The Hanky Panky? Henry, or Pete?

Puzzle Corner
by
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It is hard for me to believe that this is the start of the ninth year of "Puzzle Corner" in *Technology Review*—plus one year in *Tech Engineering News*, the student engineering magazine. But it is—and so welcome back regular readers, and a welcome to newcomers, too.

For the latter, here are the ground rules: Each month we publish five problems and several "speed problems," selected from those suggested by readers. The first selection each month will be either a bridge or a chess problem. We ask readers to send us their solutions to each problem, and three issues later we select for publication one of the answers—if any—to each problem, and we publish the names of other readers submitting correct answers. Answers received too late or additional comments of special interest are published as space permits under "Better Late Than Never." And I cannot respond to readers' queries except through the column itself.

Here goes.

Problems

O/N 1 This month we begin with a chess problem from Frank Rubin. Black and White are to cooperate to checkmate White in the fewest possible moves, starting from the standard beginning position. What are the moves if Black is constrained to move only one piece with which he may neither capture nor give check (he may, of course, mate with the piece)?

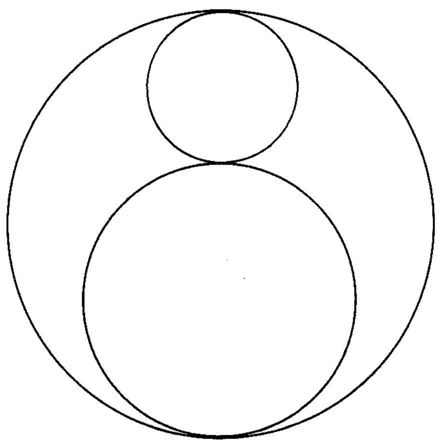
O/N 2 The following problem is from Harry Zaremba: From Pascal's triangle of binomial coefficients arranged in rectangular form, find a formula which yields the value of any element in the array.

n↓ \ m→	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	4	5
3	1	3	6	10	15
4	1	4	10	20	35
5	1	5	15	35	70

In other words, what is $P_{m,n}$?

O/N 3 Lars H. Sjodahl has submitted the following baseball problem: Two players on a baseball team each hit a foul ball in every inning of a nine-inning game in which their team was shut out. If neither was the lead-off hitter, what were their positions in the batting order, and how did this happen?

O/N 4 The following geometry problem was given to me by my York College colleague Gerald Stoodley: Given three circles of radii 1, 2, and 3 as in the diagram, how large a circle can be drawn inside the biggest circle and outside the other two?



O/N 5 An anonymous reader wants a proof that

$$\int_{-\infty}^{\infty} H_{4n+2}(x)\ \mathrm{sech}\left(\frac{1}{2}x\right) e^{-\frac{1}{2}x^2}\ dx = 0$$

where H_m, the mth Hermite polynomial, may be defined by

$H_m(x) = (-1)^m e^{x^2} d^m/dx^m\,(e^{-x^2})$.

Speed Department

O/N SD1 Bob Baird wonders in how many trailing zeros does 100! end?

O/N SD2 The following is from Frank Rubin: One pipe fills a tank in a hours, a second fills the same tank in b hours. When working together, they fill the tank in c hours. Find all sets of positive integers a, b, and c satisfying these conditions.

Solutions

The following are solutions to problems published in the May issue.

MAY 1 Given the following hands,

♠ K 3 2
♥ 7 5 2
♦ J 5 3
♣ K Q 7 2
♠ A Q J 10 5
♥ K 6 4
♦ A Q 7
♣ A J

North-South, playing the precision club system, arrive at an optimistic contract of six spades:

N	E	S	W
		1C	1H
DBLE	P	1S	P
2S	P	3H	P
4C	P	4D	P
6S	P	P	P

The opening lead is ♥A, followed by ♥Q. East follows to the second heart as South wins with the ♥K. How can South make the contract?

Everyone agrees that there are possible distributions for which the contract cannot be made. Each person made various assumptions about card placements. In light of the bidding I rejected the possibility that the ♦K is finessible. As I have said, I am not an expert in bridge; but Michael Kay (the proposer) seems to me to have the best solution, especially in light of the "historical" information supplied. His solution is as follows:

West probably has the ♦K for his (light) overcall, so the diamond finesse is not a percentage play. However, if West held five hearts for his overcall (not unreasonable), and if he holds any four clubs or exactly ♣10, ♣9, and ♣8 if he holds only three, the hand can be made. Five rounds of spades are played, leading to this position as the last one is led:

♠ –
♥ 7
♦ J 5
♣ K Q 7 2

♠ –		♠ –	♠ –		♠ –
♥ J	or	♥ J	♥ –	or	♥ –
♦ K x		♦ K x x	♦ x x x x		♦ x x x
♣ 10 x x x		♣ 10 9 8	♣ 9 8 x		♣ x x x x

♠ 5
♥ 6
♦ A Q 7
♣ A J

West is squeezed on the ♠5 if he holds four clubs; a heart or diamond pitch subjects him to a diamond-club or heart-club squeeze. A club discard is played exactly like the ♣10-♣9-♣8 holding: North discards the ♦5. South cashes the ♣A and overtakes the ♦J with the ♦Q. The ♣K clears the suit (or establishes the ♣7), and the lead of the ♣7 squeezes West in the red suits—South holding the ♦A and ♦Q and dummy and the ♥7 and ♦J. West actually held the ♣10, ♣9, and ♣8 when this hand was played in a duplicate tournament. North-South scored a top board for making six spades." (The proposer was South.—Ed.)

Also solved by Ernest Bivans, Winslow Hartford, and R. Robinson Rowe.

MAY 2 Prove that **PERM 1** (Use each of the four digits 1, 9, 7, and 3 exactly once, and using any mathematical symbols, construct expressions yielding as many numbers, beginning with 1, as possible.) can be solved for all integers; or, letting n be any integer greater than 2, prove that the set of numbers

$$\sqrt{\sqrt{\cdots\sqrt{n!!\cdots!}}}$$

is dense on the interval $[1, \infty)$.

There is a great deal to report on this one. First of all, I received the following letter from Juan Maran which proves that it is possible to construct all numbers in **PERM 1** if one uses greatest integer; thus I will cease to publish such numbers unless a flaw is found in solution c below. Juan Maran writes:

My family has come up with the following solutions:

a. My wife Juanita has decided that the solution

1, 9—7, 3, . . .

is elegant, terse, and ideal since it exhibits all the solutions simultaneously.

b. My own answer uses only the universally recognized Maran-Peano successor function (or the interrogative factorial as many of you know it): x? which is defined as the least integer greater than x. Then:

$$1 = 1 \cdot (3 + 7 - 9)$$

$$2 = 1? \cdot (3 + 7 - 9)$$

$$3 = 1?? \cdot (3 + 7 - 9) \text{ etc.}$$

These expressions have the advantage of needing but the number one to generate any other, and it is certainly "algorithmic." The only disadvantage I can see is that expressions of large integers tend to be overly inquisitive.

c. My son Juan Jr. claims that he can approximate *any number* greater than (or equal to) one using only the three numbers 1, $a \geq 0$, and $b > 1$. His expressions do not use the (much too exuberant) factorial sign or the greatest integer function. Due to the paper shortage, he represents the admissible expression

$$\sqrt{\sqrt{x}}$$

(iterated n times) by $S_n(x)$. (I think the use of the symbol $\sqrt{\ }$ is cheating but he insists I include his effort.)

First note that $(\sqrt{y} - 1)^2 \geq 0$ so that

$$\frac{1}{2}(y - 1) \geq \sqrt{y} - 1, \text{ and}$$

$$\frac{1}{2}(\sqrt{y} + 1) - 1$$

$$= \frac{1}{2}(\sqrt{y} - 1) \leq \frac{1}{4}(y - 1)$$

$$= \frac{1}{2}\left(\frac{1}{2}(y + 1) - 1\right).$$

Iterating, $\frac{1}{2}(S_n(b) + 1)$

$$- 1 \leq 2^{-n}\left(\frac{1}{2}(b + 1) - 1\right).$$

A positive term series Σa_n converges if and only if the infinite product $\Pi(1 + a_n)$ converges. Applying this theorem with

$b > 1$ to $a_n = \frac{1}{2}(S_n(b) + 1) - 1,$

we see that $\prod_{n=1}^{\infty} \frac{1}{2}(S_n(b) + 1)$

converges since the series

$\Sigma\left(\frac{1}{2}(S_n(b) + 1) - 1\right)$ is bounded

above by the series

$$\left(\frac{1}{2}(b + 1) - 1\right)\Sigma 2^{-n} = \frac{1}{2}(b - 1).$$

Thus, $\frac{a}{S_n(b) - 1}$

$$= \frac{a}{b - 1}(S_n(b) + 1)(S_{n-1}(b) + 1)$$

$\cdots (S_1(b) + 1) = K_n 2^n$ where $K_n > 0$ is an increasing, bounded sequence. This yields:

$$(*)\ S_{m+k}\left(\frac{a}{S_{n\cdot 2^k}(b) - 1}\right)$$

$$= (S_{m+k}(K_{n\cdot 2^k}))2^{\frac{n}{2^m}}.$$

We first approximate $\log_2 c$ as close as we want by the 2-adic fraction $\frac{n}{2^m}$, and then we let k be appropriately large so that $S_k(S_m(K_{n\cdot 2^k}))$ is sufficiently close to one. For these choices,

$(*) \sim 1 \cdot 2^{\log_2 c} = c.$

For **MAY 2**, an exact expression for an arbitrary positive integer n is called for, so that one application of greatest integer brackets is needed. Let $a = 97$, $b = 3$, and $c = n + \frac{1}{2}$ in (*) above. Then $n = [(*)]$.

Several readers have noticed that if logs are allowed instead of greatest integer, all integers may be generated fairly easily. The following from Jack C. Fiore is a good example: In order to generate the positive integer n (or zero), use

$$\frac{\log(\log\sqrt{\sqrt{\sqrt{3}}}/(\log\sqrt{9})}{-\log(7 + 1)}$$

For $-n$, omit the minus sign in the denominator.

Finally, Frank Rubin has a sketch of how one might prove that

$$\sqrt{\cdots\sqrt{n!!\cdots!}}$$

is dense. He wisely denotes the expression with p factorials and q square roots by $E(n,p,q)$ and proceeds as follows: For p greater than 0 we note that $E(n,p,0)$ is the factorial of a large number, M, and express it as 2^{2^x} (x real). Now $E(n,p,q) = 2^{2^{(x-q)}}$. So for every integer i not greater than x and fixed n and p there is a member of the sequence for which

$$2^{2^{i-1}} \leq E(n,p,q) \leq 2^{2^i}$$

Now M! is a product of several prime integers. Therefore, $\log_2$ M! is the sum of $\log_2 p_i$ for p_i prime and not all $p_i = 2$. For $p_i \neq 2$, each such $\log_2 p_i$ is a transcendental number and the fractional part of the sum of several of these terms is essentially random. In particular the sum lies at a random position between 2^r and 2^{r+1}. Thus $x = \log_2 \log_2 M!$ consists of an integer plus a random fraction.

Since M!! and M!!!, etc., involve new primes not in M!, their fractional portions will be independent. Thus for fixed i there are $q_1, q_2, \ldots$ such that $E(n,1,q_1)$, $E(n,2,q_2)$, . . . lie in the interval $I_i = 2^{2^{i-1}}, 2^{2^i}$ at independent random positions. Thus the sequence is dense in I_i for all i. Since the union of the I_i's gives all positive reals, the sequence is dense everywhere.

Responses were also received from Emmet Duffy and Ralph Beaman (see also **PERM 1**, below).

MAY 3 Can any square matrix composed only of zeros and ones of size n by n have determinant no greater than F_n (F for Fibbonacci), where F_n is defined by $F_1 = F_2 = 1$ and for n at least three $F_n = F_{n-1} + F_{n-2}$?

John Prussing and R. Robinson Rowe have algorithms for generating n by n matrices with determinant F_n. But neither has presented a rigorous proof that the matrix so generated has maximal determinant, so the problem is still officially open.

MAY 4 How many different possible bridge auctions (legal sequences of bids) exist?

Amazingly enough the three people who computed the answer obtained the same result. I had expected this to be another case of "majority rules." The following is from Eric Jamin: Call a "true bid" any bid which is not pass (P), double (D), or redouble (RD). Any auction except PPPP includes j true bids, j varying from 1 to 35; 35 is the total number of true bids from one club to seven no-trump. For a given j there are C^j_{35} possible sequences of true bids. Inclusion of P, D, and R gives four possible sequences before the first true bid (—, P, PP, PPP), seven possible sequences after the last true bid (PPP, DPPP, PPDPPP, DRPPP, DPPRPPP, PPDRPPP, PPDPPRPPP), and 21 possible sequences between two consecutive true bids (the above seven with PPP changed to either —, P, or PP). Thus for a given j we have $4 \times 7 \times 21^{j-1} \times C^j_{35}$ possible auctions. Summing over j and adding 1 for PPPP gives A =

$4(22^{35} - 1)/3 + 1$, or A = 12874565034703068312023192611160-9371363122697557.

Frank Rubin and the proposer, Neil Cohen, also obtained this result, and estimates were received from Emmet Duffy, Winslow Hartford, and R. Robinson Rowe. The latter owes me 25 cents since he bet two bits that "Neil Cohen doesn't know the answer to his own problem."

MAY 5. There was this picnic attended by Belinda the wife, Henry her husband, Joe their son, Mimi their daughter, and Pete, Belinda's brother. At some time during the picnic one of the members poured a can of beer over the head of another member. At that time:

1. A man and a woman were at the table.
2. The victim and the guilty one were at the beach.
3. One of the children was in swimming.
4. Belinda and her husband were not together.
5. The victim's twin was not the guilty one.
6. The guilty one was younger than his victim.

Who done it?

Henry gave it to Pete, as the following solution from Alan Faller (he says the problem is "easily solved") illustrates:

Denoting the cast by their initials H, B, J, M, and P, after simply applying clues (1), (3), and (4) and noting that P could be a child, the following six possibilities remain:

	At table	At beach	Swimming	Result
(a)	H, M	B, P	J	x
(b)	H, M	B, J	P	x
(c)	J, B	H, P	M	yes
(d)	J, B	H, M	P	x
(e)	P, B	H, M	J	x
(f)	P, B	H, J	M	x

Clue (6) rules out (d) and (e). Now we must assume that clue (5) solves the problem and that the victim's twin is one of the other characters. The victim must be B, J, M, or B. Case (a) is ruled out by clue (5). If (b) were true then by clue (6) J would be guilty, but then P (cousin of B) would not be a child. Cross that out. Case (f) is ruled out because J is the younger and H has no twin in the scenario. The answer, therefore, is case (c), and Henry (the cad) dumped the beer on his older brother-in law Pete, his wife's twin. The real problem is: Why did he do it? Perhaps your readers could provide some interesting answers to that. My guess is that he was making advances toward Mimi, who was in the water alone while the rest of the family was at the picnic table and that Henry arrived just in time. So Henry wasn't the cad after all. It was Pete!

Also solved by Ernst Bivans, Richard Chapman, Joseph Haubrich, Josh Jaffee, Eric Jamin, Harold Groot, Neil Hopkins, Mary Lindenberg, George and Margaret Marcov, Theodore Mita, Russell Nahigian, Edward Moore, John Prussing, R. Robinson Rowe, Frank Rubin, Don Tymchuck, Avi Ornstein, and the proposer, Jim Cassidy.

Better Late Than Never

PERM 1 As mentioned above (see **MAY 2**), we now have a proof that all numbers are possible if one uses the greatest integer function. Thus I will no longer print solutions using this function. I must, however, mention an astounding effort submitted by Edward and William Wong: they gave solutions for each number from 408 through 1,834; I am truly impressed. The proof of **MAY 2** is fortunate indeed; otherwise my editor would have faced the prospect of printing over 1,400 solutions. I am saving the Wongs' work in case a flaw is found in **MAY 2.**

Meanwhile, the only interest left in **PERM 1** is in solutions not using the greatest integer. Through the July/August issue, we published answers up to 256 with 22 exceptions. Below are listed last year's gaps and improvements submitted this summer:

135	$187 = 7 + (3!!/(1 + \sqrt{9}))$
149	202
152	205
155	206
163	212
166	227
172	229
178	230
	235
$181 = 1 + (3!!/(7 - \sqrt{9}))$	$245 = 7(3!\sqrt{9}! - 1)$
184	254

A few solutions beyond 256 are here and will be published in the December *Review*; meanwhile, let's fill in the gaps above. These results were taken from letters from Eric Jamin, Andrew Seager, Harold Groot, E. W. Kelley, Greg Girolami, Alfred A. Aburte, Jr., M. Kaufman, Jim Marlin, David Mallenbaum, Emmet Duffy, and Stuart D. Casper.

Several suggestions for possible **PERM 2**'s are in hand, one of which may soon be adopted.

FEB 4 Joseph Haubrich writes that Mary Youngquist's solution seems to have left out err, "which would give her a ten-fold homonym, beating out what the *Guinness Book of World Records* gives as the greatest, roz, which is everything from the flower to the plural of the Greek letter."

J/A SD1 John E. Gerli disagrees with the solution as given. He feels that the cream should be added early so that it will float to the top and act as an insulator.

Harold Groot has responded to **JAN 1**, George Uman to **JAN 3**, and Eric Jamin to **FEB 1, M/A 1, M/A 3, M/A 4,** and **M/A 5.**

Proposer's solution to **O/N SD2**, above: Let m and n be arbitrary integers; then $a = m(n + 1)$, $b = mn(n + 1)$, and $c = mn$.

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