Matrices and a Monkey's Uncle

Hi. My name is Allan Gottlieb, and I shall once again in 1969-70 have the pleasure of editing "Puzzle Corner" for *Technology Review.* Each month from among submitted problems five will be selected for inclusion in the column. Three issues later answers—as submitted by readers—will appear. Easier "speed problems" remain unanswered. This column depends upon outside support. When no problems are submitted, I can—reluctantly—supply some. Unfortunately, my creations are generally inferior to yours, so keep them coming. When you submit answers, please refer to the problems *by number;* don't just select a few words from the first sentence of the problem to identify the puzzle you've worked on. For nearly each issue I find such "title" solutions, and often—after an unsuccessful search for the problems to which they apply—the solutions are filed . . . for good. Currently I have a rather large backlog of submitted problems, so don't be discouraged if your creation does not make an early appearance. Your patience shall be rewarded . . . as shall neat handwriting. Now for the fun:

Problems

1 Smith D. Turner would like you to obtain rational factors of
$x^8 - 4x^4y^4 + 16y^8$

Our second problem comes from Edward J. Dudewicz, Assistant Professor of Statistics at the University of Rochester; it concerns positive semi-definite matrices:

2 For the real symmetric matrix Q, let $\Delta = \det(Q)$ and $\Delta_o = 1$. Let Δ_{n-t} be the determinant of Q with its last t rows and columns deleted. It is well known that Q is positive semi-definite if all principal minors of Q are $\geqq 0$ and $\Delta = 0$. Is this last condition equivalent to some (seemingly weaker) condition? (For example, one might conjecture a condition such as "$\Delta_o, \Delta_1, \ldots, \Delta_n \geqq 0$ and $\Delta = 0$," which involves only the leading principal minors. This, though, is clearly insufficient.) Such a weaker condition could be of use in determining whether a matrix is PSD, especially if it uses little more than $\Delta_o, \Delta_1, \ldots, \Delta_n$.

Number three is from a close friend but terrible first baseman (I was shortstop behind him on the M.I.T. Baker House team), John P. Rudy; he says it is "the bridge hand you've been waiting for, played correctly by Harold S. Vanderbilt in 1929:"

3 The contract is six spades, and the opening lead is ♦3. Play to make it.

North
♠ A K 8
♥ —
♦ A K 8 6
♣ A K J 10 8 2

West
♠ Q 9
♥ K J 10 6 2
♦ Q 7 4 3
♣ 7 5

East
♠ J 6 3
♥ 8
♦ J 10 9 4 2
♣ Q 9 6 4

South
♠ 10 7 5 4 2
♥ A Q 9 7 5 4 3
♦ —
♣ 3

This one is from John C. Maier:

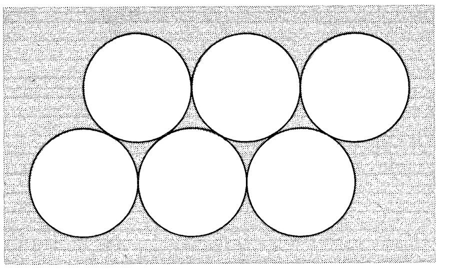
4 A monkey and his uncle are suspended at equal distances from the floor at opposite ends of a rope which passes through a pulley. The rope weighs four ounces per foot. The weight of the monkey in pounds equals the age of the monkey's uncle in years. The age of the uncle plus that of the monkey equals four years. The uncle is twice as old as the monkey was when the uncle was half as old as the monkey will be when the monkey is three times as old as the uncle was when the uncle was three times as old as the monkey. The weight of the rope plus the weight of the monkey's uncle is one-half again as much as the difference between the weight of the monkey and that of the uncle plus the weight of the monkey. How long is the rope? How old is the monkey?

Our last problem is from J. Karl Justin. I am not sure I understand it, but perhaps the readers will do better.

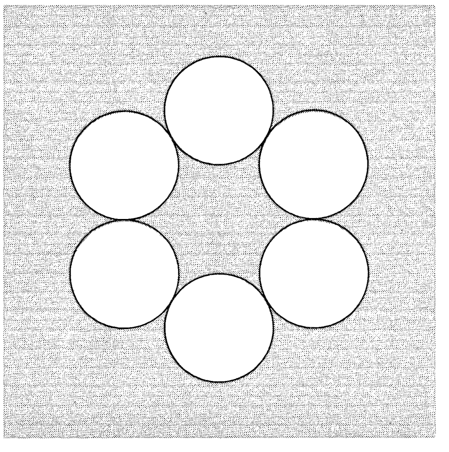
5 Three thespians came on a cache of bright, shining obols and decided to share it. Silimon took some coins, and Stupidas also helped himself. What is the probability that at most one-third of the coins were left for Preposteros?

Speed Department

SD1 Russell A. Nahigian wants you to start with six coins arranged as below:



Then, making two moves such that after each move coin always touches two others, arrive at the following:



SD2 This is a little tough for a speed problem, so take five minutes and help Frank Rubin find nine points in the plane such that 10 straight lines pass through exactly three of them and no points are colinear.

Solutions

This month we publish solutions to problems which appeared in the May issue. June solutions and a backlog of material on earlier problems next month.

31 Although this puzzle relates to a farmer, his family, and his land, it involves a good deal of engineering mathematics and logic. The problem is to find the age of Mrs. Grooby, Farmer Dunk's mother-in-law, and you must not assume the puzzle was invented this year. You'll need to know that there are 20 English shillings to the pound sterling, that an acre is 4,840 square yards, and that a rod is a quarter of an acre. Also, these hints help: One number in the puzzle is the area of Dog's Mead in rods, but it relates to something in the puzzle quite different from that area. Here's the puzzle.

[1] 3	8	[2] 7	[3] 2	0	■	[4] 1
5	■	[5] 3	2	■	[6] 4	4
5	■	9	■	[7] 3	5	2
■	[8] 1	6	[9] 1	0	■	■
[10] 6/7	2	■	[11] 1	9	[12] 1	[13] 3
8/9	■	■	■	[14] 7	9	2
[15] 2	7	■	[16] 1	6	■	5

Across
1. Area of Dog's Mead in square yards.
5. Age of Farmer Dunk's daughter, Martha.
6. The difference between the length and breadth of Dog's Mead in yards.
7. Number of rods in Dog's Mead times number nine down.
8. The year when Little Piggly came into occupation by the Dunk family.
10. Farmer Dunk's age.
11. The year Farmer Dunk's youngest child, Mary, was born.
14. Perimeter of Dog's Mead in yards.
15. The cube of Farmer Dunk's walking speed in miles per hour.
16. Number fifteen across minus number nine down.

Down
1. The value of Dog's Mead in shillings per acre.
2. The square of Mrs. Grooby's age.
3. The age of Mary.
4. The value of Dog's Mead in pounds sterling.
6. The age of Farmer Dunk's first-born, Edward, who will be twice as old as Mary next year.
7. The square, in yards, of the breadth of Dog's Mead.
8. The number of minutes Farmer Dunk needs to walk one and one-third times around Dog's Mead.
9. See number ten down.
10. Ten across times nine down.
12. One more than the sum of the digits in the second column down.
13. Length of tenure, in years, of Little Piggly by the Dunk family.

The following is from Lawrence S. Kalman:

15A (across): The walking speed must be either 3 or 4 mi./h., the only two numbers with two-digit cubes.

11A:The year must start with digit 1; so 9D = x1 (x is an unknown digit).

Try 4 mi./h.: 15A is 64, 10D is xx6, 10A = 10D/9D = xx6/x1 = x6; 8D is 16 (since 8A is also a year). Now, Farmer Dunk walks 4 mi./h. = 117 1/3 yards/min.; in 16 min. he walks 1877 1/3 yards = 1 1/3 perimeter: so the perimeter of Dog's Mead is 1408 yards. But 14A

Rods	*Sq. Yds.*	*Length*	*Breadth*	*Difference*	*Remarks*
32	38720	220	176	44	Only solution
31	37510	239 +	157 −		Not integers
30	36300	251 +	145 −		Difference > 99; so any area < 30 rods

states that the perimeter is a three-digit number, so 4 mi./h. cannot be correct.

Try 3 mi./h.: 15A is 27, 10D is xx2, 10A is x2, and 8D is 12. Now Farmer Dunk walks 3 mi./h. = 88 yards/min.; in 12 min. he walks 1056 yards and the perimeter of Dog's Mead is 792 yards (14A).

Now, if Dog's Mead were square (it isn't: the difference in dimensions must be at least 10 yards), the area would be a maximum of $198^2 = 39204$ sq. yards. Try the integral number of rods = 39204 sq. yards. (See results above.)

So 1A is 38720, 6A is 44, 12D is 8 + 1 + 2 + 7 + 1 = *1*9, and 7D is 30976 (italicized digits have previously been determined).

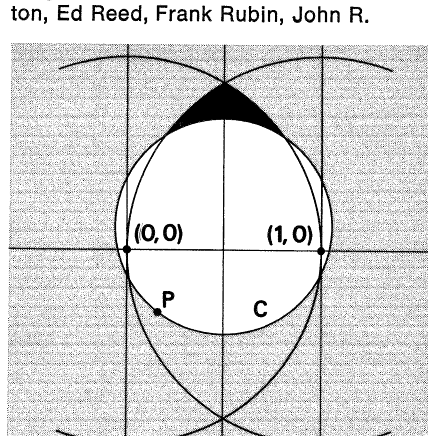
Now 7A is 32 × 9D = 32 × x1 = 3x2; so 9D is 11, 7A is 352, and 16A is 27 — 11 = 16. The only solution for 300 ≤ 1D ≤ 399 and 4D = x42 is 1D = 355 s. = £ 17.75 per acre; the area is 8A; 4D = £ 142.

Since Edward (6D) is 45, Mary (3D) is 22, and Martha (the middle child) is 32 or 42 (5A), so 2D ≥ 7300 and ≤ 7499. Therefore 2D is 7396, the only square in this range, and Mrs. Grooby is 86. Also Martha (5A) is 32 (also the number of rods, as per hint).

Now, Little Piggly came into occupation by the Dunk Family in 1610 (8A). Mary was born in 191x (11A); therefore, the current year is in the 1930's or 1940's. Therefore 13D is 32x, 11A is 1913, the current year is 1935, and 13D is 325.

That leaves only Farmer Dunk's age to be determined. His age (10A) ends in 2; 52 would make him 7 years old when Edward was born (impossible, we assume); 82 would make 10D = 82 × 11 = 902, but the first digit must be an 8 so this is impossible. However, either 10A is 62 and 10D is 682, or 10A is 72 and 10D is 792 are valid solutions, although we admit that the former solution is unlikely.

Also solved by Alan Baum, Robert A. Bender, James M. Field, William T. Frangos, P. Richard Jones, Thomas P. Kennedy, John F. Mandl, Norman D. Megil, Fram C. Minshew, Victor J. Newton, Ed Reed, Frank Rubin, John R. Schaeffer, Sudarshan P. Singh, Smith D. Turner, and Samuel S. Wagstaff, Jr.



32 Let $p_1, p_2, \ldots, p_n$ be points in the plane such that distance $(p_i, p_j) \leq 1$ for $1 \leq i \leq j \leq n$. Prove that these points lie within a circle of radius $1/3\sqrt{3}$.

Mr. Wagstaff submitted the following solution with the drawing below:

With no loss of generality, we can assume that there are two points whose distance apart is the maximum of 1. (If not, change scale.) Rotate and translate the point [set such that these two points] are (0, 0) and (1, 0). Then all the points lie in the intersection of the disks of radius 1 about (0, 0) and (1, 0). Let C be the highest circle with center on the line $x = 1/2$ and radius $\sqrt{3}/3$ such that none of the n points is below the lower semicircle of C. Such C exists, is unique, and has some point p of the n points on its lower semicircle. No points are omitted below C, so either C does the job or there are points left out above C but inside both disks (solid region). Let p = (a, b). With no loss of generality, $0 \leq a \leq 1/2$. (If not, reflect point set in the line $x = 1/2$.) Then the center of C is $(1/2, b + \sqrt{1/3 - (a - 1/2)^2})$. Clearly the nearest point of the shaded region to p is the intersection of C with the circle with center (1, 0), and a short calculation shows that the distance of this point from p is greater than or equal to 1. So none of the n points can be in the solid region because is would be too far from p.

Also solved by John E. Prussing. Frank Rubin, and Mark Yu.

34 a and A are the surface areas, v and V the volumes of a smaller and a larger sphere, respectively. If A = (a + 10) square inches and V = (v + 10) cubic inches, what are the corresponding radii? Solve to two decimals!

The following solution is from William R. Osgood: With R and r the radii of the larger and smaller spheres, respectively, write the relations between the surface areas and volumes as

$4\pi R^2 = 4\pi r^2 + 10$

$4/3\pi R^3 = 4/3\pi r^3 + 10.$

Substitution of r from the first into the second gives, after some simple algebraic manipulation:

$R^4 + 15/2\pi + 25/12\pi^2$

$= 2R^3 + (5/2\pi)R^2$, or

$R^4 + 2.598 = 2R^3 + .7958R^2$.

Solution of this equation by trial yields R = 1.22 in. The first equation then gives r = .83 in.

Also solved by Frederick Cleveland, Richard Hanau, Thomas Kennedy, Cornel Lomogy, John E. Prussing, Edward Reed, R. Robinson Rowe, Frank Rubin, Donald E. Savage, Smith D. Turner, and Samuel S. Wagstaff, Jr.