Puzzle Review

Allan J. Gottlieb, '67

Boston weather has been really amazing this winter. Out here in the sticks of Waltham (10 miles inland) we had one evening of —20°F. (Now that's cold, man!), and there was snow on the ground continually from December 23 to March 21. When the snow finally melted on the 21st another treat was in store for us. Have you ever seen a two-and-a-half day rainfall? Several dams burst and lowland communities were flooded. Damage was in the neighborhood of $100 million—and that's a pretty ritzy neighborhood.

All previous records for the most puzzles proposed at one time have been surpassed. Russell A. Nahigian, '57, has sent in 12 problems with solutions to all but one.

I must end this introduction with a combination of a request, plea and threat. Please, please enclose the *number* of the problem which you are solving along with your solution. Don't name the problem, *number* it. I reserve the right to disregard solutions without problem numbers. As always, please address all correspondence to me at the Department of Mathematics, Brandeis University, Waltham, Mass. 02154.

Problems

John M. Sandor, a graduate student in metallurgy, sent in this crossword puzzle. Note that most of the clues are in two parts, one part a definition of the word itself and the other an aid in constructing the word—i.e., by an anagram, hidden letters, puns and so on. A college dictionary and/or a familiarity with the London *Times* is recommended.

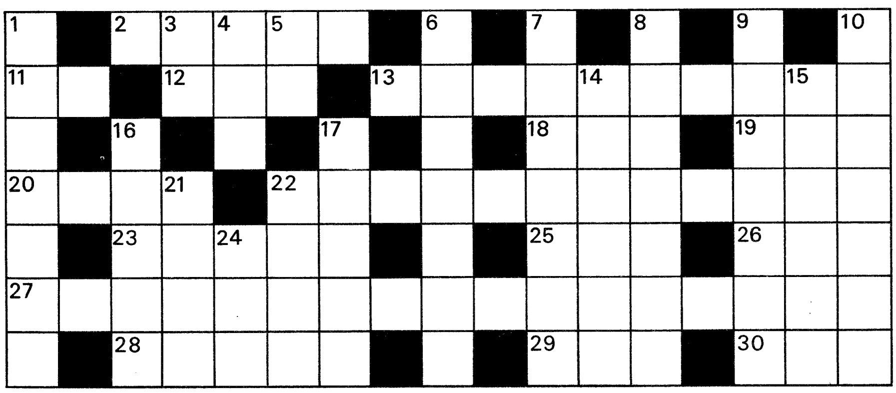
30

Across clues:

2. It scans both ways. (5)
11. Fluctuating charge account. (2)
12. He may be crazy, but his language is logical. (3)
13. It's not the best thing . . . using mixed oil at bumps! (10)
18. They electrify the country in this confused era. (3)
19. Charge! But without moving a unit . . . (3)
20. Tuned in? This will tell if you're not. (4)
22. In essence, COBOL should be reformulated—it may be getting out of date. (12)
23. and 21. down. . . . where and when? That should be self-evident. (5), (4)
25. This collection is assigned. (3)
26. Terminal member. (3)
27. This principle of the universe may give every little grain a total upset. (7), (10)
28. Fish in the frying pan, gleaming. (5)
29. Body of women militarily disposed. (3)
30. This generally means the end of a protein. (3)

Down clues:

1. Dream girl gets the sack, giving way to the famous English innovator. (7)
3. Present before noon. (2)
4. It flies after Jack frequently but at sunrise loses its direction. (3)
5. Publicity since the New Testament? (2)
6. It's noised as violent murder about nothing. (7)
7. Predicted content of chief ores—a winner! (7)
8. A stratagem reveals chains in disorder . . . (7)
9. . . . while mental disorder fits the last word to a T, with first-class backing. (7)
10. This sweet-toothed Australian has a strange color to his cheeks. (7)
14. In the past, electrical engineering has been disturbed about giving coin. (6)
15. Neckwear paradoxically rarely seen at their place of origin, though popular elsewhere. (6)
16. A climbing insect comes after father in a city on the Ganges. (5)
17. White willow tree developed from experiments in the lab—electromagnetically! (5)
21. See 23. across.
22. In general, a Tech grad is subjected to this special torture. (4)
24. Does a minimum of work; therefore has nothing to lose. (3)



Here's a problem so good that even I worked on it; it has been sent in by Douglas J. Hoylman, '64:

31 Place some or all of the chess pieces belonging to one side on the board in such a way that none of them can legally move. It must be a possible position: bishops on different colors, no pawns on the eighth rank. It can be done.

Frank G. Smith, '11, sent a letter stating: "Most of the puzzles in the *Review* are for the birds—those clever mathematicians. How about one for us elderly alumni who just squeaked by the math in our day?" And he submitted this:

32 A cow is tethered to the corner of a square barn in a level field. The length of the tether equals the perimeter of the barn. The cow can graze over just *one acre.* What is the size of the barn, "give or take" a small decimal fraction?

33 John P. Rudy, '67, wants you to prove that:

$$\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$$

34 Smith D. Turner, '26, wants you to "make 7 with two 2's."

Speed Department

Here's another football problem from R. Robinson Rowe, '18:

SD11 Many games are won by a one-point margin. Excluding large scores which are improbable in playing time for a game, what one-point margin scores are impossible? No negative numbers, please. The answer is just one combination. What is it?

The following is a cryptogram by A. NON PLUS. A cryptogram is a message which has been enciphered. This message, with the name of its author, is an elementary substitution cipher:

SD12 NRG RH Z FMREVIHRGB DRGS Z KIRNZIB ULXFH LM VMOZITRMT GSV SLIRALMH LU HXRVMXV ZMW GSV WVEVOLKNVMG LU GVXSMLOLTB ZMW RGH ZKKORXZGRLMH GL LFI HLXRVGB. —SLDZIW D QLSMHLM

Solutions

15 The rationals in [0,1] have measure zero because if we order them like (0,1,1/2,1/3,2/3,1/4,3/4,1/5,2/5,3/5, . . . 1/n,2/n, . . . (n — 1)/n,1/(n + 1), . . .) we can cover them with open intervals of length e, (1/2)e, . . . $(1/2^n)$e, so measure of cover is $\leq$ 2e. The problem is: Suppose e = 1/10; then the covering has length $\leq$ 1/5. Exhibit a real number in [0,1] which is not covered.

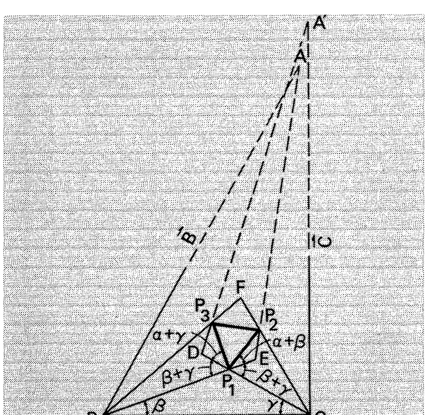
No one solved problem 15, so rather that let the cat out of the bag I'll offer a free subscription to *Tech Engineering News* as an added inducement.

16 Show that every positive integer n divides some number of the form 111 . . . 1100 . . . 00, i.e., a string of 1's followed by a string of 0's.

Mr. Hoylman sent in this solution: Consider the n + 1 numbers 1, 11, 111, . . . up to a string of n + 1 one's. By the Pigeonhole Principle (which asserts that if n + 1 pigeons are in n holes, then there is at least one hole with at least two pigeons in it), two of these must be congruent modulo n. Then their difference is a multiple of n, but this is a number of the required form.

Also solved by Mark Yu, '70, and Jed Stein, '71.

17 Show that the intersections of the trisectors of the angles of any triangle form an equilateral triangle.



Eric Rosenthal (son of Meyer S. Rosenthal, '47), gives a reference to a generalization, the statement of which I find unintelligible. The nicest solution to the given problem is from Mr. Yu:

The result of this problem is known as Morley's theorem. The easiest way to approach it is to construct the equilateral triangle using two of the vertices of triangle ABC.
Let the angle β equal one-third of the angle at B, γ equal one-third of C, and α equal one third of 180° — B — C. The trisectors of angles B and C are drawn in the diagram. First construct isosceles triangles P_1DP_3 by setting angle DP_1P_3 equal to $\alpha + \gamma$; and triangle P_1EP_2 by setting angle EP_1P_2 equal to $\alpha + \beta$. Therefore, we have angle $P_3P_1P_2 = 60°$, immediately. Furthermore, angle $BP_3P_1 = 2\alpha + \gamma + \beta =$ angle CP_2P_1, and since P_1 is the incenter of triangle BFC, $\overline{P_1P_3} = \overline{P_1P_2}$. Clearly this follows from the fact that the perpendiculars from P_1 to $\overrightarrow{P_3B}$ and $\overrightarrow{P_2C}$ are equal. Therefore, triangle $P_1P_2P_3$ is equilateral. Now we must prove that $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{DP_3}$, and $\overrightarrow{EP_2}$ are concurrent. Let A and A′ denote the intersections of $\overrightarrow{B}$ and $\overrightarrow{EP_2}$, and $\overrightarrow{C}$ and $\overrightarrow{DP_3}$, respectively. First $\overrightarrow{EP_3}$ and $\overrightarrow{DP_2}$ bisect angles E and D, respectively. This makes P_3 and P_2 incenters of triangles EAB and DA′C, respectively. In either case, it is obvious that $\overrightarrow{AP_3}$ (bisector) coincides with $\overrightarrow{A'P_3}$ and that $\overrightarrow{A'P_2}$ coincides with $\overrightarrow{AP_2}$. Hence A = A′. Q.E.D. (more or less).

18 Let a be an integer greater than 1. Prove that there exist integers x, y: $1 < x < (2a - 1)$, $1 < y < (2a - 1)$ such that $xy \equiv 2a - 1 \bmod 2a$ if and only if $a > 3$.

The following solution was sent in by Mr. Hoylman; also solved by Mr. Yu:

If $a > 3$, then $\phi(2a) > 2$. (Proof: If x/y, then $\phi(y) \geq \phi(x)$. Now any number greater than 6 must have as a factor either 8, 9, or a prime greater than 3, or 4p where p is an odd prime. But $\phi(8) = 4$, $\phi(9) = 6$, $\phi(p) = p - 1$, and $\phi(4p) = 2p - 2$, all greater than 2.) This means that there is some integer x with $1 < x < 2a - 1$ such that x is relatively prime to 2a. Hence the congruence $xy \equiv 2a - 1 \pmod{2a}$ has a solution y, and it is easily seen that y cannot be congruent to 0, 1, or —1 mod 2a, so it may be taken between 1 and 2a — 1. For a = 2, the only possible value for x or y is 2, but $4 \not\equiv 3 \pmod 4$. For a = 3, x and y must be 2, 3, or 4, but then xy is not relatively prime to 6, while 5 is. So in these cases the congruence has no solution.

19 N natives gather coconuts. The first native secretly divides the coconuts into N equal piles and has one coconut left over which he discards. He takes one pile and pushes the rest of the coconuts back into one big pile. Native number two does the same with the remaining coconuts, discarding one coconut and taking one pile. This continues through native N. When the last native is through, the number of coconuts left is divisible by N. How many coconuts were gathered in the beginning?

A partial solution was sent in by the proposer, George H. Ropes, '33. However, we are going to print a solution sent by Edward L. Friedman, '50 (space precludes using his print-out), who says: "One of our adjunct faculty members at the University of Hartford, Albert A. Bordonaro, devised a closed form solution as follows: Let n=number of natives and x=the minimum number of coconuts. Then, for n>2,

$$x = \begin{cases} n^n - (n - 1), & n \text{ odd} \\ (n^n - 1)(n - 1), & n \text{ even} \end{cases}$$

We are unable to describe *briefly* the derivation of the above formulas, so details are omitted. You may, however, enjoy knowing that seven natives gathered 823,537 coconuts.

SD5 As promised, problem 5 will be answered. The best was the original sent by Charles L. Sandberg, Jr., '67:

A dozen, a gross, and a score,/ Plus three times the square root of four/ Divided by seven/ Plus five times eleven/ Equals nine squared and not a bit more.

Allan J. Gottlieb, '67, is a graduate student in mathematics at Brandeis University. "Puzzle Review" is written for *Technology Review* and *Tech Engineer-News,* the M.I.T. undergraduate professional magazine.