Welcome to a French Ally

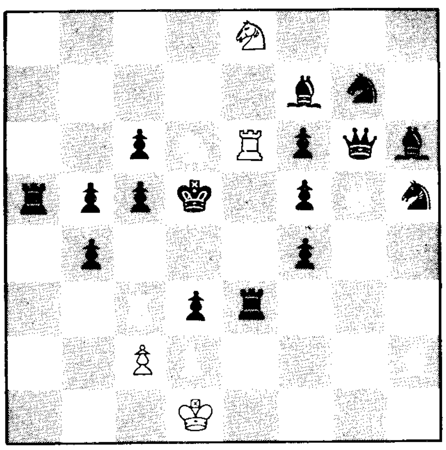
Puzzle Corner
by
Allan J. Gottlieb

Spring rugby season ended, and my first year is highlighted by two impressive statistics:
— I finished every game in which I started.
— I ate (and retained) lunch prior to each game.
Next year I hope we will have a winning record and that I will score.

PERM 1 die-hards will be glad to know that the problem has been taken up by Pierre Berloquin in *Science et Vie*. I am sure that he would appreciate correspondence from any interested reader (11, rue Perronet, 92200 Neuilly, France). Speaking of problems involving large numbers, anyone knowing how to balance a budget might submit a solution to Mr. A. Beame, Gracie Mansion, New York City.

Problems

J/A 1 Breaking the usual alternation, we begin with the following chess problem from Paul Reeves; although not in great supply, chess problems have an edge over their bridge counterparts. The problem: Starting from the beginning position, what is the minimum number of moves required to reach the following (the setting for **JAN 1**):



J/A 2 The following former Putnam examination question was submitted by Ermanno Singorelli: Find integers $0 < a < b$ such that for all pairs of non-negative integers m and n the linear combinations na + nb fail to include exactly 35 positive integers, one of which is 58.

J/A 3 It is well known that there are the same "number" of points in the unit interval $0 \leq x \leq 1$ as in the unit square $0 \leq x, y \leq 1$. Robert M. Saenger would like you to produce an explicit one-to-one correspondence.

J/A 4 As a take-off on **M/A 5**, Fritz Olenberger wants you to find four distinct positive integers such that the sum of any three is a perfect square.

J/A 5 Our final problem this month is an acrostic from Dawn Friedell Jacobs (at the top of the next page).

Speed Department

J/A SD 1 A Bicentennial question from Emmet J. Duffy: How many years elapsed between the signing of the Declaration of Independence and the delivery of the Gettysburg Address?

J/A SD 2 A traffic light near R. E. Crandall's home is alternately red for 30 seconds, then green for 30 seconds. What is the expected wait at the light?

Solutions

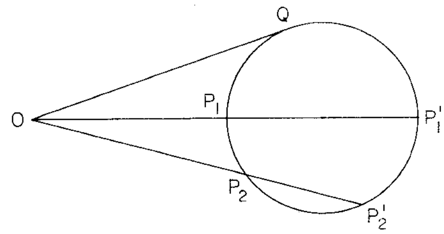
M/A 1 Black and White are to cooperate to checkmate White in the fewest possible moves, starting from the standard beginning position. What are the moves, if Black is constrained to move only one pawn and White to move one pawn and one other piece only once?

The following five-move solution from Al Gregory is typical:

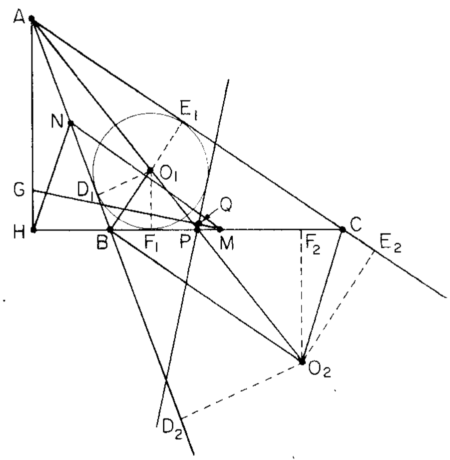
1. P-QB3 P-QN4
2. Q-R4 P x Q
3. P-QB4 P-R6
4. P-B5 P x P
5. P-B6 P x B (mate)

Also solved by F. Steele Blackall, William Butler, Neil Hopkins, Winthrop Leeds, Harry Movitz, and Paul Reeves.

M/A 2 Consider a triangle; prove that the three midpoints of the sides, the three basepoints of the altitudes, and the three midpoints of the segments joining the vertices to the orthocenter (the common intersection of the three altitudes) all lie on one circle. Show that the "nine-point circle" defined above is tangent to the inscribed circle and to the exinscribed circles.



Only the proposer was willing to tackle this one. Eric Jamin's solution:
It suffices to show the theorem for one exinscribed circle. The whole idea is to cook up an inversion (which keeps the properties of tangency) leaving circles 0_1 and 0_2 invariant and inverting the nine-point circle into the second common tangent to 0_1 and 0_2. We recall what an inversion is: an inversion of center 0 and power a^2 is to map a point P into the point P′ on the ray OP satisfying $OP \cdot OP' = a^2$. It is based on the theorem of the power of a point vs. a circle, viz.: $OP_1 \cdot OP'_1 = OP_2 \cdot OP'_2 = OQ^2$. Note that circles are inverted into circles except that a circle through the center of the inversion is inverted into a line and, conversely, lines passing through the center of the inversion are inverted into themselves.



To prove Feuerbach's Theorem, given triangle ABC, construct bisectors AO_1, BO_1, CO_1 (internal), BO_2, and CO_2 (external) and perpendiculars on the sides O_1D_1, O_1E_1, O_1F_1, O_2D_2, O_2E_2, and O_2F_2. AO_1O_2 cuts BC in P, AH is perpendicular

1-B	2-F		3-A	4-H	5-Q		6-A	7-C	8-N		9-D	10-E
	11-K		12-Q	13-O	14-S	15-C	16-P	17-L	18-M	19-I		20-C
21-E		22-Q	23-K	24-G	25-J	26-P	27-S	28-I	29-D	30-F	31-H	
32-J	33-B	34-P		35-I	36-A	37-C	38-E	39-Q	40-K	41-R		42-D
43-P	44-C	45-F	46-J	47-E		48-D	49-S	50-O		51-D	52-R	53-M
54-J	55-Q	56-N	57-K	58-I	59-C	60-E		61-G	62-K		63-S	64-A
65-M	66-C	67-E	68-I	69-R	70-O	71-H		72-D	73-G	74-O	75-S	76-Q
77-N	78-L	79-J	80-I	81-K	82-O	83-D	84-P		85-G	86-R	87-C	88-M
89-H	90-F	91-L		92-F	93-Q	94-D	95-N		96-P		97-E	98-R
99-O	100-D	101-S	102-A	103-I	104-B		105-J	106-Q	107-C	108-H	109-L	

A. Highly excited	64	3	6	36	102				
B. Creature of genus Bubo	33	1	104						
C. Interpretation	87	44	37	107	66	20	15	7	59
D. Type of data often misused	51	94	29	72	9	100	42	83	48
E. Ultimate victor	38	10	60	67	21	97	47		
F. Dryer	30	45	92	2	90				
G. Better than none	73	61	24	85					
H. Swiss mathematician, 1707-83	31	89	71	108	4				
I. Absolute scale	28	103	80	35	68	19	58		
J. Author of best-known law	105	25	46	54	32	79			
K. Neither vegetable nor mineral	11	62	40	57	81	23			
L. Set	109	91	17	78					
M. Magnificent display	65	18	88	53					
N. Suffragette, 1819-1910	95	77	8	56					
O. Freeholder	99	74	13	82	70	50			
P. Scrap or tag	26	34	96	16	84	43			
Q. Rude	93	76	12	106	55	22	39	5	
R. Sofa	41	86	52	69	98				
S. High comedy	14	49	27	101	75	63			

to BC, N is the midpoint of AB, and M is the midpoint of BC. M is the midpoint of F_1F_2; indeed, $D_1D_2 = E_1E_2$ and $D_1D_2 = D_1B + BD_2 = BF_1 + BF_2 = 2BF_1 + F_1F_2$; $E_1E_2 = E_1C + CE_2 = F_1C + CF_2 = 2CF_2 + F_1F_2$. Thus $BF_1 = CF_2$ and, from BM = MC, we get $F_1M = MF_2$. Consider an inversion of center M and power $\overline{MF_1}^2$. Obviously, circles O_1 and O_2 are invariant. The nine-point circle, passing through M, is inverted into a line. P is the inversion of H; thus the inverted nine-point circle passes through P. We have indeed $AO_1/AO_2 = HF_1/HF_2$ = Radius O_1/Radius $O_2 = PF_1/PF_2$. From $HF_1/HF_2 = PF_1/PF_2$ we have

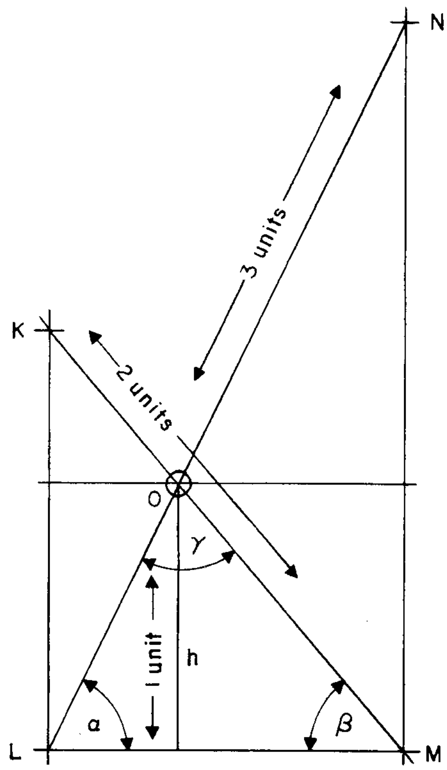
$$\frac{HF_1}{HF_2 - HF_1} = \frac{PF_1}{PF_2 - PF_1}$$

or

$$\frac{HF_1}{F_1F_2} = \frac{PF_1}{[(MF_2 + MP) - (MF_1 - MP)]}$$

Since $F_1F_2 = 2MF_1 = 2MF_2$, $HF_1/2MF_1 = PF_1/2MP$. Thus $(HF_1 + MF_1)/MF_1 = (PF_1 + MP)/MP$ or $MH \cdot MP = \overline{MF_1}^2$. Let G be the midpoint of the segment from A to the orthocenter. G is a point of the nine-point circle on AH. Let PQ be perpendicular to MG. From the similarity of the triangles GHM and PQM, we have $MQ \cdot MG = MP \cdot MH$. So $MQ \cdot MG = \overline{MF_1}^2$. Hence Q is the inversion of G and PQ the inversion of the nine-point circle. Claim that PQ is the second tangent to circles O_1 and O_2, or $\sphericalangle O_1PB = \sphericalangle O_1PQ$ Equivalently, $\sphericalangle O_1PB = \sphericalangle BPQ - \sphericalangle O_1PB$ or $2 \sphericalangle O_1PB = \sphericalangle BPQ$. Let $\sphericalangle A = \sphericalangle BAC$ and $\sphericalangle B = \sphericalangle ABC$. From triangle ABP we must show that $2\ (180 - \sphericalangle B - \tfrac{1}{2}\sphericalangle A) = 180 - \sphericalangle QPM$, or $\sphericalangle HGM = 2 \sphericalangle B + \sphericalangle A - 180$. This is so since $\sphericalangle HGM = \sphericalangle HNM$ (H, G, M, and N are on the nine-point circle and the two angles subtend the same arc) $= \sphericalangle HNB + \sphericalangle BNM = \sphericalangle HNB + \sphericalangle A$ $(MN \| AC)$ $= 180 - 2 \sphericalangle NBH + \sphericalangle A$ (triangle HNB is isosceles) $= 2 \sphericalangle B - 180 + \sphericalangle A$. Conclusion: Since PQ is tangent to circles O_1 and O_2, the inversion gives the nine-point circle tangent to circles O_1 and O_2.

M/A 3 Determine the sides and angles of the triangle LMO (uniquely defined) in the drawing. The two parallel lines KL and MN are perpendicular to the base LM. The height (h) is 1 unit, KM is 2 units, and LN is 3 units.



The following solution is from Paul A. Reeves: In the published triangle and diagram, LK and MN are both perpendicular to LM and therefore parallel. Angle LKO = angle OMN (alternate interior angles). Angle KOL = angle NOM (vertical angles). Triangle LOK is similar to triangle MON (two like angles). OK/OL = OM/ON. OM = OK (ON/OL); OK + OM = OK + OK(ON/OL) =

OK(OL + ON)/OL. OK + OM = KM = 2; OL + ON = LN = 3; 2 = 3 · OK/OL; OK = 2 · OL/3. MO = 2 − OK = 2 − 2 · OL/3 = (6 − 2 · OL)/3 . LO = $1/\sin\alpha$; MO = $1/\sin\theta$ (trigonometrically obvious). $\sin\beta = 1/MO = 3/(6 - 2\cdot OL) = 3/(6 - 2/\sin\alpha) = 3\sin\alpha/(6\sin\alpha - 2)$. $\sin^2\beta = 9\sin^2\alpha/(36\sin^2\alpha - 18\sin\alpha + 4)$. LM = KM $\cos\beta = 2\cos\beta$ or LN $\cos\alpha = 3\cos\alpha$. $\cos^2\beta = 9\cos^2\alpha/4$; $(1 - \sin^2\beta) = (9 - 9\sin^2\alpha)/4$; $\sin^2\beta = (9\sin^2\alpha - 5)/4$. $(9\sin^2\alpha - 5)/4 = 9\sin^2\alpha/(36\sin^2\alpha - 18\sin\alpha + 4)$. $81\sin^4\alpha - 54\sin^3\alpha - 45\sin^2\alpha + 30\sin\alpha - 5 = 0$. $\sin\alpha = 0.91191$ (by Horner's method). $\cos\alpha = (1 - \sin^2\alpha)^{\frac{1}{2}} = 0.41039$. $\cos\beta = 3\cos\alpha/2 = 0.615595$. $\sin\beta = (1 - \cos^2\beta)^{\frac{1}{2}} = 0.78806$. α = Arc sin 0.91191 = 65 46′14″. β = Arc sin 0.78806 = 52 0′17″. $\gamma = 180 - \alpha - \beta$ = 62 13′29″. LO = $1/\sin\alpha$ = 1.09660. OM = $1/\sin\beta$ = 1.26894. LM = $3\cos\alpha$ = 1.23118.

Also solved by William Butler, Emmet Duffy, Winthrop Leeds, John E. Prussing, Dura Sweeney, Norman Wickstrand, Harry Zaremba, and the proposer, Walter G. Walker.

M/A 4 Devise a simple scheme for deciding if a binary number (i.e., a number expressed in base 2) is divisible by 3.

Many readers submitted algorithms for solving this problem. Most were based on congruence mod 3. Walter Penney's solution was selected because he supplied a clear explanation of why the algorithm works: A binary number is divisible by 3 if and only if the sum of the bits in the odd positions minus the sum of the bits in the even positions is divisible by 3. For example, 1 1 0 1 0 1 1 1 0 1 (=861) is divisible by 3 since the bits in the odd positions sum to 5 and the bits in the even positions sum to 2. That this is so can be seen by writing the number as $a + 2b + 4c + 8d + \ldots + 2^n k$, where a, b, c, . . . k are either 1 or 0. This is equivalent to $3(a - b + c - d + \ldots \pm k)$, since powers of 2 are alternately 1 more and 1 less than multiples of 3. Therefore the number will be divisible by 3 if and only if $a - b + c - d + \ldots \pm k$ is divisible by 3 — i.e., if the sum of the bits in the odd positions minus the sum of the bits in the even positions is divisible by 3. This is simply an extension of the rule for divisibility by 11 in base 10, or — for that matter — divisibility by N + 1 in base N. Thus an octal number is divisible by 9 if and only if the sum of the digits in the odd positions minus the sum of the digits in the even positions is divisible by 9. For example, 3241676 in base 8 (=869310) is divisible by 9 since the sum of the digits in the odd positions is 19 and the sum of the digits in the even positions is 10.

Also solved by William Butler, Emmet Duffy, Ed Gershuny, P. V. Hefter, Neil Hopkins, Paul Reeves, and the proposer, D. J. Huntley.

M/A 5 Find three distinct positive integers such that the sum of any two is a square.

Let me pool everyone's results. For integer x, the triple $6 - x$, $19 + x$, and $30 + 13x + x^2$ works. Any triple multiplied by a square will also be a solution. If a + b is a square and b − a is odd, then a, b, and $[\frac{1}{2}(b - a - 1)]^2 - a$ yield another set of solutions. Finally, $2n^2 - 4n - 6$, $2n^2 + 4n + 6$, and $2n^2 + 12n + 10$ $(n > 3)$; and $2n^2 + 4n$, $2n^2 - 4n$, and $2n^2 + 1$ $(n > 2)$ give more solutions.

Solutions received from William Butler, P. V. Heftner, Neil Hopkins, Winthrop Leeds, Fritz Olenberger, John E. Prussing, Paul Reeves, Harry Zaremba, and the proposer, R. E. Crandall.

Better Late Than Never

1974 M/A4 Walter Sadler has responded.

O/N 3 Frank Rubin and Bob Lutton noticed that a batter was omitted. The batter numbered 8 should be 9 and batter 8, who gets on base in each even-numbered inning, should be inserted.

O/N 4 Emmet Duffy and Joseph Stockert have submitted generalized solutions.

DEC 5 Llewellyn Dougherty, Peter Groot, and Steve Winkler have sent in solutions.

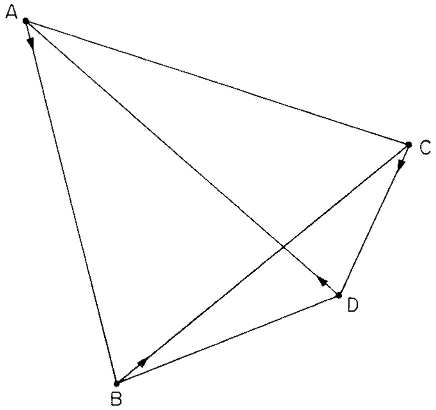
JAN 1 Frank Rubin has sent a solution.

JAN 3 R. B. Stambauch and Vonn Feldman have solved it.

JAN 4 Thomas Warner has responded.

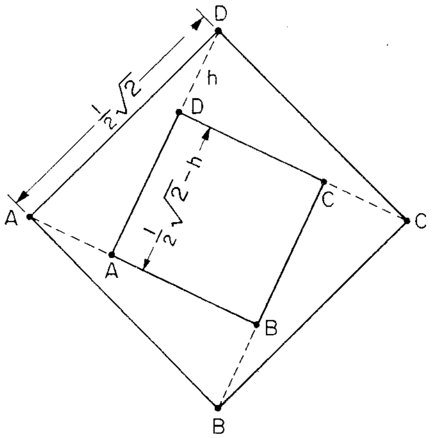
DEC 1 Both Richard Rubin and A. T. Lewis noted that if East covers one spade lead and ducks the other, West can discard the ♠8 and no squeeze results.

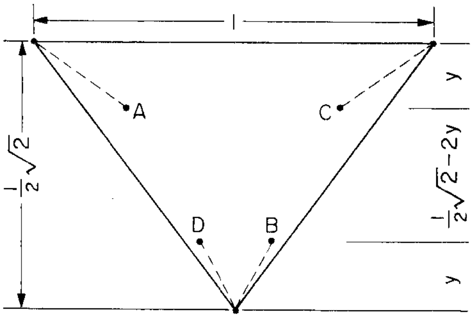
DEC 2 Emmet J. Duffy and the proposer point out that the printed solution fell into the "symmetry trap" of assuming that cos θ is constant. Mr. Duffy's solution follows:



Assume that two birds, A and C, are at the start in an upper horizontal plane one unit apart, and the other two birds, B and D, are in a lower horizontal plane one unit apart as shown in the diagram. Assume that A moves toward B, B toward C, C toward D, and D toward A. At the start the birds are equidistant, so if lines were drawn connecting all birds, they would form the outline of a tetrahedron with equilateral triangles for faces. If A moves an incremental distance ds toward B then, because all angles are are 60°, B moves an incremental distance 0.5 ds toward A and total decrease in distance from A to B is 1.5 ds. The same decrease occurs between B and C, C and D, D and A. A also moves an incremental distance 0.5 ds toward C and C moves an incremental distance 0.5 ds toward A. The total decrease from A to C is ds. The decrease from B to D is also ds. Thus AC and BD decrease less than the other four lines and the four faces change from equilateral to isosceles triangles. The differential equation of motion will now be developed.

HORIZONTAL PROJECTION





VERTICAL PROJECTION

Lines AC and BD will always be equal and they will be at right angles in their projection on a horizontal plane. Lines AB, BC, CD, and DA at any instant will make the same angle with a horizontal plane; hence the projection of A, B, C, D on a horizontal plane will place them at the corners of a square. At the start, the side of the square will be $\frac{1}{2}\sqrt{2}$. In the horizontal plane each bird moves directly toward a bird which has no motion either away from or toward the pursuer. Hence total horizontal distance for each bird is $\frac{1}{2}\sqrt{2}$. If a bird has moved a distance h in the horizontal plane, then the side of the square will be the horizontal distance to be traveled or $\frac{1}{2}\sqrt{2} - h$. At the start the distance between the upper and lower planes is $\frac{1}{2}\sqrt{2}$. If, during the horizontal travel h, the lower birds move up a distance y, then due to symmetry of motion the upper birds will move down a distance y, and the distance between the two planes is $\frac{1}{2}\sqrt{2} - 2y$. The line between a pursuing bird in the lower plane and a pursued bird in the upper plane will make an angle with the lower plane whose tangent is $(\frac{1}{2}\sqrt{2} - 2y)/(\frac{1}{2}\sqrt{2} - h)$. The differential equation of motion is then:

$dy/dh = (\frac{1}{2}\sqrt{2} - 2y)/(\frac{1}{2}\sqrt{2} - h)$. The solution of this equation is $y = h - \frac{1}{2}\sqrt{2}h^2$. The total distance travelled will be:

$$S = \int_{h=0}^{h=\frac{1}{2}\sqrt{2}} (dy^2 + dh^2)^{\frac{1}{2}}$$

$$= \int_{0}^{\frac{1}{2}\sqrt{2}} [(1 - \sqrt{2}h)^2 + 1]^{\frac{1}{2}} dh\,.$$

Letting $z = 1 - \sqrt{2}h$,

$$S = \int_{1}^{0} -\tfrac{1}{2}\sqrt{2}(z^2 + 1)^{\frac{1}{2}} dz\,.$$

From a table of integrals, the integration is:

$$S = -\tfrac{1}{4}\sqrt{2}\, z(z^2 + 1)^{\frac{1}{2}} + \log_e[z + (z^2 + 1)^{\frac{1}{2}}]\Big]_{z=1}^{z=0}$$

$$S = \tfrac{1}{2} + [\log_e (1 + \sqrt{2})]\tfrac{1}{4}\sqrt{2} = 0.8116\,.$$

DEC 5 What a coincidence. The third December problem in which two readers made the same comment. William Butler and Hal Vose point out that we could have saved 14.5 gallons had we carefully eliminated the 0.0172648 "gallons to spare."

M/A SD 2 Joseph Horton points out that, allowing negative powers, one has the solution 0124 (or 1024 if no leading zeros) which "contains" ¼ ½ 1 2 4.

Responses have come to four problems published in February:

FEB 1 Jeffrey C. MacGilliuray and Captain John Woolston.

FEB 2 Robert Lutton, Soo Tang Tan, and Captain John Woolston.

FEB 3 and **FEB 5** Robert Lutton and Captain John Woolston.

Proposers' Solutions to Speed Problems

SD 1 Four score and seven (naturally).

SD 2 7.5 seconds.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973); he is now Assistant Professor of Mathematics at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, 150-14 Jamaica Ave., Jamaica, N.Y., 11432.

Letters

Continued from p.4

late, somewhat, the operation of a steam engine has been shown to have no significant beneficial effect on fuel economy. In actual practice one has to be concerned with handling the water and emulsions in cold climates, the cold-start problem with emulsions, and the effect of water on NO, CO, and hydrocarbon emissions.

The effect of water addition will be to