \sqrt{2}/2$. The total minimum area of the circles

$$A = 4\pi(\sqrt{2}/4)^2 = \pi/2;\ A = 1.570796.$$

The result is the same for two and eight circles.

Also solved by Neil Cohen, R. Robinson Rowe, Winslow H. Hartford, Ralph Menikoff, Abe Schwartz, Joseph Horton, William J. Butler, Jr., Gerald Blum, A. Stephen Tepper, and Richard I. Hess.

DEC 3 A problem concerning words containing sequences of letters in alphabetical order: What is the fewest number of words needed so that the consecutive strings use all 26 letters?

I did not accept the names of organic chemicals; given this exclusion, the minimal solutions used five words; the following list is from Richard I. Hess:

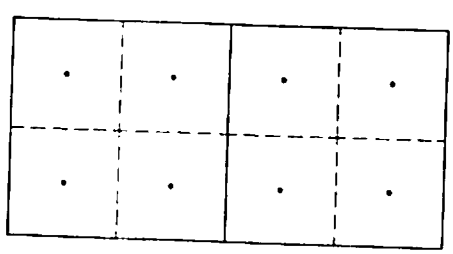
Bright-faced	(A to I)
Lumberjack	(J to M)
Propinquities	(P to U)
Vow	(V, W)
Oxygenize	(X to Z)

Also solved by Gerald Blum, William J. Butler, Jr., Abe Schwartz, Ralph Wanger, Winslow H. Hartford, S. J. Warner and Virginia S. Glessner, R. Robinson Rowe, Harvey Elentuck, William F. Nornick, George H. Ropes, Avi Ornstein, Dave Rabinowitz, Hugh W. Thompson, Emmet J. Duffy, Harry Zaremba, and the proposer, Don Forman.

DEC 4 Take two unit squares and place them side by side so that they form a rectangle two units wide and one unit high. Now choose a point A at random in the first square and, again at random, a point B in the second square. Then A and B will be a certain distance apart. Question: If we repeat the random choice of A and B many times, how far apart will they be on the average?

Before presenting a solution, let me clarify the origin of this problem by reprinting a letter I received from Art Schact: "I was a little startled to see my name in "Puzzle Corner" as having submitted the problem. Association of my name with this problem stems presumably from the news release I prepared about it, a copy of which is enclosed. The placing of my name in the upper right corner, no doubt the cause of the confusion, was intended simply to provide editors with a point of contact. You will see from the text of the release that the problem was actually proposed by Charles R. Johnson and solved by Hans J. Oser, both of National Bureau of Standards. Perhaps you could give credit to these gentlemen in the March/April issue, when you plan to publish the answers."

Several readers gave approximate answers obtained by drawing a grid on each square, connecting the center of each subsquare of square A to the center of each subsquare of square B, and averaging the lengths. For example, an order two grid is



Note that one has 2^2 dots to connect to 2^2 dots giving 2^4 lines. William J. Butler, Jr., carried this out to order 12 ($12^4 = 20,736$ lines!) and obtained the approximation 1.08750. This took ten *hours* on an HP55 calculator. Dr. Oser solved the problem by integrating

$$\int_0^1\int_{-1}^0\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}[(x1 - x2)^2 + (y1 - y2)^2]^{1/2}\,dx1\,dx2\,dy1\,dy2.$$

His solution appears in *SIAM Review*. Several readers set up the above quadruple integral but none solved it analytically. Steve Hirshman and John E. Prussing used probability theory to solve the problem. Dr. Prussing's answer is

$$29/30 - \sqrt{2}/15 - \{4 \log [(\sqrt{5} - 1)/2]\}/3 + \{\log [(1 + \sqrt{5})/(3 - \sqrt{5})]\}/6 + \log [(\sqrt{2} - 1)/(\sqrt{2} + 1)] - \sqrt{5},$$

which has the decimal approximation 1.08813825. The solution is obtained as follows: Let the cartesian coordinates of the two points be (x1,y1) and (x2,y2) where x1, y1, and y2 are in [0,1] and x2 is in [1,2]. The distance function d is then $[(x2 - x1)^2 + (y2 - y1)^2]^{1/2}$.

Define a = y2 − y1 and b = x2 − x1. Then a is in [−1,1] and b is in [0,2]. The distance function d is then $(a^2 + b^2)^{1/2}$. The probability density functions for a and b can be calculated using the Convolution Theorem and the fact that the xk and yk are uniformly distributed random variables. The result is that the density function for a, f(a), is

$$f(a) = \begin{cases} 1 + a & -1 \le a \le 0 \\ 1 - a & 0 \le a \le 1 \end{cases}$$

Similarly for b we get

$$f(b) = \begin{cases} b & 0 \le b \le 1 \\ 2 - b & 1 \le b \le 2 \end{cases}$$

Since a and b are independent, the joint density function is f(a,b) = f(a)f(b).

The expected value of d is obtained by integration of the product of d with the joint density f(a,b) over the appropriate domain:

$$\begin{aligned} E(d) &= \int_0^2 db \int_{-1}^1 da\, \sqrt{a^2 + b^2}\, f(a,b) \\ &= \int_0^1 db \int_{-1}^0 da\, \sqrt{a^2 + b^2}\,(1 + a)b + \\ &\quad \int_0^1 db \int_0^1 da\, \sqrt{a^2 + b^2}\,(1 - a)b + \\ &\quad \int_1^2 db \int_{-1}^0 da\, \sqrt{a^2 + b^2}\,(1 + a)(2 - b) + \\ &\quad \int_1^2 db \int_0^1 da\, \sqrt{a^2 + b^2}\,(1 - a)(2 - b) \end{aligned}$$

Note that the first two integrals are equal and the last two integrals are equal. These integrals can be evaluated analytically by transforming to polar coordinates: a = r · cos t; b = r · sin t. The evaluation is straightforward (but tedious) and results in the answer given at the beginning of the solution.

Also solved by R. Robinson Rowe, Winslow H. Hartford, Harry Zaremba, Neil Cohen, Ralph Wanger, Ralph Menikoff, Joseph Horton, Neil Hopkins, Norman Wickerstrand, and Richard I. Hess.

DEC 5 Show that there is an infinity of trigonometric functions of real numbers that are algebraic numbers. Specifically, given any real number a, where cos a is an algebraic number; if T is any direct trigonometric function, show that T(ma/n) is an algebraic number for all integers m and n.

This amazing result is actually rather easy. The key is that sin(nx) is a polynomial in sin(x) since sin(a + b) = sin(a)cos(b) + cos(a)sin(b). (If n is negative use sin(x) = −sin(−x)). R. Robinson Rowe supplies a polished answer:

We note first that if any trigonometric function of an angle is an algebraic number, so are all the others. Let a be a tabulated angle; then any other tabulated angle can be represented by Na/D, where N/D is a proper rational fraction. Also let S = sin a and C = cos a. Then using the addition formulae for sine and cosine, we have:

$$\sin 2a = 2SC, \qquad \sin 3a = 3S - 4S^3,$$
$$\sin 4a = 4SC(1 - 2S^2), \qquad \sin 5a = 5S - 20S^3 + 16S^5$$

and so on. Thus sin Na is always an algebraic number. If N is odd, it is expressed in powers of S alone; if N is even, it is expressed as C times powers of S. But if we square an N-even expression and substitute $1 - S^2$ for C^2, we have an expression in powers of S alone. For example, this procedure yields

$$\begin{aligned}\sin^2 4a &= 16S^2(1 - S^2)(1 - 2S^2)^2 \\ &= 16S^2 - 80S^4 + 128S^6 - 64S^8\end{aligned}$$

Now, generally, consider the character of sin Na/D = sin x = X. Depending upon the parity of N, we have either:

$$\sin Dx = \sin Na = AX + BX^3 + CX^5 + \ldots JX^N$$
$$\text{or } \sin^2 Dx = \sin^2 Na = AX^2 + BX^4 + CX^6 + \ldots JX^{2N},$$

where A, B, C, etc. are positive or negative integers. In either case we have an algebraic equation in X and its roots will be algebraic numbers.

Also solved by: William J. Butler, Jr., Richard I. Hess, Winslow H. Hartford, Philip O. Martel, Neil Cohen, and the proposer, Eugene W. Sard.

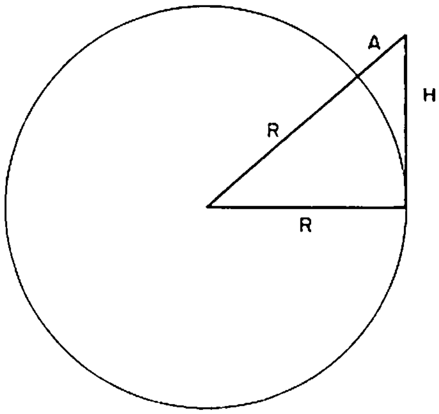
Proposer's Solutions to Speed Problems

M/A SD1 The answer is shorter than the question! Each family has one son; since boys and girls are born with equal probability, each family has *an average* one daughter. Hence: reproduction rate: 1 for 1 (Z.P.G.!!!). Male-female ratio: constant at 1.

M/A SD2

R = Radius of earth

$H = (2AR + A^2)^{\frac{1}{2}}$



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Book Reviews

Continued from p. 13

temporal order of variables is clear. For example, a city's socio-economic conditions precede a mayor's adoption of an urban renewal plan; i.e., "city" is prior to "agenda." Over longer periods, it is true, a more dynamic relationship may be seen, and here the systems perspective is useful. But surely no one, including mayors, can avoid making causal inferences about the world, and these are better made explicit.

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