Gosubs, Foul Balls, and Concentric Circles

Puzzle Corner
by
Allan J. Gottlieb

Hi. Perhaps I can make use of my large captive and talented audience. A colleague of mine, Jerry Stoodley, and I were discussing photography last week, and an interesting question in "practical" optics arose. One can use a bellows to convert a normal camera lens into one suitable for close-up work, and one can achieve essentially any magnification ratio by using a long enough bellows or an extension tube. I was told by a professional photographer, whose ignorance of optics I could easily document, that one "should" mount the lens in reverse position when magnification ratios exceed 1:1. This fact seems to be verified by checking some charts put out by Nikon for use with their bellows. Of course, no explanation is offered. Is there some legitimate reason for this, or is it just a myth started by photographic companies to entice people into buying the necessary adapters to mount their lenses in reverse position? Your enticed editor would most appreciate a clear explanation.

Problems

FEB 1 This month we start with an unsolved bridge problem from Charles E. Blair: In *Bridge in the Menagerie,* Victor Mollo gives a "curious hand: Both sides can make four hearts." Actually, the deal is one in which both sides can make four hearts with more or less plausible misdefense. I wonder how close one can get if one insists on best defense. A deal on which both sides make seven no-trump is easy, but can one deal a hand on which both sides make even two hearts?

FEB 2 Frank Rubin wants you to find the set of positive real numbers whose sum is 100 and whose product is maximal.

FEB 3 I suppose this could be called a wordy number problem: If the numbers from 1 to 5,000 are listed in equivalence classes according to the number of written characters (including blanks and hyphens) needed to write them out in full in correct English, there are exactly 40 such nonempty classes. For example, class "4" has three elements (4, 5, and 9) and class "42" has nine (3373, 3377, 3773, 3777, 3378, 3778, 3873, 3878, and 3877). There is a class with exactly one number; what is it?

FEB 4 Jerry Stoodley and I enjoyed solving this one. A little practice with computers or recursive functions would be helpful. Thank you, Dave Kaufman: In the programming language BASIC, each line is numbered and the subroutine call is called GOSUB. It transfers control to a specified line number, as in
10 GOSUB 20.

Control continues as usual from there until a RETURN instruction is read, when control is passed back to the line *following* the GOSUB. When several GOSUBs are executed without intervening RETURNs, they are *stacked*; that is, a RETURN returns to the line following the *latest* pending GOSUB which is then removed from the stack. The next RETURN encountered refers to the previously pending GOSUB, which is then removed. And so on. Assuming a RETURN without pending GOSUB is illegal, can you prove the legality or illegality of this program:

```
10 GOSUB 20
20 GOSUB 30
30 GOSUB 40
40 GOSUB 50
50 GOSUB 60
60 GOSUB 70
70 GOSUB 80
80 GOSUB 90
90 RETURN
99 END
```

Another question: How many GOSUBs were executed?

FEB 5 A magic hypercube problem from Eric Jamin: Everyone knows the 3 x 3 magic square with integers 1 through 9 used once each. Can you build a 3 x 3 x 3 magic cube using the integers 1 through 27 once each? How about a magic hypercube using the integers 1 through 81 once each?

Speed Department

FEB SD 1 Our first speed problem this month is from John T. Rule: Two ferry boats ply back and forth across a river with constant speeds, turning at the banks without loss of time. They leave opposite shores at the same instant, meet the first time 700 ft. from one shore, continue on their way to the banks, return and meet for the second time 400 ft. from the opposite shore. As an oral exercise, determine the width of the river.

FEB SD 2 How far can you lower an endless rope into a bottomless hole? (Think of a real rope, not a theoretical one.)

Solutions

The following are solutions to problems published in October/November.

O/N 1 Black and White are to cooperate to checkmate White in the fewest possible moves, starting from the standard beginning position. What are the moves if Black is constrained to move only one piece with which he may neither capture nor give check (he may, of course, mate with the piece)?

The famous fool's mate is not possible, as Black moves two pieces (King's Pawn and Queen). Most readers (and the editor) submitted three-move solutions similar to the following from Mark Sinz:

1	P-KN3	N-QB3
2	P-K3	N-Q5
3	N-K2	N-B6.

The proposer had a different meaning in mind for the phrase, "may neither capture nor give check," from that which I understood. I added the portion of the problem in parentheses, because that was the only way I could see to understand it. Dr. Rubin intended, however, that the checkmate would have to be by discovery — i.e., discovering check is not giving check. This makes the problem substantially more challenging, so I will not supply an answer to this modified version until the May issue, to give readers a chance to respond.

Three-move solutions were received from John Epstein, Milton Grossberg, Warren Heller, Frank Model, Anne Taft, and William Wise.

O/N 2 From Pascal's triangle of binomial coefficients arranged in rectangular form, find a formula which yields the value of any element in the array (see the top of the next column). In other words, what is $P_{m,n}$?

n: \ m:	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	4	5
3	1	3	6	10	15
4	1	4	10	20	35
5	1	5	15	35	70

This was an easy problem. All respondents agreed to this. Curiously, however, there was less agreement in their answers. The following solution is from Gerald J. Roskes, who includes an interesting observation on the relationship of the given matrix to another one also derived from Pascal's triangle:

By examining diagonals running from the lower left to the upper right, we see that $P_{m,n}$ is the binomial coefficient defined by

$$P_{m,n} = \binom{m+n-2}{n-1} = \frac{(m+n-2)!}{(n-1)!(m-1)!}$$

Another interesting formula can be obtained by considering the identity

$$(1+x)^{m+n-2} = (1+x)^{m-1}(1+x)^{n-1}.$$

Comparing coefficients of x^{n-1} on both sides of this equation, we find that

$$P_{m,n} = \binom{m+n-2}{n-1} = \sum_{i=0}^{n-1}\binom{m-1}{i}\binom{n-1}{i}, \ (m \geq n).$$

Let B be the upper triangular Pascal matrix defined by

$$B_{i,m} = \begin{cases} 0 \text{ if } i > m, \\ \binom{m-1}{i-1} \text{ if } m \geq i. \end{cases}$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, the above equation for $P_{m,n}$ indicates that

$$P_{m,n} = \sum_{i=0}^{n-1} B_{i+1,m} B_{i+1,n} = \sum_{i=1}^{n} B_{i,m} B_{i,n}, \text{ or } P = B^T B.$$

Thus the Pascal matrix P can be factored into an upper triangular Pascal matrix multiplied by a lower triangular Pascal matrix.

Batters:	Innings: 1	2	3	4	5	6	7	8	9
1	On base	Out		Out		Out		Out	
2	On base	* →	On base	* →	On base	* →	On base	* →	On base
3	On base		On base		On base		On base		On base
4	Out		Out		Out		Out		Out
5	Out		Out		Out		Out		Out
6	* →	On base	* →	On base	* →	On base	* →	On base	Out
7		On base		On base		On base		On base	
8		Out		Out		Out		Out	

* = At bat when runner is thrown out stealing.

Also solved by Theodore Edison, David Green, Milton Grossberg, Winslow Hartford, Mary Lindenberg, Robert C. Lutton, John Prussing, Ben Rouben, R. Robinson Rowe, Frank Rubin, and the proposer, Harry Zaremba.

O/N 3 Two players on a baseball team each hit a foul ball in every inning of a nine-inning game in which their team was shut out. If neither was the lead-off hitter, what were their positions in the batting order, and how did this happen?

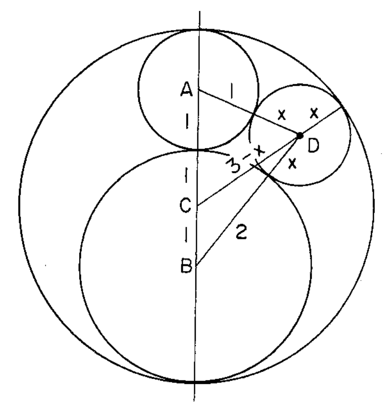
The following solution, in fine diagrammatic form, was submitted by Robert C. Lutton:

The two batters are second and sixth in the batting order. They alternate innings in which one or the other is at bat and has hit a foul ball when a base runner is subsequently thrown out trying to steal, and the same batter leads off the following inning.

The particular order of "Outs" and the "On base" is immaterial, as is the manner of getting on base. However, the runners must not be driven home. Also, the two batters must first hit a foul ball each time before they do whatever is indicated above.

Also solved by: Theodore Engel, Seville Chapman, Milton Grossberg, David Green, M. Kaufman, Rich Rosen, Frank Model, R. Robinson Rowe, Frank Rubin, Leo Sartori, and the proposer, Lars H. Sjodahl.

O/N 4 Given three circles of radii 1, 2, and 3 with the two smaller circles inside the larger, how large a circle can be drawn inside the biggest circle and outside the other two?



For a start, there is a "normal" geometric calculation from Robert Pogoff: The required circle must be tangent to the three given circles. A, B, C, and D are the centers, respectively, of the circles with radii 1, 2, 3, and x. Draw the lines of centers, which intersect the circles at the points of tangency. The segment lengths are:

AD = 1 + x	AC = 2
BD = 2 + x	BC = 1
CD = 3 − x	AB = 3

The area of a triangle with sides a, b, and c and perimeter p (Hero's formula) is

$$A = \sqrt{(p/2)[(p/2) - a][(p/2 - b][(p/2) - c]}$$

In triangle ACD, p = 6; in triangle BCD, p = 6; and in triangle ABD, p = 6 + 2x. Thus,

$$A_{ACD} = \sqrt{3(3-2)(3-1-x)(3-3+x)} = \sqrt{3x(2-x)}$$
$$A_{BCD} = \sqrt{3(3-1)(3-3+x)(3-2-x)} = \sqrt{6x(1-x)}$$
$$A_{ABD} = \sqrt{(3+x)(3+x-3)(3+x-1-x)(3+x-2-x)} = \sqrt{2x(3+x)}$$
$$A_{ABD} = A_{ACD} - A_{BCD}.$$

Thus

$$\sqrt{2x(3+x)} = \sqrt{3x(2-x)} + \sqrt{6x(1-x)}$$
$$2x(3+x) = 3x(2-x) + 6x(1-x) + 2\sqrt{18x^2(2-x)(1-x)}$$
$$6 + 2x - 6 + 3x - 6 + 6x = 2[3\sqrt{2(2-3x-x^2)}]$$
$$(11x-6)^2 = 72(2-3x+x^2)$$
$$49x^2 + 84x - 108 = 0$$
$$(7x-6)(7x+18) = 0$$

Hence x = 6/7.

Now we get fancy. Henry Paynter writes, "Being a geometer by preference, I enclose a solution via geometrical inversion, which can at least be visualized, if not actually executed, entirely mentally. Also, such methods generalize readily and have practical applications to physics and engineering." He then proceeds: suppose we take as the circle of inversion that designated IC in Fig. 1 (at the top of the next page) centered at the point 0 common to the two larger circles (A, B) and having radius R = 4. Then under inversion, using $r' = R_2/r$, these circles map into two parallel lines (A', B') and the small circle (C) outside IC inverts into a smaller circle (C') inside IC, all as indicated in Fig. 2

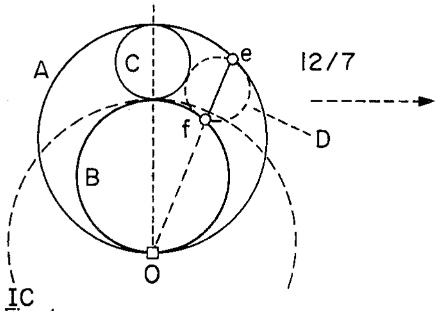


Fig. 1

INVERSE PLANE

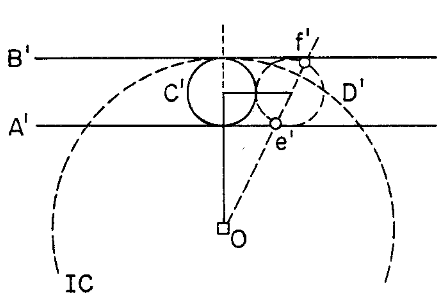


Fig. 2

(see above, right). Clearly in this inverse plane, the solution to the problem is the circle D′, mutually tangent to A′, B′, C′. The diametrically opposite points (e′, f′) along the ray through the center of D′ invert into diametric points (e, f) of the solution circle D, one half of whose separation is the desired answer.

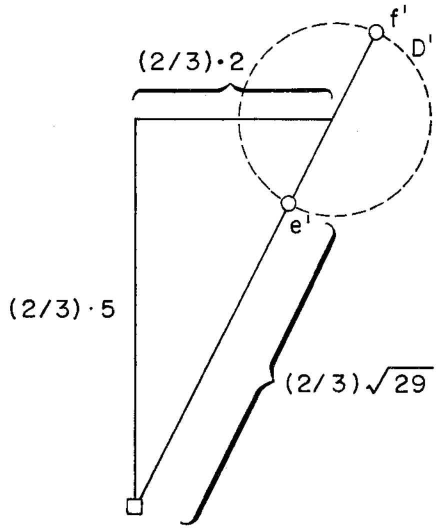


Fig. 3

With reference then to Fig. 3, Pythagoras teaches us that the radial distances of (e′, f′) are $(2/3)(\sqrt{29} - 1)$ and $(2/3)(\sqrt{29} + 1)$, respectively, so that under inversion the corresponding radial distances (e, f) become $(24)\ [1/(\sqrt{29} - 1) - 1/(\sqrt{29} + 1)] = 12/7$ giving us the radius 6/7.

Also solved by Norman Spencer, Daniel Tynan, R. Robinson Rowe, Theodore Engel, Frank Rubin, Milton Grossberg, Robert C. Lutton, Winslow Hartford, Mary Lindenberg, Ben Rouben, Anne Taft, Paul Kaschube, Lars Sjodahl, F. R. Morgan, Raymond Gaillard, David Green, William Cooper, Adam Reed, Hal Vose, Dick Boyd, Norman Wickstrand, Eugene Sard, Harry Zaremba, Waller Moore, Ermanno Signorelli, and Theodore Edison.

O/N 5 Prove that

$$\int_{-\infty}^{\infty} H_{4n+2}(x)\ \mathrm{sech}\,(\tfrac{1}{2}x)\ e^{-\frac{1}{2}x^2}\,dx = 0$$

where H_m, the mth Hermite polynomial, may be defined by

$$H_m(x) = (-1)^m e^{x^2} d^m/dx^m (e^{-x^2}).$$

No solutions have been received for this one.

Better Late Than Never

The solution to the bridge problem published as number **51** (in January, 1972!) is apparently incorrect. Allan Truscot of the *New York Times* has given a defense against which the contract cannot be made. This was pointed out by Frank Model.

DEC 5 (1973) Lars Sjodahl has a counterexample. Of course, he "cheats" by not using a "usual" square but rather one tilted to form a diamond:

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  2
 1 1
2 – 12
 1 1
  2
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M / A 1 Eric Jamin has responded that the minimum boards required are 10 x 10 for even boards and 15 × 15 for odd boards. In addition, he claims that no *closed* tour is possible on a 12 × 12 board but that one is possible on a 14 × 14 board.

MAY 5 An anonymous reader points out that it does not follow that Pete is either younger or older than Henry. Thus it is possible that Pete is at the table with Belinda and Henry is with Joe.

JUN 2, JUN 3 Ben Rouben has responded.

J / A 3 Craig Presson and Kenneth Horton have responded.

O / N SD 2 The solution given is not the most general. The general solution is a = mp(n + p), b = mn(n + p), and c = mnp, with m, n, and p positive integers.

PERM 1 shall rest in peace until the arrival of **Y1975** (see last issue).

Proposers' Solutions to Speed Problems

FEB SD 1 1,700 ft.

FEB SD 2 The distance obtained by dividing the breaking strength by the weight per unit length.

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