Puzzle Corner

Allan J. Gottlieb, '67

A new era in my life has started. I now have a bunch of freshmen who thrice weekly call me "Mr. Gottlieb." I am really enjoying teaching calculus and would appreciate helpful suggestions any of you more experienced teachers could send along.

A whole month of school without pressure has really turned me into a cheerful person. So cheerful that I will make sure that at least one problem this month will be easy. My "vast" experience in teaching (one month) has taught me the advantages of instilling confidence.

My girl friend, a physics major, is currently taking an electronics lab. Somehow seeing a miniskirt and a red bow playing with a Tektronix oscilloscope strikes me as very funny. I feel that she is more suited for working over an oven (where all women belong, yeah!). Also, the results turn out better that way.

There will be no solutions printed this issue or next. We'll use this chance to catch up on the mountain of late replies to last year's problems which have accumulated over the summer. The February issue will contain solutions to the problems given in October/November, the March issue to the problems given below.

Problems

Here's one from the Russell A. Nahigian ('57) Collection:

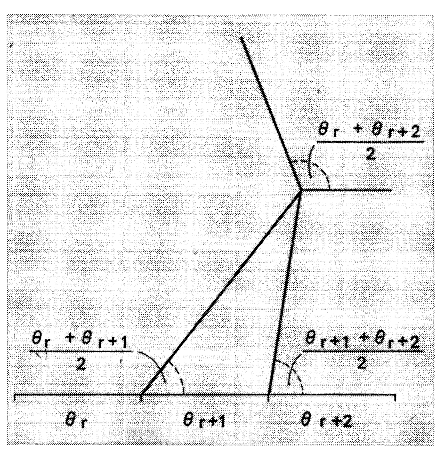
6 A man dies leaving a will stating that one-half of his horses go to the eldest son, one-third to the next oldest, and one-ninth to the youngest. Alas, at the time of death the man has 17 horses. However, a shrewd lawyer solves the problem without killing any of the horses. How?

Donald R. Oestreicher, '67, has a take-off on a chess problem from last year:

7 Problem 31 (May, 1968) was quite easy, but it suggests a more interesting problem. Consider an infinite chessboard having one edge 8 squares long and all 32 pieces. What is the largest number which can be on the board and there still be no legal moves for either side?

This mathematical problem submitted by S. G. Ellis actually arose from honest scientific research. (See, Virginia, math *is* useful.) Mr. Ellis writes: "I have enjoyed reading your column in *Technology Review* and have wondered if you would be interested in a problem which none of my mathematical friends at RCA Laboratories has been able to solve. The problem arose during some speculations on the growth of crystalline films."

8 Take rectangular co-ordinates 0x, 0y, and divide the 0x into equal segments of length d. With each segment associate an angle θ, circularly random in the range $0 \leqq \theta \leqq \pi$. Where the segments meet, lines are erected at angles to 0x which are the arithmetic mean of the θ's in the adjoining segments. As these lines meet, new lines are continued from the intersection according to the rule illustrated below.

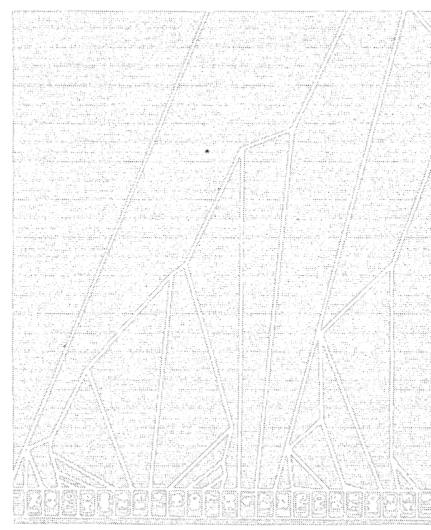


That is, each line makes an angle $\propto$ with the 0x axis which is the arithmetic mean of the θ's of the segments between which it forms a common boundary. It is required to find an analytic expression

$$\overline{X} = \overline{X}(y,d)$$

for the average length of intercept, $\overline{X}$, of these lines on a horizontal line y = constant.

As a supplementary comment, Mr. Ellis writes: "Jules Levine [Ph.D.'63] of RCA Laboratories has obtained a solution for 'y' small by treating the problem as a "bi-molecular collision" where the colliding molecules stick together. At large 'y' the contributing segments no longer have a random distribution of angles; indeed, they cluster around 0 and π. This is illustrated in the drawing below which was worked out in our drafting department using a cyclical boundary condition."



John F. Claerbout, '60, has submitted a specific problem which he feels leads to an interesting field of study.

9 Given the quadratic polynomial with matrix coefficients

$$P(Z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Z \begin{pmatrix} -3 & -1 \\ 14 & -11 \end{pmatrix} + Z^2 \begin{pmatrix} -4 & 4 \\ -58 & 28 \end{pmatrix}$$

Factor it. One solution is:

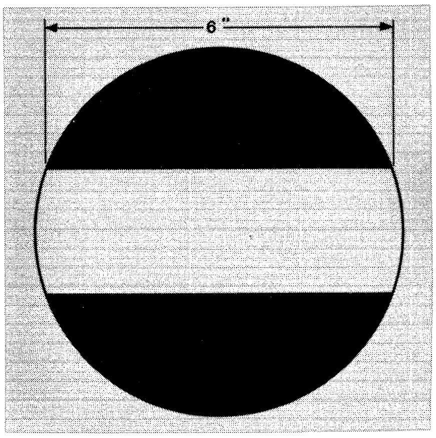
$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Z \begin{pmatrix} 2 & -1 \\ 20 & -7 \end{pmatrix}\right] \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Z \begin{pmatrix} -5 & 0 \\ -6 & -4 \end{pmatrix}\right]$$

There are five other solutions.

Jon writes as follows about this problem: "I worked up the theory and showed that for order p and degree n there will be $(np)!/(p!)^n$ factorizations. One reason mathematicians may have avoided writing this up is that it is hard to describe what happens in the case of degeneracies. Has anyone ever seen a write-up of this type of problem?

The following familiar problem and amazing solution is from James Barton, Head of the Mathematics Department at Pennsylvania State University.

10 A cylindrical hole of length 6 inches is drilled through the center of an ivory ball. What is the volume of the ivory remaining after the hole is drilled?



Speed Department

SD3 Francis A. Packer, Jr., '51, wants you to find the number of diagonals that can be drawn in an n-sided polygon.

The last problem for this month comes from Russell L. Mallett,'57.

SD4 How can you break a two-way tie using a coin which may be biased and still give both competitors an equal chance? In a three-way tie, how can you decide on first, second and third places in an unbiased manner?

Better Late Than Never

The following solutions and correspondence pertain to problems numbered in the 1967-1968 series, in Volume 70 of the *Review.*

5 Robert Scott comments and proposes a new problem which will appear later. And William T. Moody, '31, offers a solution in the form

$V = (15 + 7/5)\, a^3/4.$

10 Solutions have come from Kenneth B. Blake, '13, Marshall Greenspan, '61, and Frank G. Smith, '11.

15 A beautiful solution by Edward L. Cohen, S.M.'66, earns him a free subscription to *Tech Engineering News.* The problem:

The rationals in [0, 1] have measure zero because if we order them like (0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, . . . 1/n, 2/n, . . . (n — 1)/n, 1/(n — 1), . . .) we can cover them with open intervals of length e, (1/2)e, . . . $(1/2^n)e$, so measure of cover is $\leqq 2e$. The problem is: Suppose $e = 1/10$; then the covering has length $\leqq 1/5$. Exhibit a real number in [0, 1] which is not covered.

Mr. Cohen writes that the problem has one solution, $\sqrt{2}/2$. His method: "0, 1, and 1/2 are no problem, for even with $e = 1$, $\sqrt{2}/2$ is not covered. For p/q, we take the interval cover of length $e/2^q$, which gives a larger cover than was sought. Therefore, 1/3 has the cover $e/2^3$, 2/3 has the cover $e/2^3$, 1/4 has the cover $e/2^4$, etc. The sum of the covers is equal to or less than

$$\sum_{n=3}^{\infty} (ne/2^n),$$

which converges (by the ratio test). Returning to the problem, if $\sqrt{2}/2$ occurs within these intervals, then
$|p/q - \sqrt{2}/2| < 1/2^q \cdot 1/10 = 1/2^{(q+1)} \cdot$
1/5
But /2 is irrational; so for each pair of integers p, q (with q = 1), $2p^2 - q^2$
1. Hence

$$|p/q - \sqrt{2}/2| = \frac{|(p/q)^2 - 1|}{(p/q) + \sqrt{2}/2}$$

$$= \frac{|2p^2 - q^2|}{2pq + q^2\sqrt{2}} > 1/4q^2 \geqq 1/2^{(q+1)} \cdot 1/5$$

The last fact can be proved by induction. Therefore, we have a contradiction and the problem is solved. We do not even assume that any of the rational points lie in the middle of their covering sets. Could the proposer be familiar with a relevant passage in Kamke's book *Das Lebesgue-Stieltjes Integral* for his inspiration?"

George P. Wachtell, '43, gives an existence proof and a constructible sequence of which the desired point is the l.u.b. But alas this is not giving the point itself.

19 A solution has come from Charles Volkstaff.

20 A solution has come from Smith D. Turner, Jr., '26.

21 The problem was this end-game double-dummy problem called the "Whitfield Six." Given:

|---|---|---|
	♠ —	
	♥ 6 3	
	♦ A 9	
	♣ 8 2	
♠ 7 3		♠ 6 2
♥ —		♥ —
♦ K 10		♦ 8
♣ 9 5		♣ 7 4 3
	♠ 5 4	
	♥ —	
	♦ Q	
	♣ J 10 6	

South to lead; hearts trump; North-South to make all the tricks against any defense.

David W. Ulrich, '52, writes: "I hate to keep picking on Peter J. Davis, Jr. (he gave the wrong answer to last year's problem 81, also a bridge problem), but his solution to problem 21 published in June, 1968, is incorrect. He has the right idea but the wrong timing. Here's how his solution breaks down: South ruffs spade in dummy and leads last trump. East discards ♦ 8, South discards ♦ Q and West discards ♣ 5. Now North-South can only make five tricks—two hearts, *two* clubs, and one diamond. Remember, after East discards ♦ 8, South *must* discard ♦ Q to save threat card in spades. This gives West a safe discard of ♣ 5. Now South cannot cash three club tricks as the ♣ 8 blocks the suit. Also there is now no hope for a squeeze.

"The correct solution is to cash the ♣ J or ♣ 10, discarding the ♣ 8 from North. *Now* trump a spade in North and lead the remaining trump. East's best discard is ♦ 8; South discards ♦ Q and West must discard ♠ 7. If West discards ♣ 9, North can cash ♦ A and finesse East for ♣ 7 (thus reason for discarding ♣ 8 on the first club trick). When West makes the ♠ 7 discard, North cashes ♦ A to squeeze East in spades and clubs.

"As for the comments regarding the missing ace, king, and queen of clubs, the 'Whitfield Six' is better constructed if the club suit is as follows:

|---|---|---|
	♣ J x	
♣ Q x		♣ 10 x x
	♣ A K 9	

But I still haven't solved the seven-spade problem!"

25 A solution has come from David D. Terwilliger, '35.

26 Solutions have come from Frank Rubin, '62, and Susan Prytherch.

31 Solutions have come from Mark H. Yu, '70, and Mr. Volkstaff.

33 Solutions have come from Messrs. Volkstaff and Yu.

34 Solutions have come from Messrs. Turner and Yu.

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