The Famous School Girl Problem

Puzzle Corner:
Allan J. Gottlieb

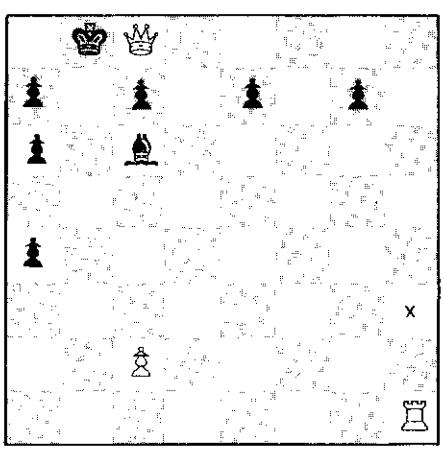
Everything worked out well with my Ph.D. The degree was officially awarded to me on February 1; to celebrate I developed a case of the 'flu; I can only be thankful I didn't get sick a month earlier, when I hadn't a moment to spare. It's hard to believe I'm not a student any more. Oh, well, we all get old sometime.

To answer a popular question: my thesis was in dynamical systems, and my adviser was Mike Shub.

Problems

One of my favorite chess problems comes from Harry Nelson; he recalls that he first encountered it, or one similar to it, in the early 1950s. This is the way it came out when he reconstructed it for some friends in the 1960s:

M/A1 Two neophyte players at the local chess club, under the watchful eye of their teacher who made sure that every move was legal, adjourned their game for lunch, leaving the pieces on the board. When they returned a member of the club standing by their board apologized: "While I was walking by my sleeve happened to brush against one of the men on your board, and it was knocked off amongst the captured men. I don't know which piece it was, but it was on White's KR3 square." No one could remember what the piece was, and they were about to start another game when an old master player happened by. After the situation was explained, he studied the board for a few moments and then said, "There is only one man which could possibly have been on that square." He put it back and the game continued. Here is the position with "x" denoting the unknown man; what is it?



A fairly hard problem comes from my old colleague Mike Rolle:

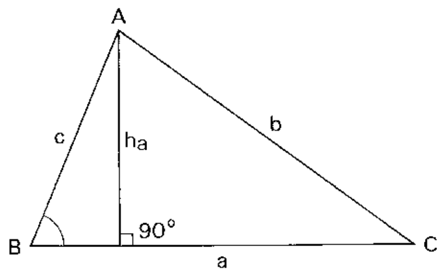
M/A2 Prove the following:
If (1) F is continuous from $(0, \infty) \to (0, \infty)$; and (2) for all $t > 0$, the sequence $F(t), F(2t), F(3t), \ldots \to 0$; then $F(x) \to 0$ as $x \to \infty$ $[x \in (0, \infty)]$.

This combinatorial question comes from Edward J. Sheldon, who calls it "the famous school girl problem." A reference in *Scientific American* in 1959 attributes it to the 19th century:

M/A3 Every day for a week (of seven days) a class of 15 school girls went for a walk. They walked in five rows of three girls each. Each day, each girl had two new "row" mates. How did they do this?

Here is one from Fereidoun Farassat which he says requires "a little knowledge of conic sections (parabola)":

M/A4 Draw a triangle given lengths a, (b — ha), and a and (b — h_a), given angle b, and given that h_a is perpendicular from A to BC.



A number problem from John Hughes:

M/A5 Define $n\Delta = n(n + 1)/2$. When does $n\Delta + 1 = m^2$? Does the following algorithm work for finding the n and m? $n(i) = m(i - 2) * 8 + n(i - 4)$.
Given: n and m are required to be integers.

Speed Department

Here is one from Alan D. Whitney:

SD1 Given the same source at the same temperature and the same cup and the same recipient, why is the second cup of coffee always hotter than the first? The third may be still hotter—but not necessarily so. It is *never* colder.

An electrical engineering problem from William W. Plummer:

SD2 Is there a way to wire three S.P.D.T. ("three-way") switches, two light bulbs, and a power source such that bulb A is on only if all three switches are on and bulb B is on if any of the switches is off? No relays, resistors, or diodes may be used.

Solutions

The following are solutions to problems published in *Technology Review* for December, 1972.

DE-1 Here is a hand actually encountered at the bridge table:

North:
♠ K 10 5
♥ K 10 7
♦ A Q J 4
♣ J 8 3

West:
♠ 8 4
♥ Q 9 8 6 5 4
♦ 6
♣ K 10 5 4

East:
♠ 9 7 2
♥ A
♦ 10 9 8 7 5 3 2
♣ Q 9

South:
♠ A Q J 6 3
♥ J 3 2
♦ K
♣ A 7 6 2

West leads ♦6. How is South to make six spades?

W. Bowman Cutter complains that we "laid out the hands wrong" or he "missed something vital, because it is very hard to find a way *not* to make six spades." Here is his solution:

Take the first trick in the closed hand and draw two trumps using dummy's ♠ K and ♠ 10. Play the ♦ A, ♦ Q and ♦ J, discarding the three small clubs from the declarer's hand. Lead the ♥ 7 to East's ♥ A, discarding any heart from the closed hand. Capture any return East can make in the closed hand and draw the last trump (assuming East did not lead it when in the lead with the ♥ A). Lead a heart toward Dummy's ♥ K and ♥ 10, and West's ♥ Q is finessed. The remaining three cards in the closed hand are high.

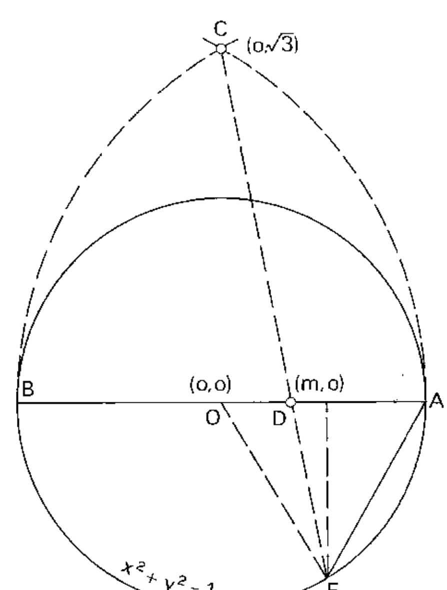
Also solved by Peter Groot, Barry Kington, John Meader, R. Robinson Rowe, Daniel Sheingold, Eugene Spiegel, and Patrick Sullivan.

DE-2 Give an algebraic proof of the following geometrical construction method for inserting within a circle a figure of any number of sides:

1. Draw the diameter of any circle you may have chosen.
2. Divide this diameter into the number of units you wish to have inscribed in the circle.
3. At the extremities of this diameter, scribe two intersecting arcs with radius equal to the diameter.
4. From the intersection of these two arcs draw a line through the second division point from the circle, extending it to the circle.
5. From this intersection with the circle, draw a line to the (zero) point of the diameter on the circle. This line will then be the side of a figure inscribed in the circle, having the number of sides into which the diameter was divided.

The following is from R. Robinson Rowe, who assumes that the inscribed figure is to be a *regular* polygon; otherwise, he says, the problem would be trivial:

The proposition is not generally true, and analytic geometry is used to find the particular number of sides, n, for which it is true.



n	m	cos AOE	AOE	AOE′	Error
3	−1/3	−1/2	120°	120°	0
4	0	0	90	90	0
5	1/5	$(15+\sqrt{73})/76$	71.953	72	−0.047°
6	1/3	1/2	60	60	0
7	3/7	$(21+\sqrt{129})/52$	51.518	51.429	+0.089
8	1/2	$(6+\sqrt{10})/13$	45.187	45	+0.187
10	3/5	$(15+\sqrt{57})/28$	36.356	36	+0.356
12	2/3	$(18+\sqrt{76})/31$	30.473	30	+0.473

Let AB be the diameter of a unit circle with center at 0, divided into n equal parts and with the second division point from A located at D = (m, 0). Thus AD = 4/n = 1 − m. The two construction arcs intersect at C = (0, $\sqrt{3}$), and the equation of line CD is

$x = m(1 - y/\sqrt{3})$

Line CD intersects the circle $x^2 + y^2 = 1$ at point E, with an abscissa which is the cosine of angle AOE; that is,

$\cos AOE = x_E = m(3 + \sqrt{3 - 2m^2})/(m^2 + 3)$.

But AOE is the central angle of chord AE, which allegedly is one side of a regular n-gon, for which the central angle should be

$AOE' = 2\pi/n = \pi/2\ (1 - m)$.

The table at the top of this page will show that AOE = AOE′, as alleged, when and only when n = 3, 4 and 6.

Also solved by Gerald Blum and Frank Rubin.

DE-3 In each of the 16 squares of the figure below place a *different* letter, selected so each row, column, and long diagonal will spell a *different* four-letter word when the letters are selected consecutively in one or the other of the only two possible directions, as we do with numbers. There will be a total of 10 different words, all of which must be defined in any one edition of Webster's dictionaries.

No solution received so far has satisfied all the conditions, although the proposer claims to know of one. Keep trying.

DE-4 The area under the curve

$y = \cos x,\ 0 \leq x \leq \pi/2$

and the line y = 0 is to be divided into four equal areas by a line parallel to the y-axis and another line. Give the equation of the two lines. (In the drawing, areas $A_1 = A_2 = A_3 = A_4$.)

The following is from Gerald Blum; who says this one "is easy if you set it up properly":

Let L_1 be defined as x = c (it is misdrawn in the published diagram), and let L_2 be defined as y = mx + b; c, m and b are constants to be determined. Define the x-coordinate of the point where L_2 intersects the curve as x = d, where d is another constant to be found. Since the total area under the curve is 1 (a trivial integral), we have $A_1 = A_2 = A_3 = A_4 = 1/4$. From $A_1 = A_3 = 1/2$ = sin c on doing the integration, we quickly find $c = \pi/6$. Substituting this into the expression found by integrating A_3 gives us

$(\pi^2/72)m + (\pi/6)b = 1/4$.

Subject to the restriction $0 \leq b \leq 1$, this equation generates a family of L_2's, all of which satisfy the conditions of the problem. Although the simplest equations for d are cos d = md + b and sin d − md²/2 − bd = ½, these are both transcendental. Taking these two with the identity $\sin^2 + \cos^2 = 1$ and the equation for m and b above, we can generate a purely algebraic expression for d in terms of b. This expression can be considerably simplified by a sort of scale change substitution as follows. Define $M = \pi^2 m$ and $B = \pi b$ and $D = d/\pi$. The result, after some algebraic simplification, is

$36(3 - 2B)^2D^4 + 24B(3 - 2B)D^3 + [(3 - 2B)^2 + 9]D^2 + 4BD - 3 = 0.$

This yields two easy solutions, B = 3/2 and D = 1/3, and B = 0 and $D = (\sqrt{\sqrt{13} - 1})/6$.

Also solved by Peter Groot, B. Rouben, R. Robinson Rowe, and Les Servi.

DE-5 Find all solutions to

sin (x + y) = sin x + sin y

The following is from B. Rouben:

Since we have the identity sin (x + y) = sin x cos y + cos x sin y, the equation becomes

sin x cos y + cos x sin y = sin x + sin y

sin x (cos y − 1) = sin y (1 − cos x)

(sin x)/(1 − cos x) = − (sin y)/(1 − cos y) (1)

unless 1 − cos x = 0 or 1 − cos y = 0.

The latter condition implies either $x = 2k\pi$ or $y = 2k\pi$, with k any integer. Returning to (1), and using $\sin 2\theta = 2 \sin\theta \cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$, we obtain $(2 \sin x/2 \cos x/2)/2 \sin^2 x/2 = -(2 \sin y/2 \cos y/2)/(2 \sin^2 y/2)$

cot x/2 = − cot y/2

which implies $x/2 = -y/2 + k\pi$, where k is any integer;

$x = -y + 2k\pi$, where k is any integer.

Thus all solutions to the problem fall under three categories:

1. $x = 2k\pi$, y = anything;
2. $y = 2k\pi$, x = anything; and
3. $x = -y + 2k\pi$, k being any integer.

Also solved by Gerald Blum, Peter Groot, Ron Moore, John E. Prussing, Henry Radoski, Robert Rogoff, B. Rouben, R. Robinson Rowe, Frank Rubin, Victor Sauer, Les Servi, and the proposer, Richard Lipes.

As promised, here is Peter Groot's solution to JN-3 as revised in the December issue:

JN-3 Show or prove that

$$\left(\frac{1-x}{1+x}\right)(2x+1)\prod_{k=1}^{\infty}\left\{[1+x^{2k}]\left[1+\left(\frac{x}{1+x}\right)^{2k}\right]\right\}=1$$

$-\frac{1}{2} < x < 1.$

Separating products,

$$\frac{(1-x^2)}{(1+x)^2}[(1+x)^2-x^2]\prod_{k=1}^{\infty}[(1+x^{2k})]\prod_{k=1}^{\infty}\{(1+[x/(1-x)]^{2k}\}\overset{?}{=}1$$

Now $\prod_{k=1}^{\infty}(1+q^{2k})$ = by expansion $1 + q^2 + q^4 + q^6 + \ldots = 1/(1-q^2)$ if $|q^2| < 1$.

For q = x, $-1 < x < 1$.

For $q = x/(1+x)$, $-\frac{1}{2} < x < \infty$.

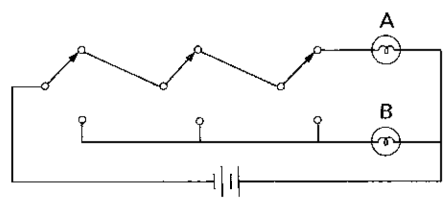
So $-\frac{1}{2} < x < 1$ converges for both.

$$(1-x^2)\frac{(1+x)^2-x^2}{(1+x)^2}\cdot 1/(1-x^2)\cdot 1/\{1-(x/[1+x])^2\overset{?}{=}1$$

$[(1+x)^2 - x^2]/(1+x)^2 \cdot (1+x)^2/[(1+x)^2 - x^2] = 1.$ 1 = 1. Q.E.D.

Solution to Speed Department Problems

The solution to **SD1** is left to our readers. The solution to **SD2** is in the diagram below; Mr. Plummer notes that it "clearly extends to any number of switches, but knowledge of that fact seems to make the solution easier to find.



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