# Word Games to the Lesser Antilles . . . and Back Again

John Forester reports that a factory-wired PC board on my amp may have an incorrect component. If further tests corroborate this thesis, Dyna can expect a rather poignant letter and I can expect some music from my system. Here's hoping. Since this month's installment is rather lengthy, I shall dispense with further small talk and get right down to the problems.

## Problems

Frank Rubin proposes the following:

**21** Define two functions, f and g recursively by

$$f(n, a) = \begin{cases} a \\ 1 - \log f(n-1, a) \end{cases}$$

$$g(n, a) = \begin{cases} 1/a \\ g(n-1, a)/f(n, a) \end{cases}$$

in each case if $n = 0$ and $n > 0$.
Then determine whether either of the following converge:

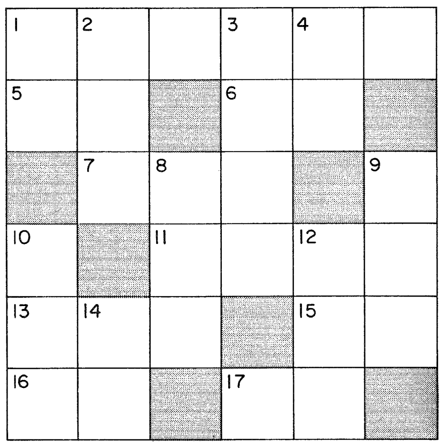
$$\sum_{n=1}^{\infty} g(n, n)$$

$$\sum_{n=1}^{\infty} g(n, 2)$$

**22** Recall problem 26 of last year:
Find the smallest integers m and n such that $m - n^3$, m, and $m + n^3$ are all perfect squares.
Robert L. Bishop wants you to show that there are infinitely many solutions to it (easy), and he wants an exhaustive description of them.

The following excellent problem appears to be unsigned, so I cannot credit anyone for it. Will the real author please stand up?

**23** So many of our friends have asked about the boat in which we cruised the Lesser Antilles and about the crew that we have prepared a simple diagram which answers most of their questions:




The crew of five were the skipper, first mate Joseph, navigator Peter, deck hand Moses, and cook Able. They all voted for Eisenhower. The total miles shown on the taffrail log was twice the number for the first nine days plus exactly 200 miles. However, we had the log carefully checked and found that for each mile registered we had sailed 6,120 feet, so the distance sailed was slightly greater than that shown on the log. As to the crew, it so happened that if Peter had been 14 years older the skipper would have been twice the average age of his crew. Also if the skipper had been 13 years older his age would have equaled the sum of the ages of the three youngest members of the crew. The dimensions of the boat, sail area, and ages of crew can now be easily ascertained by completing the diagram and using the following clues.

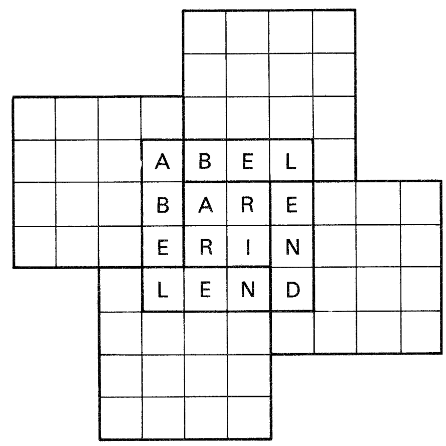
*Across*
1 Yards sailed in 9 days
5 Age of first mate
6 Twice the age of Joseph
7 Miles logged in nine days plus 1 down
11 Square of 4 down minus 2 down
13 Total miles logged
15 Age of Moses
16 Length overall
17 1 down reversed

*Down*
1 Cube of beam in yards or square of draft in feet
2 Miles logged in nine days
3 Area of mizzen times beam
4 Two times 5 across
8 Length overall times draft
9 Area of mainsail or twice area of mizzen plus sum of digits of 11 across
10 Area of mainsail plus length overall
12 Low water length plus length overall plus draft plus beam
14 Age of Able

**24** James J. Heyman wants you to construct a triangle given the three altitudes.

This problem was submitted by Robert S. Cox and interpreted by Randy Gabel:

**25** Complete the following so that all squares contain only words and the word in row i of a square is the same as the word in column i of that square.




## Speed Problems

**SD8** Leslie Shipman can punctuate the following (periods permitted); can you?
"That that is is that that is not is not is that it it is"

This speed problem is by "Sincerely" (Doesn't anybody sign letters these days?):

**SD9** A man walked into a hardware store and said, pointing to an item, "How much is one?" The answer was 20 cents. "I see," he said, "and how much for 12?" Forty cents was the answer. Nodding approvingly, the man said, "Fine, then I want 912." "That will be 60 cents," said the salesman. The question is, What was he buying?

Correction: In SD2 (October/November) we asked you to find nine points in the plane such that 10 straight lines pass through exactly three of them and no points are colinear. Now the proposer, Frank Rubin, corrects us: we should have specified that no *four* points are colinear.

## Solutions

**6** Given the equation
$(x + 4)(6 - x) = 9$, solve for x.
One method is simply $x + 4 = 9$, $x = 5$; and $6 - x = 9$, $x = -3$; both answers are correct! The problem is to find a general form of the equation for this method of solution to work.

Douglas J. Hoylman is back and has supplied the following: I interpret the problem as that of finding all equations of the form
$(ax + b)(cx + d) = f$
such that when $ax + b = f$, $cx + d = 1$, and when $cx + d = f$, $ax + b = 1$.
Eliminating the x from each pair of equations, we have
$a[(1 - d)/c] + b =$
$c(1 - b)/a + d = f$,
or
$a^2 - a^2d + abc = c^2 - bc^2 + acd = acf$.
Now if $a + c \neq 0$, then we can solve the left-hand equation uniquely for d:
$d = (a - c + bc)/a$.
But then the right-hand equation gives $f = 1$, which is not a very interesting case. So if we want $f \neq 1$, we must have $a + c = 0$. Then we may choose b and d arbitrarily, and the above equations give $f = b + d - 1$. (Assuming, of course, that $a \neq 0$, or the whole thing becomes trivial.) So the general form is
$(ax + b)(-ax + d) = b + d - 1$, $a \neq 0$

Also solved by Charles Buncher, Mrs. Martin S. Lindenberg, Roger D. Milkman, George H. Roper, Eric Rosenthal, Frank Rubin, Leslie Servi, John J. Sytek, Jr., Smith D. Turner, and Ralph Wanger, Jr.

**7** The contract is 6 no trump by South, and the opening lead is ♦Q:

North:
♠ x x
♥ K Q x x x
♦ x x
♣ A K Q J

West:
♠ A Q x x
♥ x x
♦ Q J 9 x
♣ x x x

East:
♠ J x x
♥ x x
♦ x x x
♣ x x x x x

South:
♠ K x x x
♥ A J x x
♦ A K 10 x
♣ x

Captain John Woolston had little trouble: "The opening lead is taken by South with ♦A and the clubs and hearts are run. When there are only three cards left in each hand, it is probable that West will hold ♠A, ♦J, and ♦9, while South ends up with ♠K, ♦K, and ♦10. Thus the spade lead from North is taken by West, who loses the last two diamonds. Of course, West could hold other cards, but South is able to adjust—i.e., West ♠A, ♠A and keeps three diamonds, South's monds and drop ♠K in his last discard, thus winning them all; or West ♠A, ♠Q, and ♦J, at worst South loses a spade if West's last discard is ♦9. If West drops ♠A and keeps three diamonds, South's two kings are good."

Also solved by Richard A. Bator, David A. Finnegan, Stanley Horowitz, Elmer C. Ingraham, Leon M. Kaatz, T. D. Landale, Michael Lintner, Private Michael Mann, John P. Rudy, Smith D. Turner, Ralph Wanger, Jr., and the proposer, Frank Model.

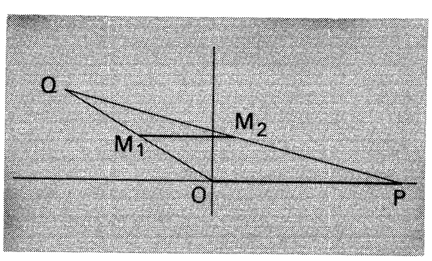
**8** If it takes an hour to work a jigsaw puzzle of 100 pieces, how long should it take to do one with 300? (Assume that the puzzles have no regular edges and are of solid color, and that pieces are of similar size and pattern in both puzzles.)

The following is from John P. Rudy: Assume that the amount of time to place the nth piece is proportional to the number of pieces remaining. This says that one looks through the remaining pieces to fill a particular hole. Assume that the first piece takes no time. For the 100-piece puzzle,
$0 + 99 + 98 + 97 + \ldots + 1$
$= (99)(100)/2 = 4{,}550$
For the 300-piece puzzle,
$0 + 299 + 298 + \ldots + 1$
$= (299)(300)/2 = 44{,}850$
The ratio is approximately 9:1.

Also solved by David A. Finnegan, Smith D. Turner, and Captain John Woolson.

**9** Connect the three midpoints of a triangle and prove that each of the resulting line segments is parallel to and equal in length to one-half of the opposite side.

Donald Morrison found this rather easy; he says "it is a simple problem found in high school geometry courses" and is most easily proved analytically.




A rectangular coordinate system may be established so that the vertices have the following coordinates: O(0,0), P(2,0) and Q(2a,2b). The midpoints of OQ and PQ are $M_1(a,b)$ and $M_2(1 + a,b)$.
$M_1M_2 = \sqrt{[a - (1 + a)]^2 + (b - b)^2}$
$= \sqrt{(-1)^2} = 1$;
$OP = \sqrt{(2 - 0)^2 + (0 - 0)^2} = \sqrt{2^2} = 2$.
Therefore $M_1M_2 = \frac{1}{2}OP$. The slope, m, of $M_1M_2 = (b - b)/(1 + a - a) = 0/1 = 0$. Therefore, $M_1M_2$ is horizontal and parallel to OP.

Also solved by David A. Finnegan, Mrs. Martin S. Lindenberg, A. Porter, John E. Prussing, Eric Rosenthal, Smith D. Turner, Ralph Wanger, Jr., and Captain John Woolson.

**10** Let G be a group with precisely two conjugacy classes (x and y are conjugate if there is an element a such that $y = axa^{-1}$). If G is assumed to be finite, what can be concluded about G?

If instead we assume that G contains a nontrivial element which is of finite order (i.e., there is an x not equal to 1 in G and that $x^n = 1$ for some nonzero integer n), what can be concluded about G?

Thomas H. Sadler has responded as follows:
I found that the order of the finite group must be the power of a prime and that if the group has a nontrivial element of finite order, every element in the group has the same order k, where k is a prime.
Let G be a group with exactly two conjugacy classes $C_1$ and $C_2$. Then the identity element $e \in C_1$, say. Now, $x \in C_1 \Leftrightarrow e = axa^{-1}$ for some $a \in G$. But $e \Rightarrow axa^{-1} \Rightarrow a^{-1}ea \Rightarrow x = a^{-1}(ae) = x \Rightarrow x = e$. So $C_1 = \{e\}$ $\therefore C_2 = G - \{e\}$; i.e., all nontrivial $x \in G$. This implies in particular that $G \neq \{e\}$.
Now suppose that G has a nontrivial element $x \neq e$ such that $x^n = e$ for some nonzero integer n. (I can safely assume $n > 0$, also.) Let m be the order of x; then $1 \leq m \leq n$, since $x \neq e$ and $x^n = e$.
Now every element $y \in C_2$ has order m.
For $y \in C_2 \Rightarrow x = aya^{-1}$ for some $a \in G$.
$\Rightarrow e = x^m = \underbrace{(aya^{-1}) \ldots (aya^{-1})}_{m \text{ times}}$
$= ay^ma^{-1}. \Rightarrow y^m = e$. Therefore, y has finite order k where $k \leq m$. Now $x = aya^{-1} \Rightarrow a^{-1}xa = y \Rightarrow a^{-1}x^ka = y^k = e \Rightarrow x^k = e \Rightarrow k \geq m$.
$\therefore k \leq m$ and $m \leq k \Rightarrow k = m$.
So every element in $C_2$ has finite order m. Furthermore, m is prime. For suppose not; then for some integer p such that $1 < p < m$, $p|m$. Since x has order m and $p < m$, $x^p \neq e$. Since p divides m, $(x^p)m/p = x^p(m/p) = x^m = e$. So $x^p$ has order m/p and $1 < m/p < m$. But every element in $C_2$ has order m. Contradiction. Therefore, every nontrivial element in G has order m, where m is a prime.
Now suppose that G is finite. Then there is an $x \in G$ of order $k > 1$ (since $G \neq \{e\}$). By the above, k must be prime and every $y \neq e$ in G has order k. Let n be the order of G. By Lagrange's Theorem, $p \mid n$. Suppose there were another prime $q \neq p$, such that $q \mid n$. Then there must be an element in G of order q. But every nontrivial element in G is of order q and $p \neq q$. Contradiction. Therefore $q \mid n$. So the order of G is $n = p^r$ where r is some positive integer.

Neil Cohen, Douglas J. Hoylman, and Dennis W. Sivers also responded.

*Allan J. Gottlieb, who studied mathematics at M.I.T. with the Class of 1967, is now on the teaching staff at Brandeis University. Send new problems and solutions to him at the Department of Mathematics, Brandeis University, Waltham, Mass., 02154.*