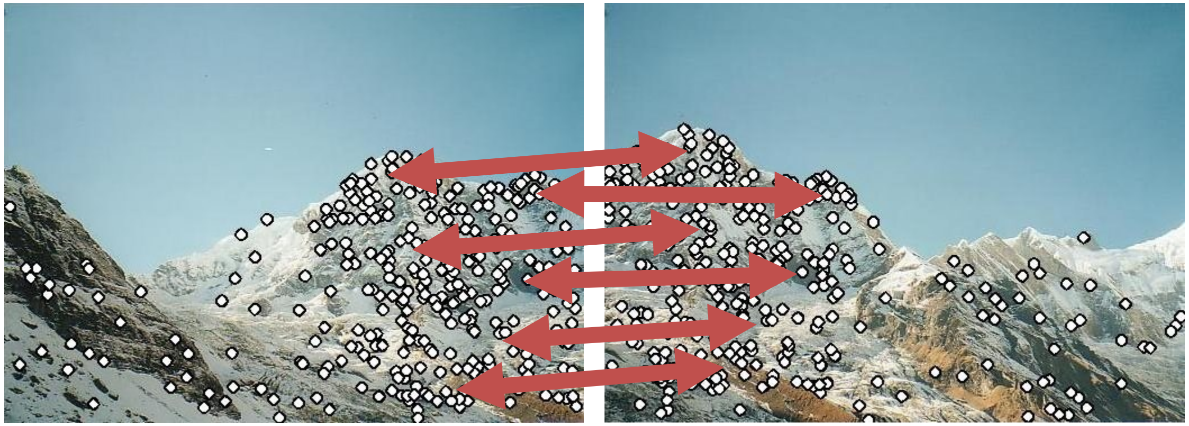
Transformations and Fitting

EECS 442 – David Fouhey

Winter 2023, University of Michigan

https://web.eecs.umich.edu/~fouhey/teaching/EECS442_W23/

So Far



1. How do we find distinctive / easy to locate features? *(Harris/Laplacian of Gaussian)*
2. How do we describe the regions around them? *(histogram of gradients)*
3. How do we match features? (L2 distance)
4. How do we handle outliers? (RANSAC)

Today

As promised: warping one image to another

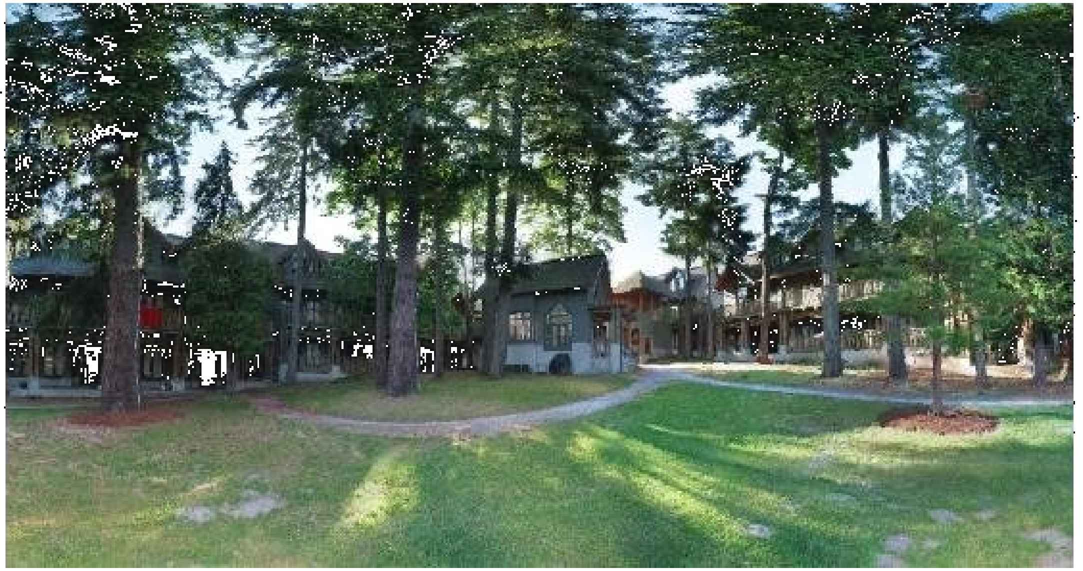
Why Mosaic?

- Compact Camera FOV = 50 x 35°

Slide credit: Brown & Lowe

Why Mosaic?

- Compact Camera FOV = 50 x 35°
- Human FOV = 200 x 135°



Slide credit: Brown & Lowe

Why Mosaic?

- Compact Camera FOV = 50 x 35°
- Human FOV = 200 x 135°
- Panoramic Mosaic = 360 x 180°

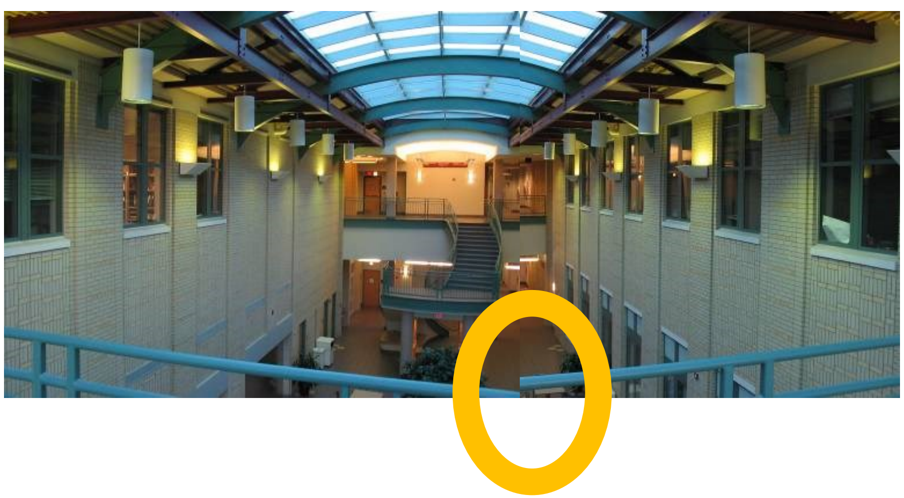
Slide credit: Brown & Lowe

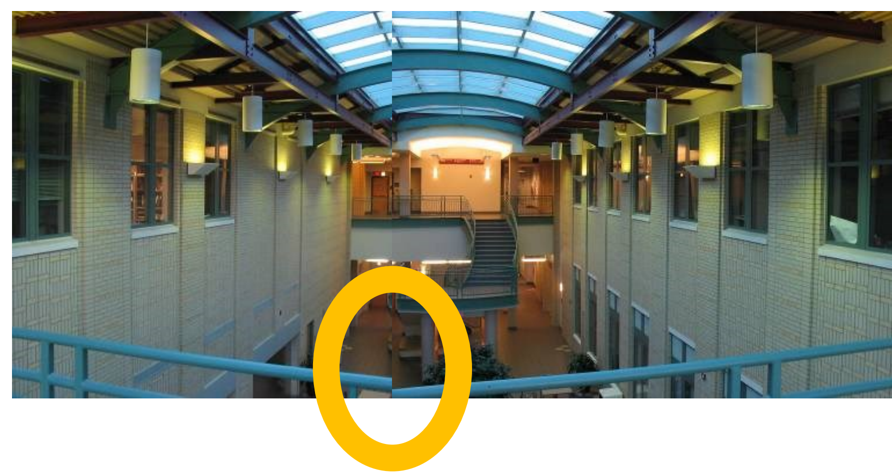
Why Bother With This Math?

Slide credit: A. Efros

Homework 1 Style

Translation only via alignment





Slide credit: A. Efros

Result

Slide credit: A. Efros

Image Transformations

Image filtering: change range of image

$$g(x) = T(f(x))$$

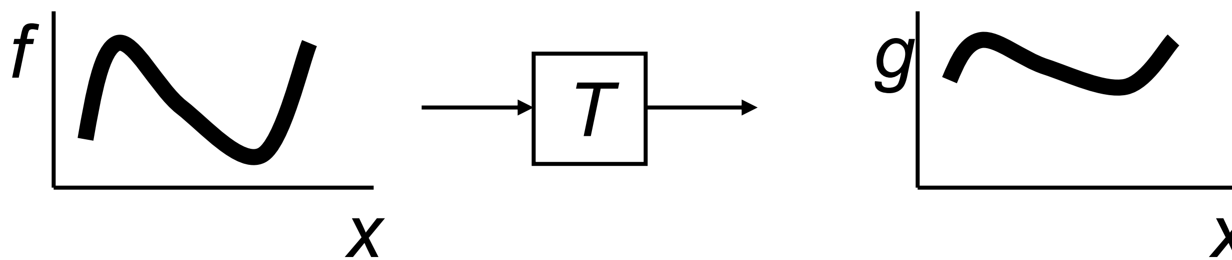
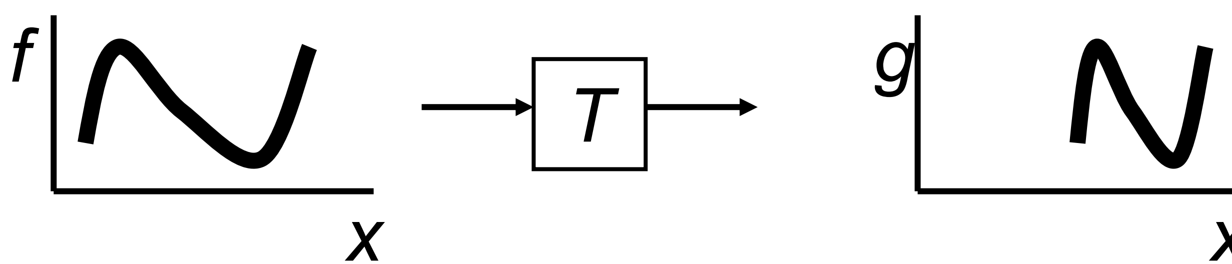


Image warping: change ***domain*** of image

$$g(x) = f(T(x))$$



Slide credit: A. Efros

Image Transformations

Image filtering: change range of image

$$g(x,y) = T(f(x,y))$$

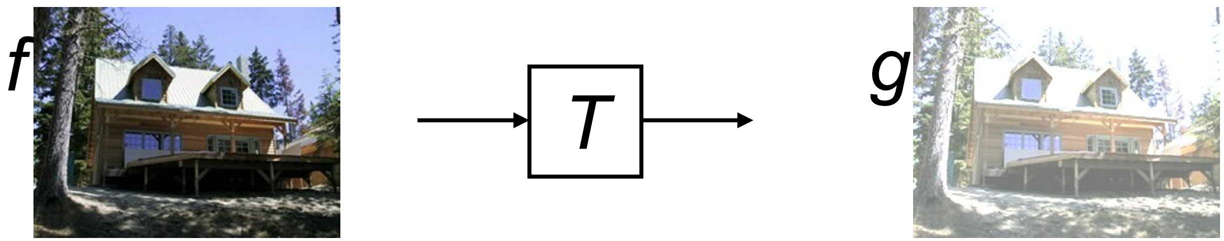
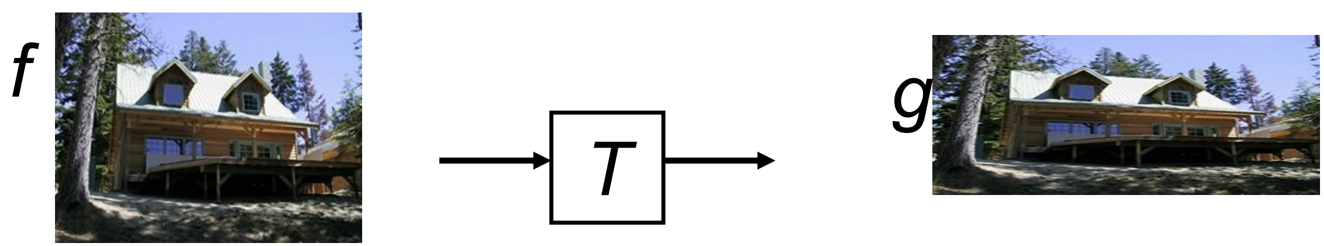


Image warping: change ***domain*** of image

$$g(x,y) = f(T(x,y))$$



Slide credit: A. Efros

Parametric (Global) warping

Examples of parametric warps


translation


rotation


aspect


affine


perspective


cylindrical

Slide credit: A. Efros

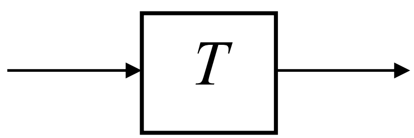
Parametric (Global) Warping

T is a coordinate changing machine

$$\boldsymbol{p}' = T(\boldsymbol{p})$$

Note: T is the same for all points, has relatively few parameters, and does **not** depend on image content

p = (x,y)



p’ = (x’,y’)

Slide credit: A. Efros

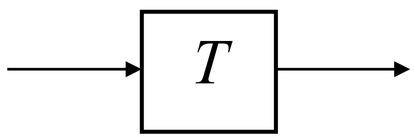
Parametric (Global) Warping

Today we'll deal with linear warps

$$\boldsymbol{p}' \equiv \boldsymbol{T}\boldsymbol{p}$$

T: matrix; p, p': 2D points. Start with normal points and =, then do homogeneous cords and ≡

p = (x,y)



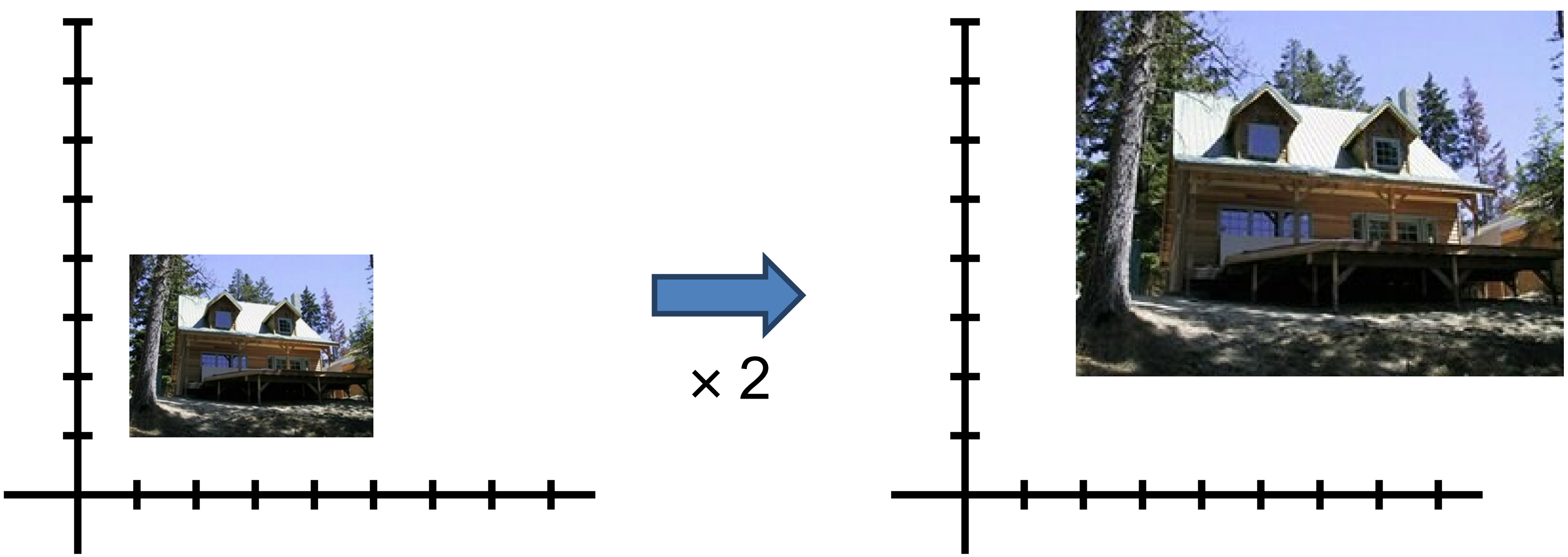
p' = (x',y')

Slide credit: A. Efros

Scaling

Scaling multiplies each component (x,y) by a scalar.
Uniform scaling is the same for all components.

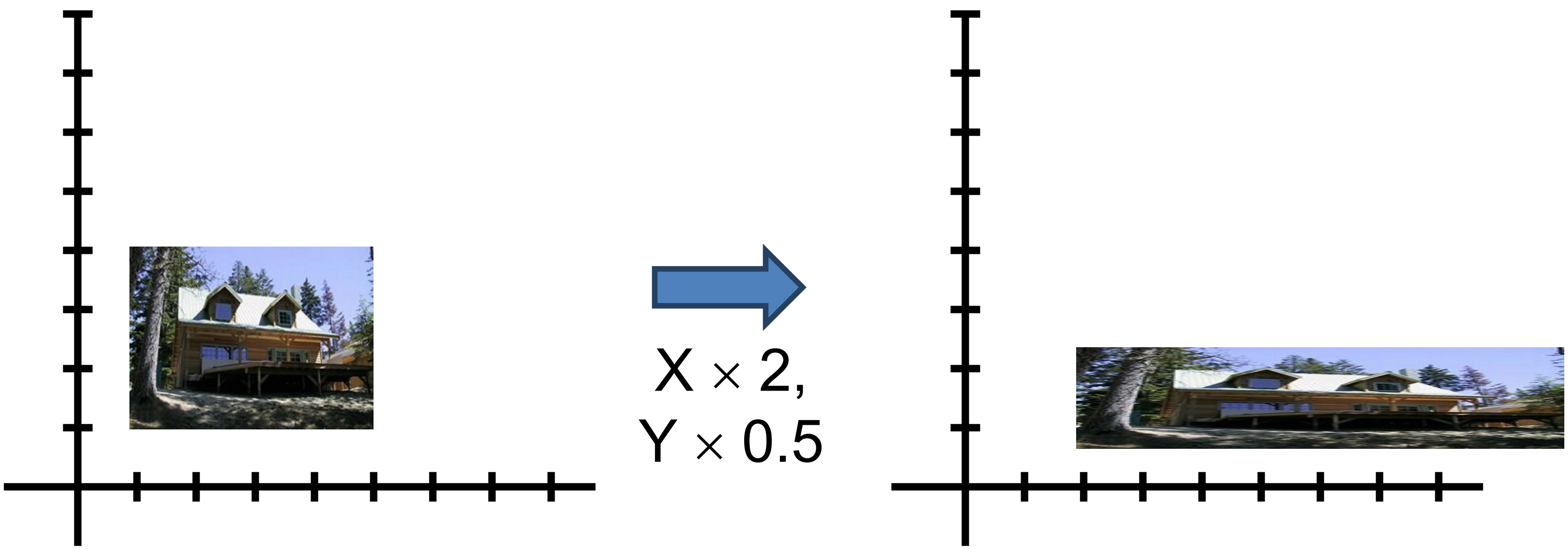
Note the corner goes from (1,1) to (2,2)



Slide credit: A. Efros

Scaling

Non-uniform scaling multiplies each component by a different scalar.



Slide credit: A. Efros

Scaling

What does T look like?

$$x' = ax$$
$$y' = by$$

Let's convert to a matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

scaling matrix S

What's the inverse of S?

Slide credit: A. Efros

2D Rotation

Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

But wait! Aren't sin/cos non-linear?

x' *is* a linear combination/function of *x, y*
x' *is not* a linear function of θ

What's the inverse of R_θ? $\boldsymbol{I} = \boldsymbol{R}_{\boldsymbol{\theta}}^{T} \boldsymbol{R}_{\boldsymbol{\theta}}$

Slide credit: A. Efros

Things You Can Do With 2x2

Identity / No Transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Slide credit: A. Efros

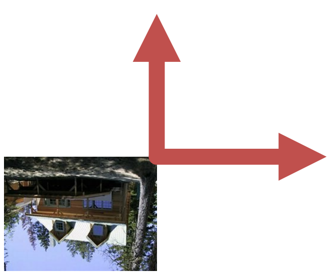
Things You Can Do With 2x2

Before

After

Before

After



2D Mirror About Y-Axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror About X,Y

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

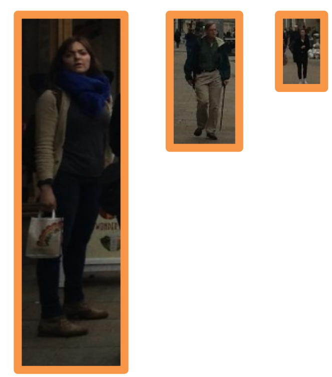
Slide credit: A. Efros

What's Preserved?

3D lines project to 2D lines so lines are preserved

Projections of parallel 3D lines are not necessarily parallel, so not parallelism

Distant objects are smaller so size is not preserved



What's Preserved With a 2x2

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$

After multiplication by T (irrespective of T)

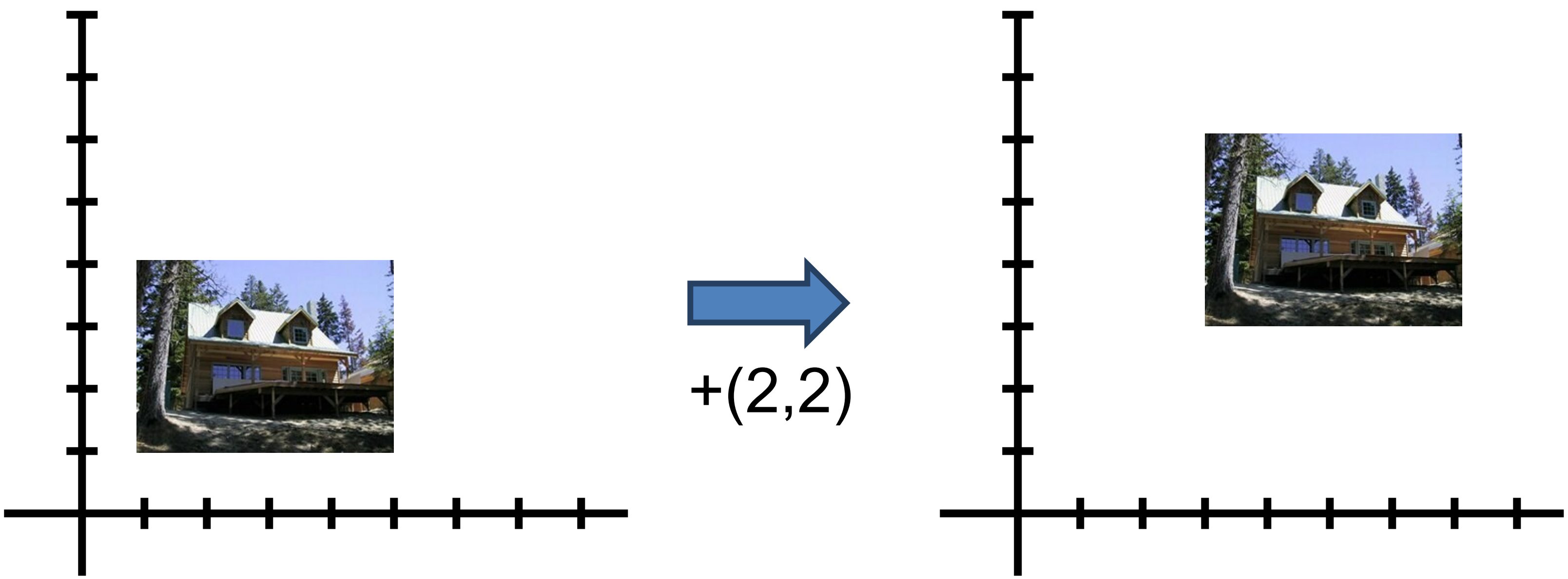
- Origin is origin: **0 = T0**
- Lines are lines
- Parallel lines are parallel

Things You Can't Do With 2x2

What about translation?

$x' = x + t_x,\ y' = y + t_y$

How do we make it linear?

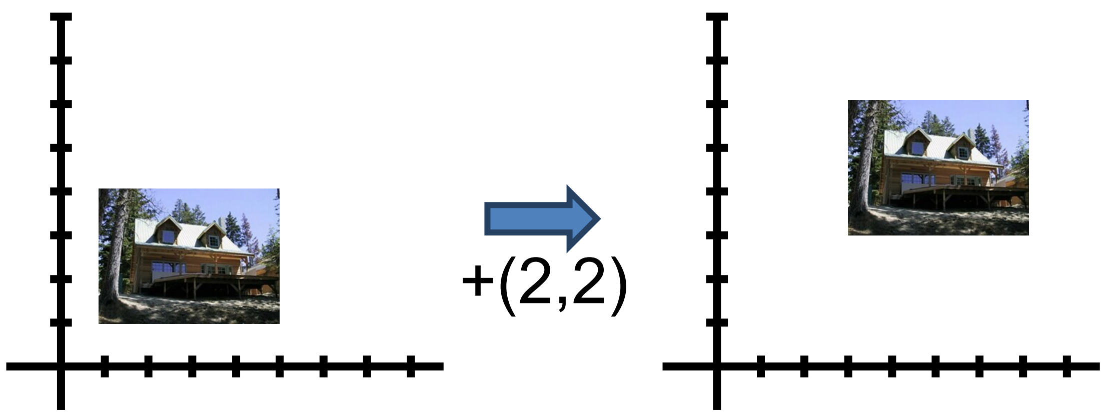


Homogeneous Coordinates Again

What about translation?

$x' = x + t_x,\ y' = y + t_y$

$$\begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} \equiv \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Slide credit: A. Efros

Representing 2D Transformations

How do we represent a 2D transformation?

Let's pick scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} s_x & 0 & a \\ 0 & s_y & b \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

What's	a	b	d	e	f
	0	0	0	0	1

Affine Transformations

Affine: *linear transformation* plus *translation*

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Will the last coordinate w' always be 1?

In general (without homogeneous coordinates)

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

Matrix Composition

We can combine transformations via matrix multiplication.

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \equiv \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{T(t_x, t_y)} \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R(\theta)} \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{S(s_x, s_y)} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Does order matter?

Slide credit: A. Efros

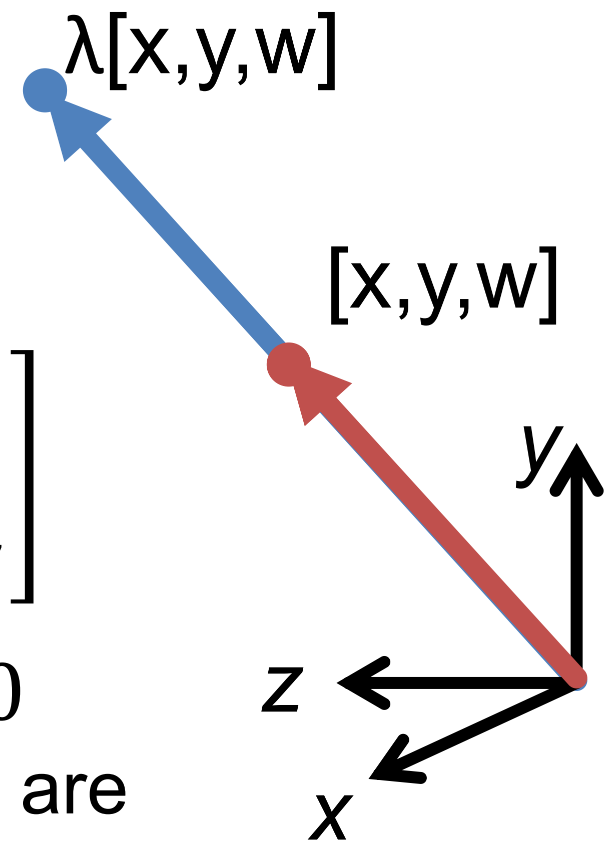
What's Preserved With Affine

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv \boldsymbol{T} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

After multiplication by T (irrespective of T)

- ~~**Origin is origin: 0 = T0**~~
- Lines are lines
- Parallel lines are parallel

Homogeneous Equivalence



Triple / Equivalent

Double / Equals

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \equiv \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \leftrightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \lambda \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

$$\lambda \neq 0$$

Two homogeneous coordinates are **equivalent** if they are proportional to each other. **Not = !**

Perspective Transformations

Set bottom row to not [0,0,1]

Called a perspective/projective transformation or a ***homography***

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Can compute [x’,y’,w’] via matrix multiplication.

How do we get a 2D point?

(x’/w’, y’/w’)

Perspective Transformations

Set bottom row to not [0,0,1]

Called a perspective/projective transformation or a ***homography***

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

How many degrees of freedom?

How Many Degrees of Freedom?

Can always scale coordinate by non-zero value

Perspective $$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \equiv \frac{1}{i} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \equiv \frac{1}{i} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \equiv \begin{bmatrix} a/i & b/i & c/i \\ d/i & e/i & f/i \\ g/i & h/i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Homography can always be re-scaled by $\lambda \neq 0$

Typically pick it so last entry is 1.

What's Preserved With Perspective

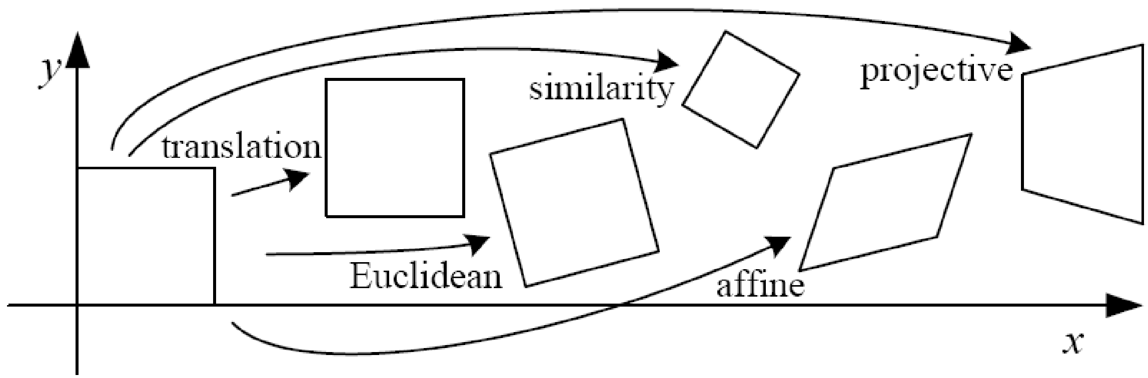
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv \boldsymbol{T} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

After multiplication by T (irrespective of T)

- ~~**Origin is origin: 0 = T0**~~
- Lines are lines
- ~~Parallel lines are parallel~~
- ~~Ratios between distances~~

Transformation Families

In general: transformations are a nested set of groups



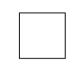
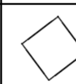
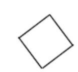
Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation $+\cdots$	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths $+\cdots$	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles $+\cdots$	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism $+\cdots$	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

Diagram credit: R. Szeliski

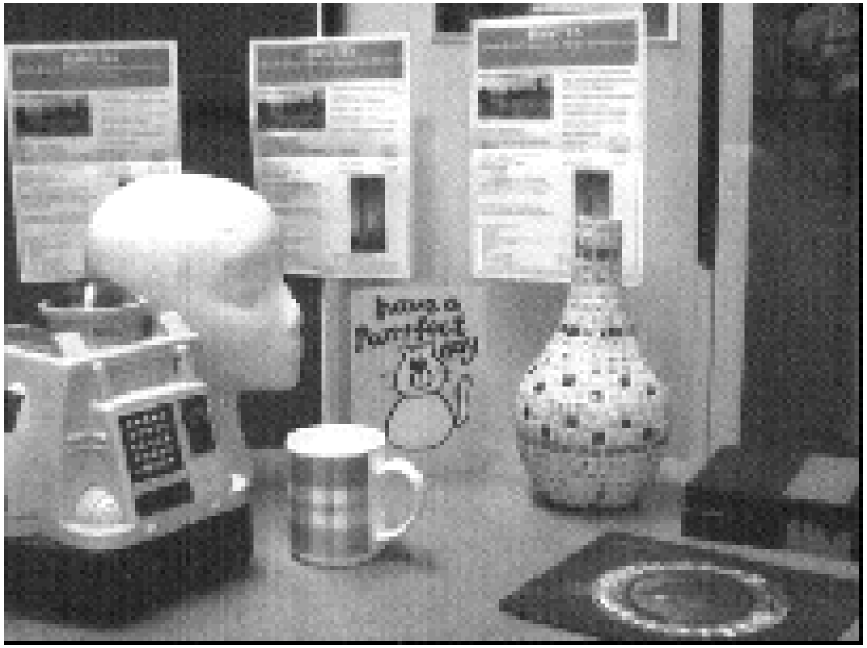
What Can Homographies Do?

Homography example 1: any two views of a *planar* surface

Figure Credit: S. Lazebnik

What Can Homographies Do?

Homography example 2: any images from two cameras sharing a camera center



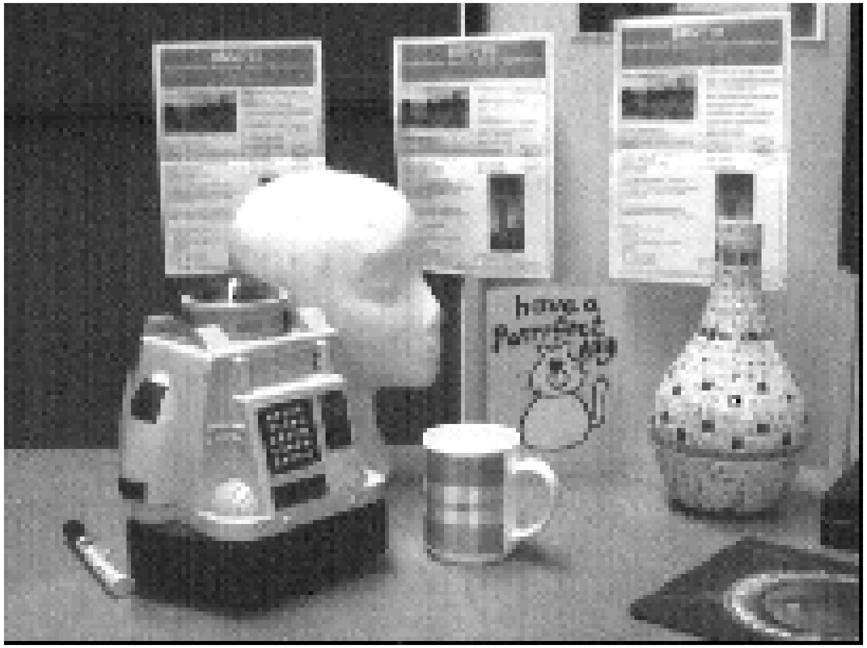
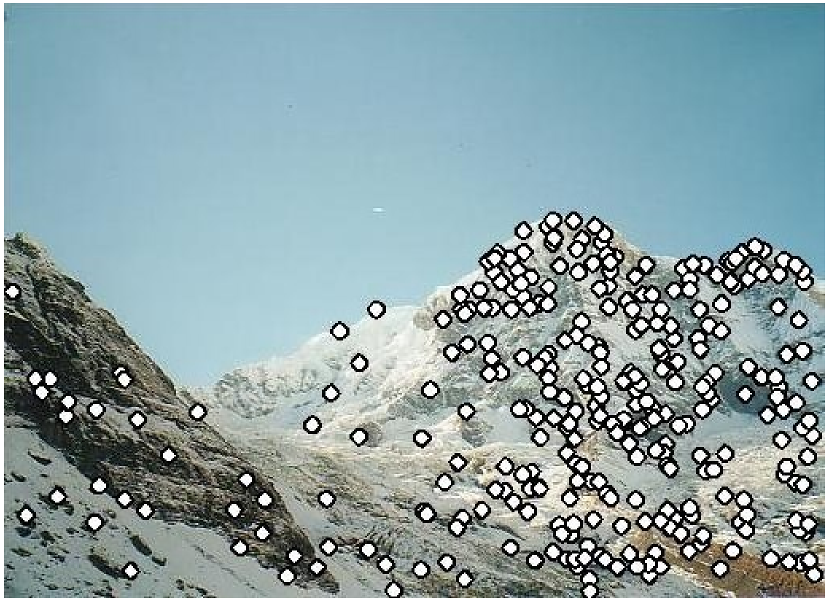


Figure Credit: S. Lazebnik

What Can Homographies Do?

Homography sort of example “3”: far away scene that can be approximated by a plane



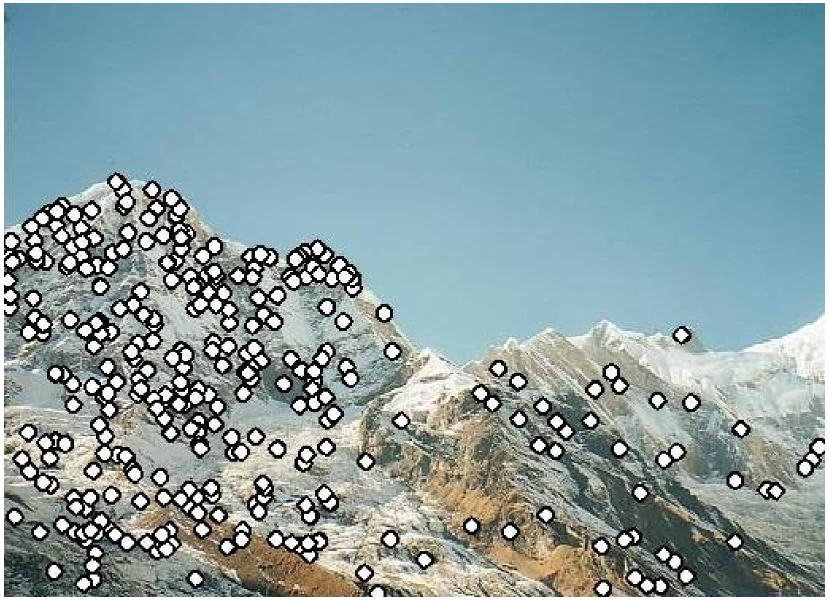


Figure credit: Brown & Lowe

Fun With Homographies

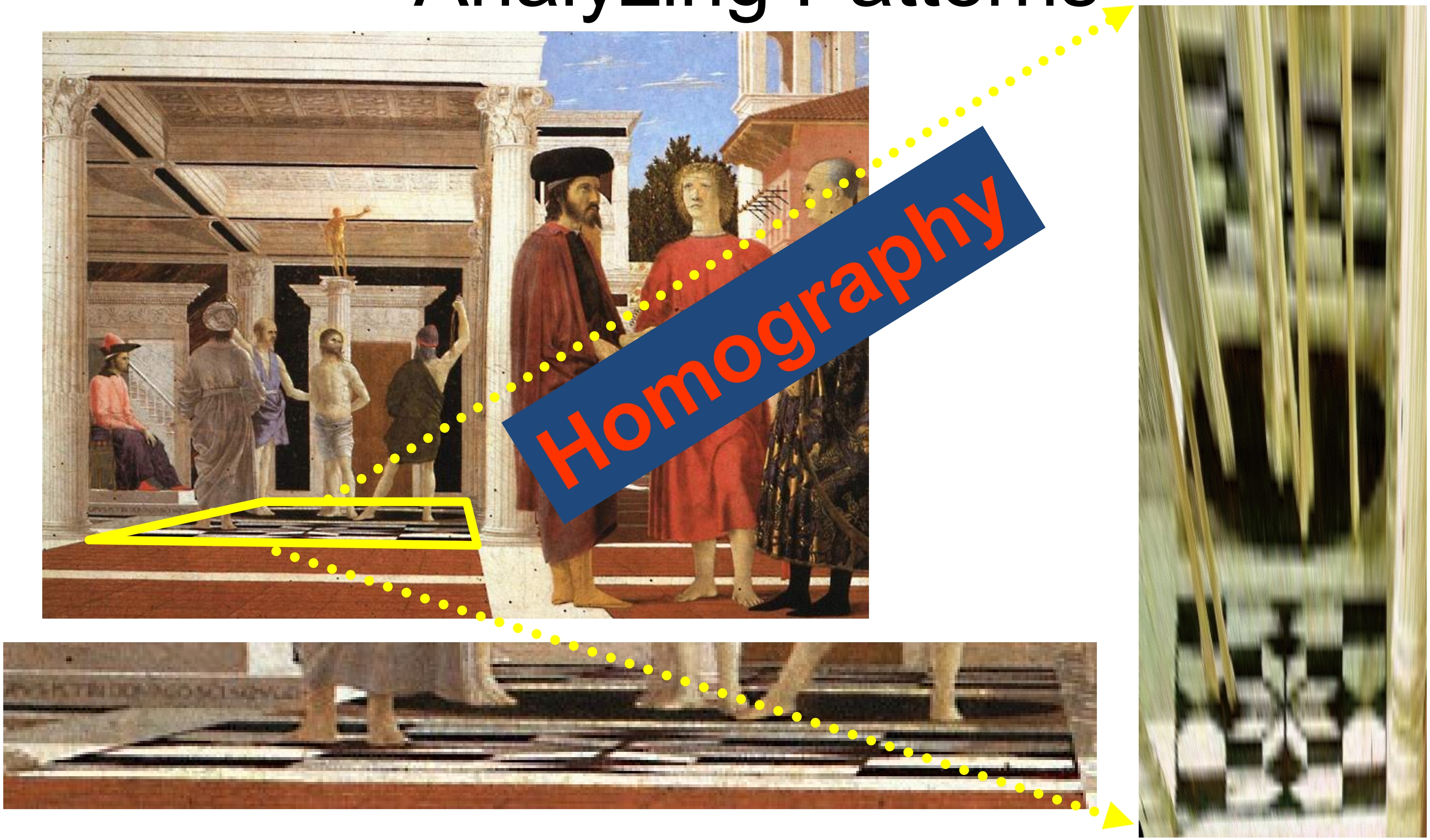
Original image

St. Petersburg
photo by A. Tikhonov

Virtual camera rotations

Slide Credit: A. Efros

Analyzing Patterns

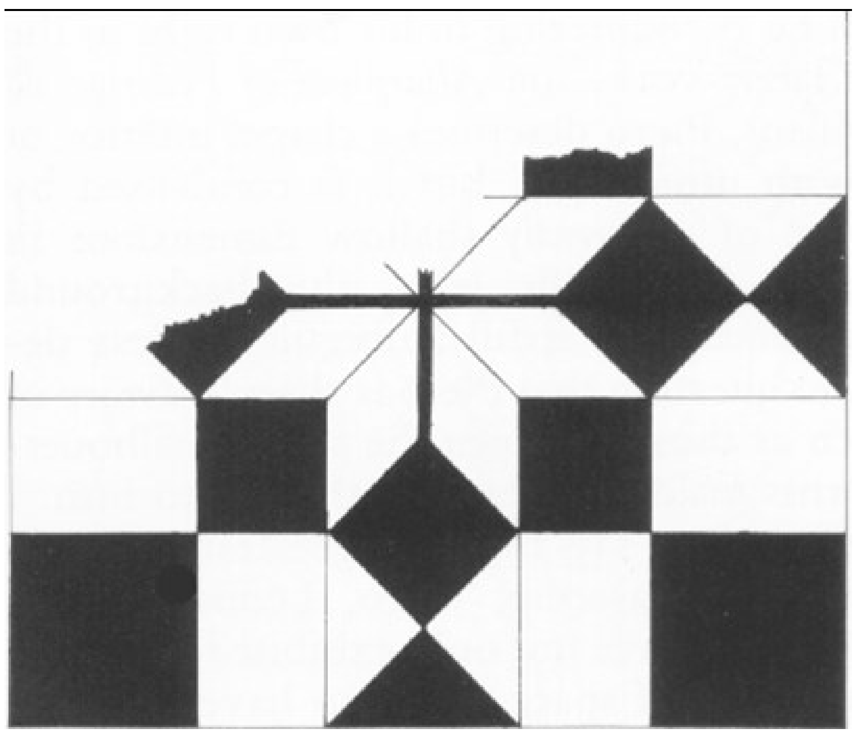


The floor (enlarged)

Automatically rectified floor

Slide from A. Criminisi

Analyzing Patterns



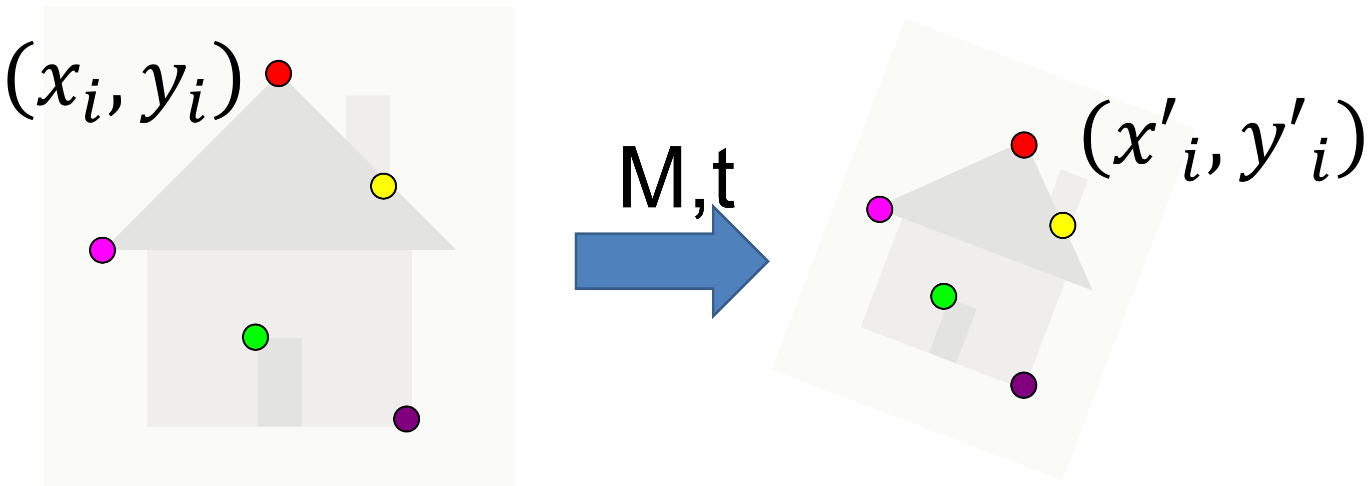
From Martin Kemp *The Science of Art* *(manual reconstruction)*

Automatic rectification

Slide from A. Criminisi

Fitting Transformations

Setup: have pairs of correspondences



$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \boldsymbol{t}$$

Slide Credit: S. Lazebnik

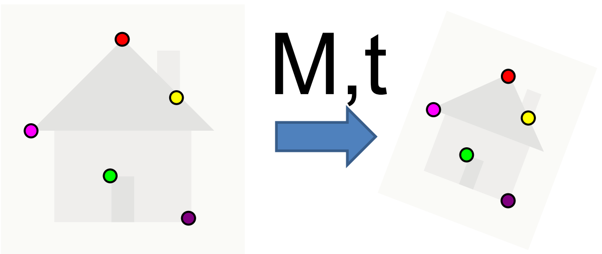
Fitting Transformation

Affine Transformation: M,t

Data: (x_i,y_i,x'_i,y'_i) for $i=1,\ldots,k$

Model:
$[x'_i,y'_i] = \mathbf{M}[x_i,y_i]+\mathbf{t}$

Objective function:
$||[x'_i,y'_i] - (\mathbf{M}[x_i,y_i]+\mathbf{t})||^2$



Fitting Transformations

Given correspondences: $[x'_i, y'_i] \leftrightarrow [x_i, y_i]$

$$\begin{bmatrix} x_i{}' \\ y_i{}' \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Set up two equations per point

$$\begin{bmatrix} \vdots \\ x_i' \\ y_i' \\ \vdots \end{bmatrix} = \begin{bmatrix} & & \cdots & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_x \\ t_y \end{bmatrix}$$

Fitting Transformations

$$\underbrace{\begin{bmatrix} \vdots \\ x_i' \\ y_i' \\ \vdots \end{bmatrix}}_{2k} = \overbrace{\begin{bmatrix} & & \cdots & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ & & \cdots & & & \end{bmatrix}}^{6} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_x \\ t_y \end{bmatrix}$$

2 equations per point, 6 unknowns

How many points do we need to properly constrain the problem?

Fitting Transformations

$$\underbrace{\begin{bmatrix} \vdots \\ x_i' \\ y_i' \\ \vdots \end{bmatrix}}_{\mathbf{b}\ (2k)} = \underbrace{\begin{bmatrix} & & \cdots & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ & & \cdots & & & \end{bmatrix}}_{\mathbf{A}\ (6)} \underbrace{\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_x \\ t_y \end{bmatrix}}_{\mathbf{x}}$$

2k

6

b **A** **x**

Want: **b = Ax** (**x** contains all parameters)

Overconstrained, so solve $\arg\min \|\boldsymbol{Ax} - \boldsymbol{b}\|$

How?

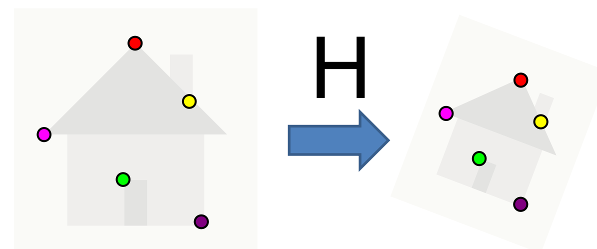
Fitting Transformation

Homography: H

Data: (x_i, y_i, x'_i, y'_i) for $i=1,\ldots,k$

Model:
$[x'_i, y'_i, 1] \equiv \mathbf{H}[x_i, y_i, 1]$

Objective function:
It's complicated



Fitting Transformation

$$\boldsymbol{p_i} = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

9

k points → 2k

$$\begin{bmatrix} \mathbf{0}^T & -\boldsymbol{p}_1^T & y_1'\boldsymbol{p}_1^T \\ \boldsymbol{p}_1^T & \mathbf{0}^T & -x_1'\boldsymbol{p}_1^T \\ & \vdots & \\ \mathbf{0}^T & -\boldsymbol{p}_n^T & y_n'\boldsymbol{p}_n^T \\ \boldsymbol{p}_n^T & \mathbf{0}^T & -x_n'\boldsymbol{p}_n^T \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \end{bmatrix} = \mathbf{0}$$

Row 1 of H

$$\boldsymbol{Ah} = \mathbf{0}$$

What do we use from last time?

$$h^* = \arg \min_{\|h\|=1} \|Ah\|^2$$

→ Eigenvector of A^TA with smallest eigenvalue

In Practice

9

k points → 2k

$$\begin{bmatrix} \mathbf{0}^T & -\boldsymbol{p}_1^T & y_1'\boldsymbol{p}_1^T \\ \boldsymbol{p}_1^T & \mathbf{0}^T & -x_1'\boldsymbol{p}_1^T \\ & \vdots & \\ \mathbf{0}^T & -\boldsymbol{p}_n^T & y_n'\boldsymbol{p}_n^T \\ \boldsymbol{p}_n^T & \mathbf{0}^T & -x_n'\boldsymbol{p}_n^T \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \end{bmatrix} = \mathbf{0}$$

Row 1 of H

$$\boldsymbol{p}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

$$\boldsymbol{A}\boldsymbol{h} = \mathbf{0}$$

Should consist of lots of {x,y,x’,y’,0, and 1}.

If it fails, **assume** you mistyped.

Re-type differently and compare all entries.

Debug first with transformations you know.

Small Nagging Detail

||Ah||^2 doesn't measure model fit (it's an *algebraic error* that's mainly just convenient to minimize)

Also, there's a least-squares setup that's wrong but often works.

Really want *geometric error*:

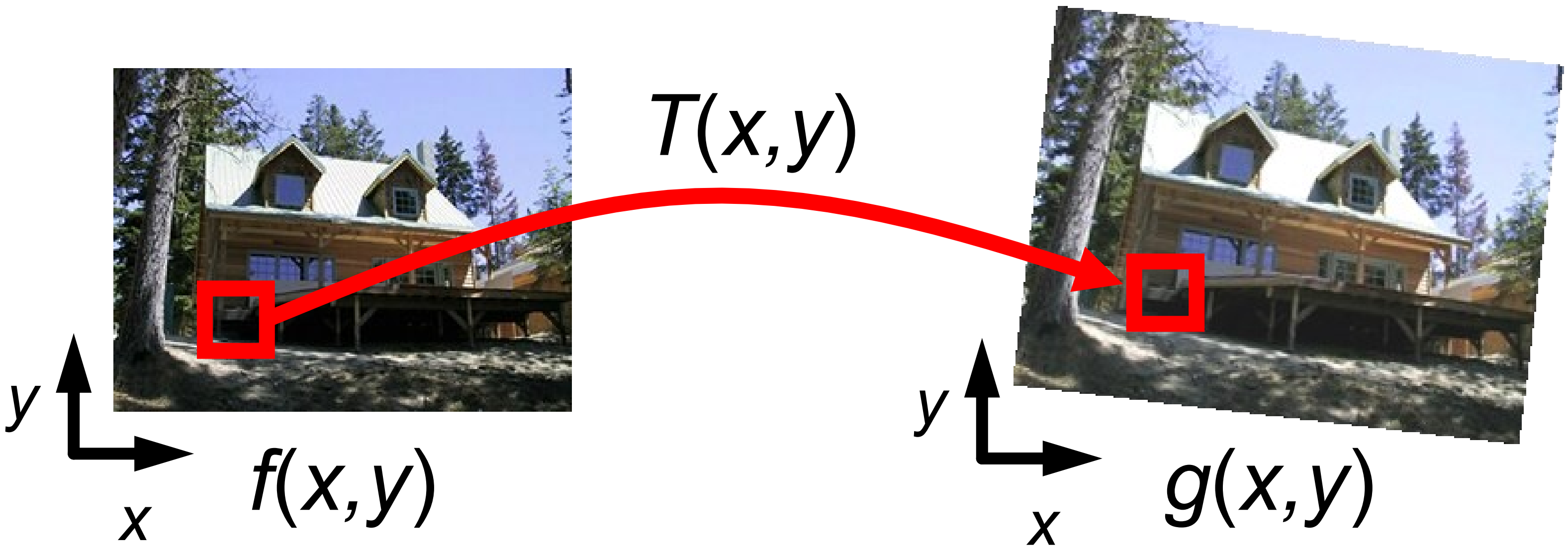
$$\sum_{i=1}^{k} \left\| [x_i', y_i'] - T([x_i, y_i]) \right\|^2 + \left\| [x_i, y_i] - T^{-1}([x_i', y_i']) \right\|^2$$

Small Nagging Detail

Solution: initialize with algebraic (min ||Ah||), optimize with geometric using standard non-linear optimizer

In RANSAC, we always take just enough points to fit. Why might this not make a big difference when fitting a model with RANSAC?

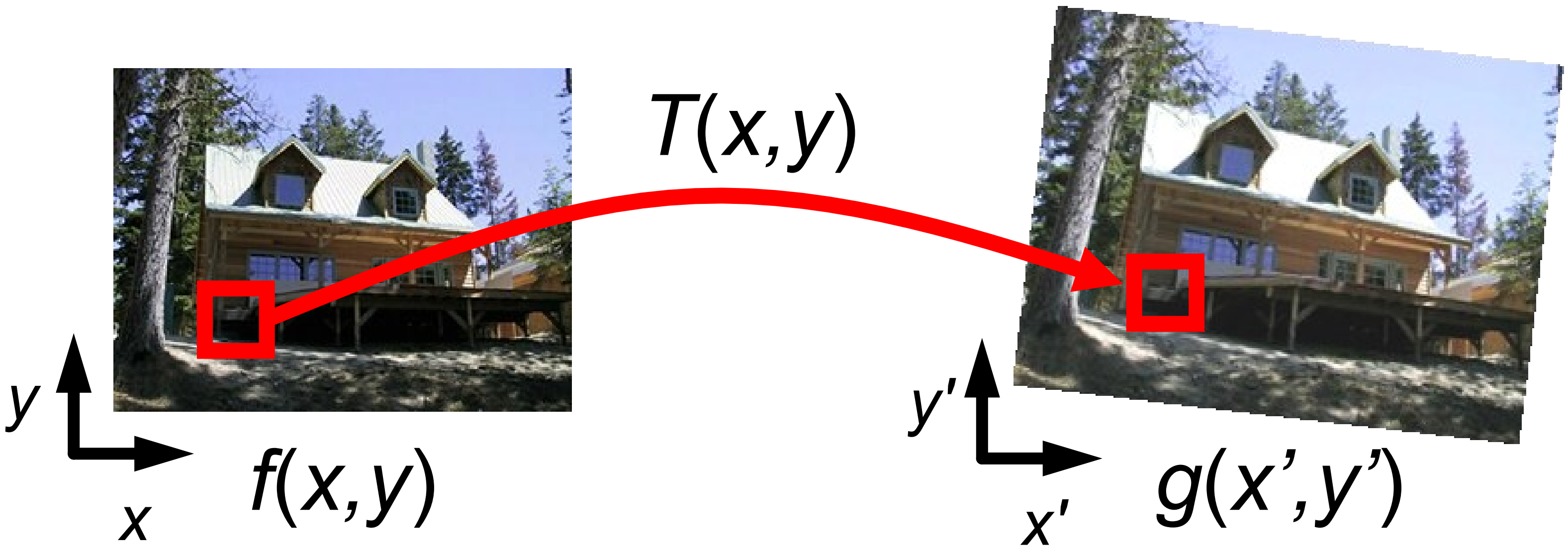
Image Warping



Given a coordinate transform $(x',y') = T(x,y)$ and a source image $f(x,y)$, how do we compute a transformed image $g(x',y') = f(T(x,y))$?

Slide Credit: A. Efros

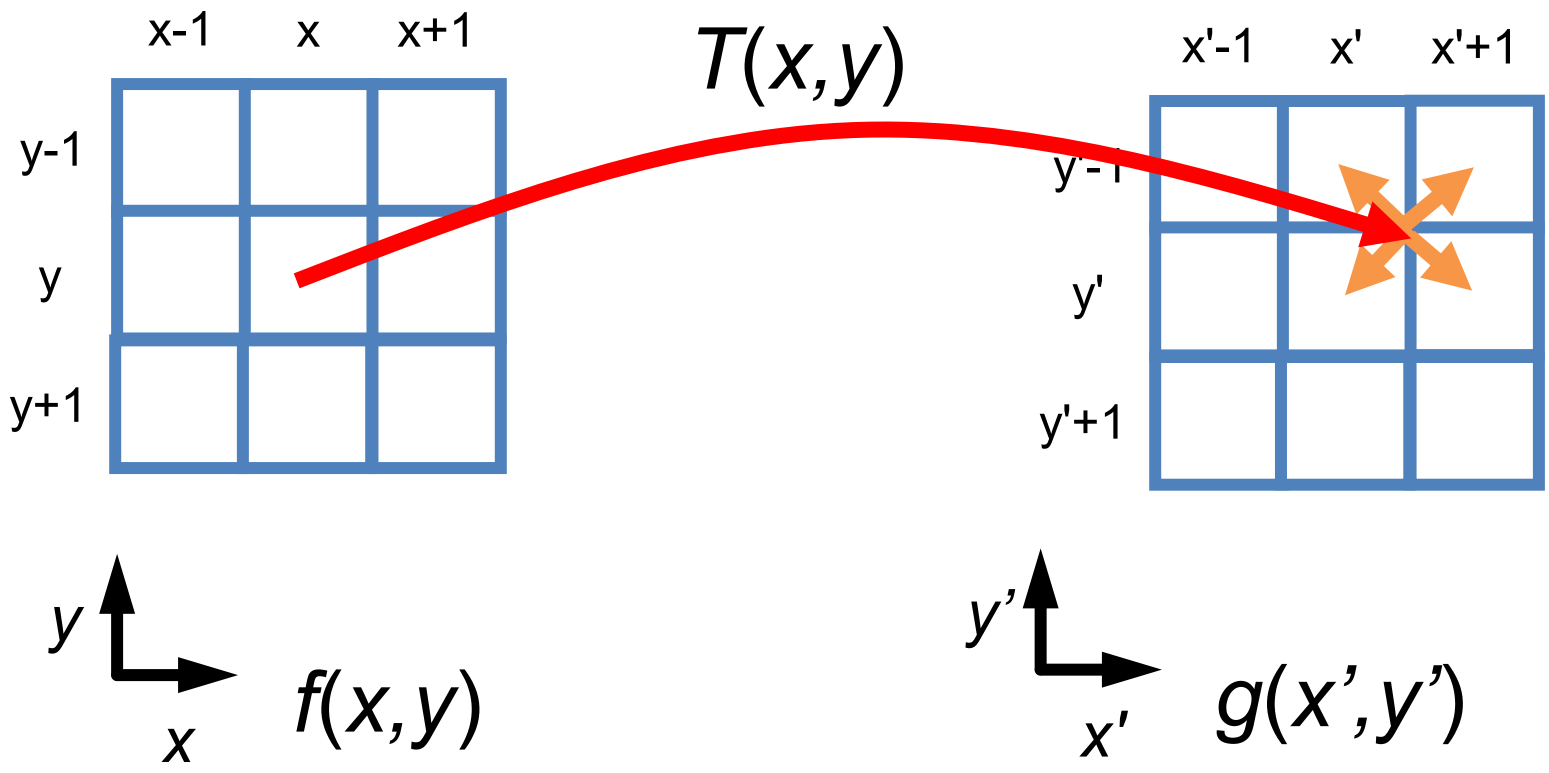
Forward Warping



Send the value at each pixel (x,y) to the new pixel (x',y') = T([x,y])

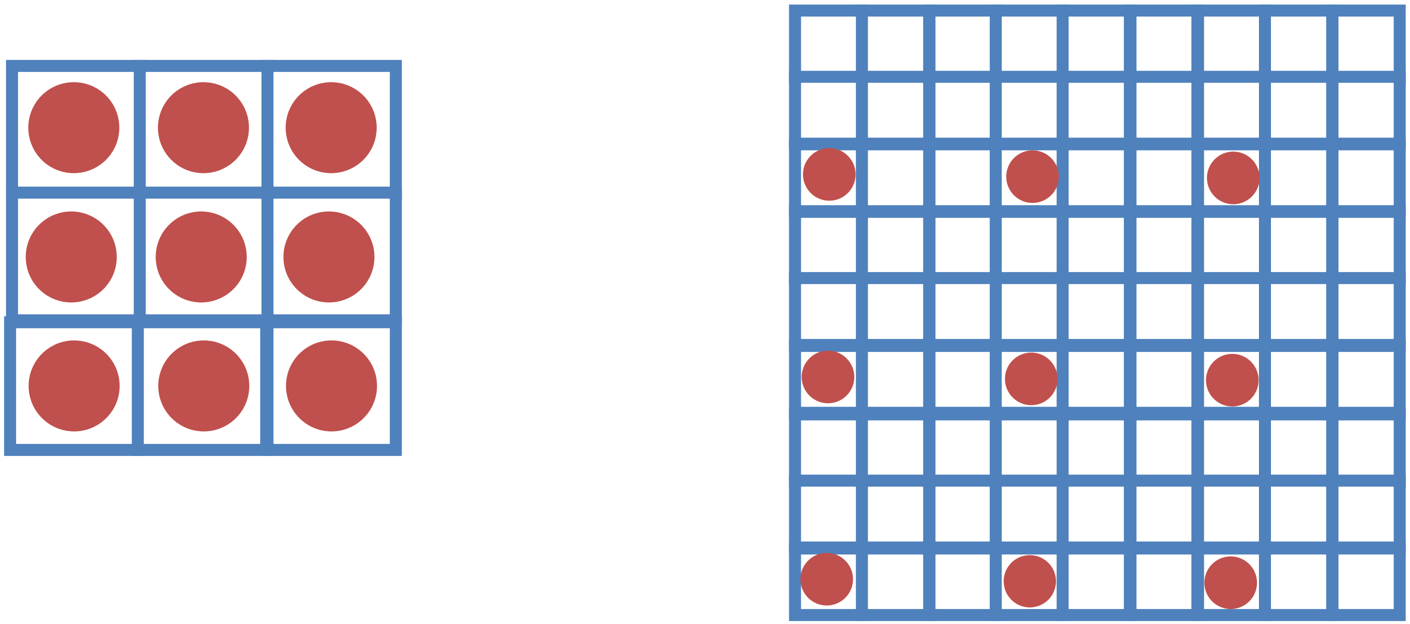
Slide Credit: A. Efros

Forward Warping



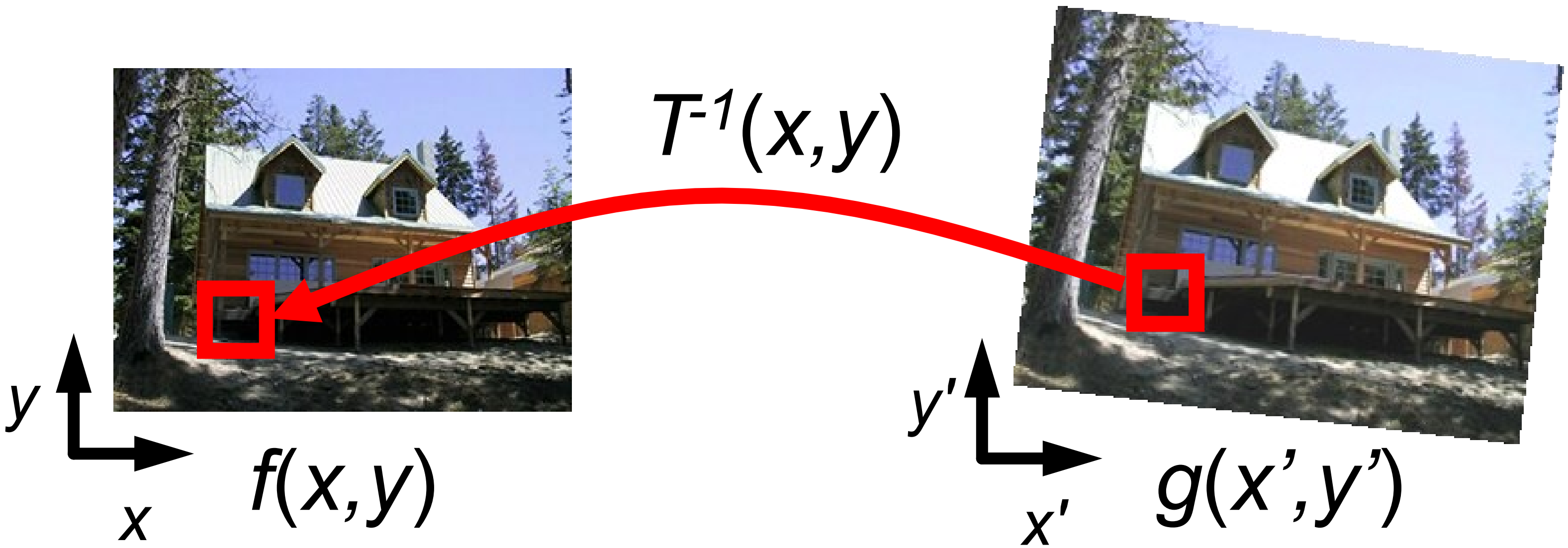
If you don't hit an exact pixel, give the value to each of the neighboring pixels ("splatting").

Forward Warping



Suppose T(x,y) scales by a factor of 3.
Hmmmm.

Inverse Warping

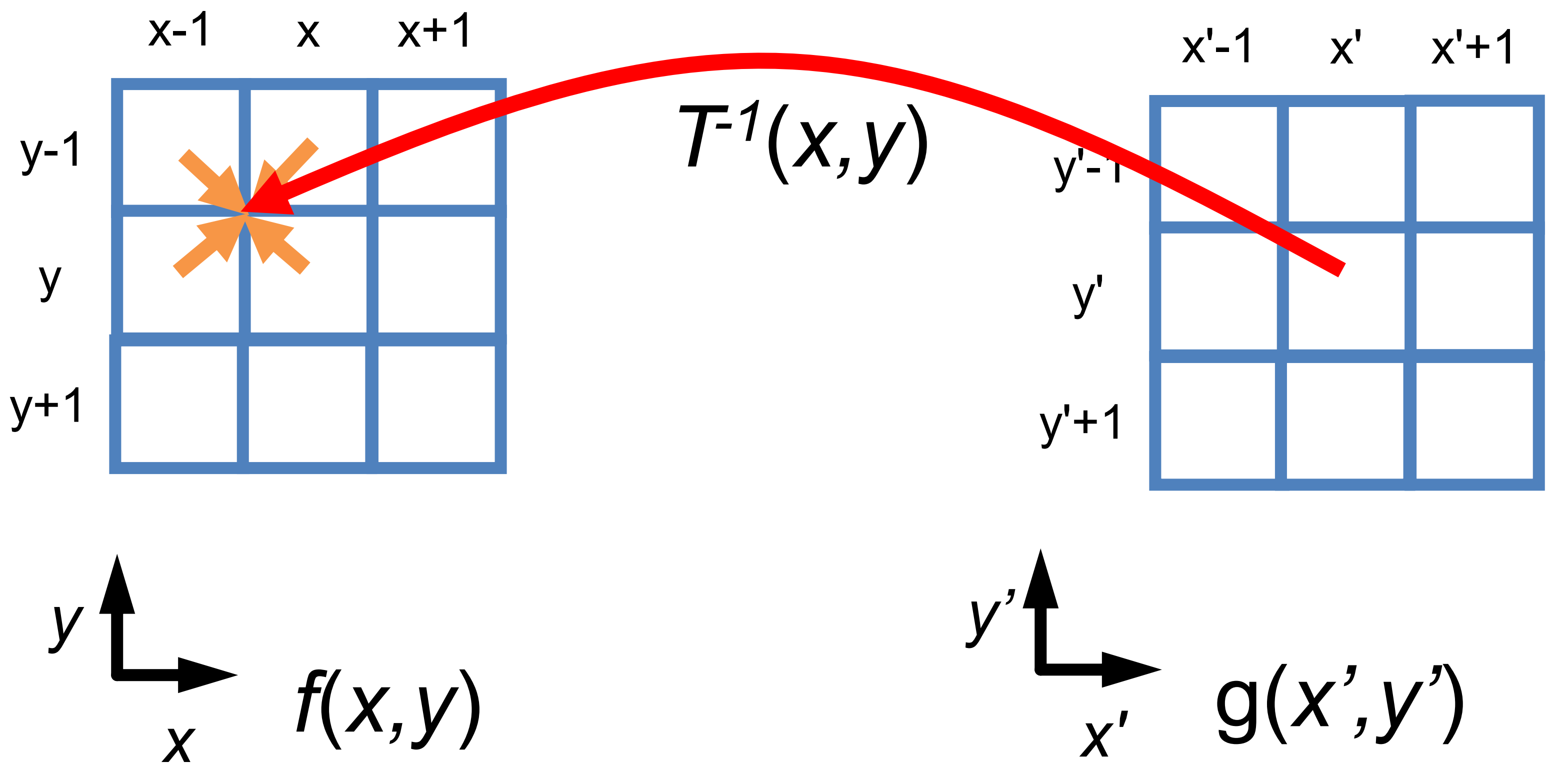


Find out where each pixel g(x’,y’) should get its value from, and steal it.

Note: requires ability to invert T

Slide Credit: A. Efros

Inverse Warping

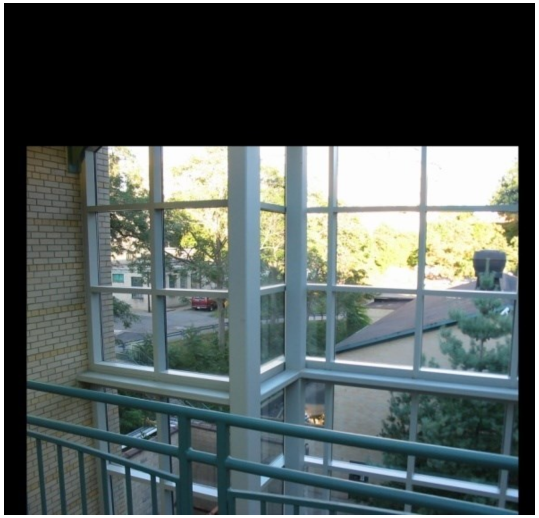


If you don't hit an exact pixel, figure out how to take it from the neighbors.

Mosaicing

Warped Input 1 I_1

Warped Input 2 I_2



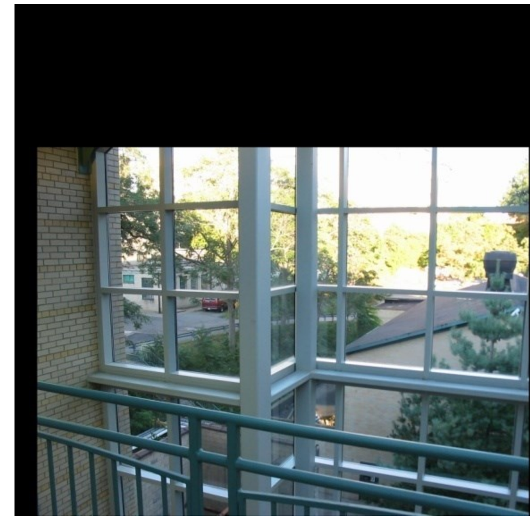
Can warp an image. Pixels that don't have a corresponding pixel in the image are set to a chosen value (often 0)

Image Credit: A. Efros

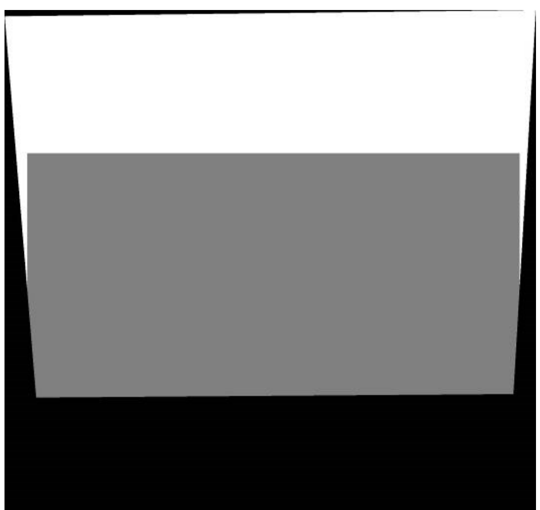
Mosaicing

Warped
Input 1
I_1

Warped
Input 2
I_2



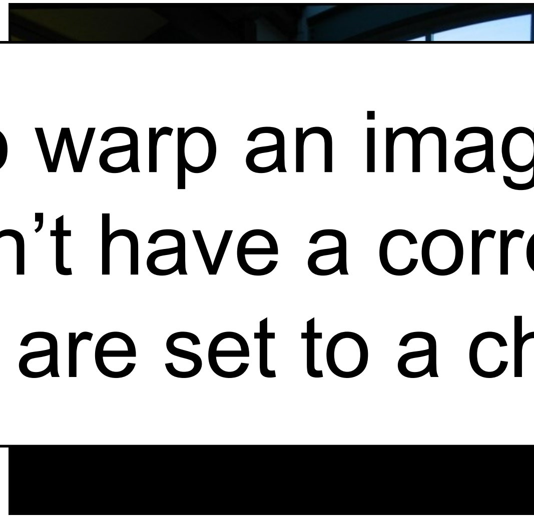
α

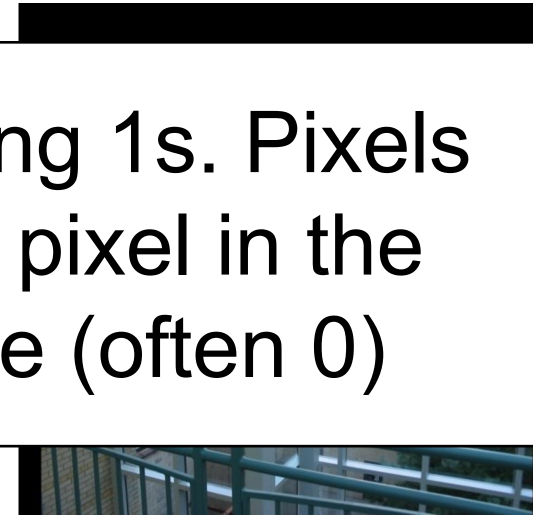


$\alpha I_1 + (1-\alpha) I_2$

Image Credit: A. Efros

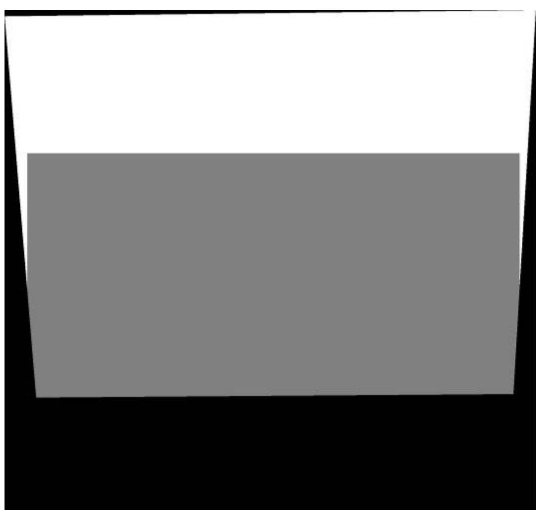
Mosaicing





Can also warp an image containing 1s. Pixels that don’t have a corresponding pixel in the image are set to a chosen value (often 0)

α



$\alpha I_1 + (1-\alpha) I_2$

Slide Credit: A. Efros

Putting it Together

How do you make a panorama?

Step 1: Find “features” to match

Step 2: Describe Features

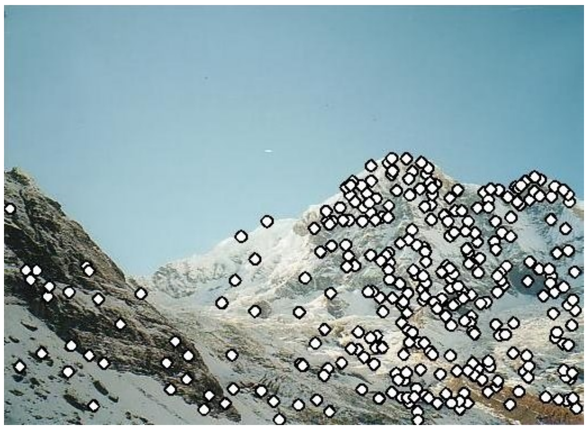
Step 3: Match by Nearest Neighbor

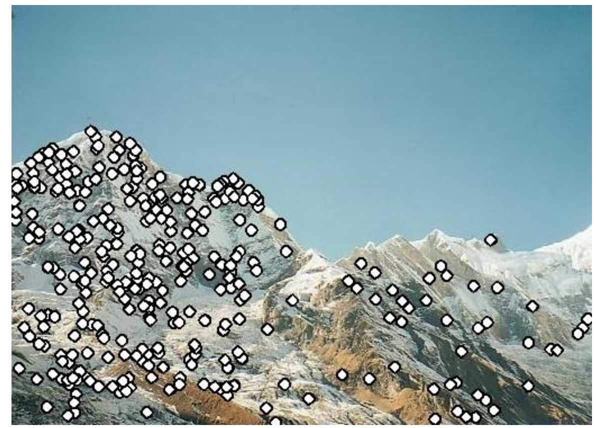
Step 4: Fit H via RANSAC

Step 5: Blend Images

Putting It Together 1

Find corners/blobs

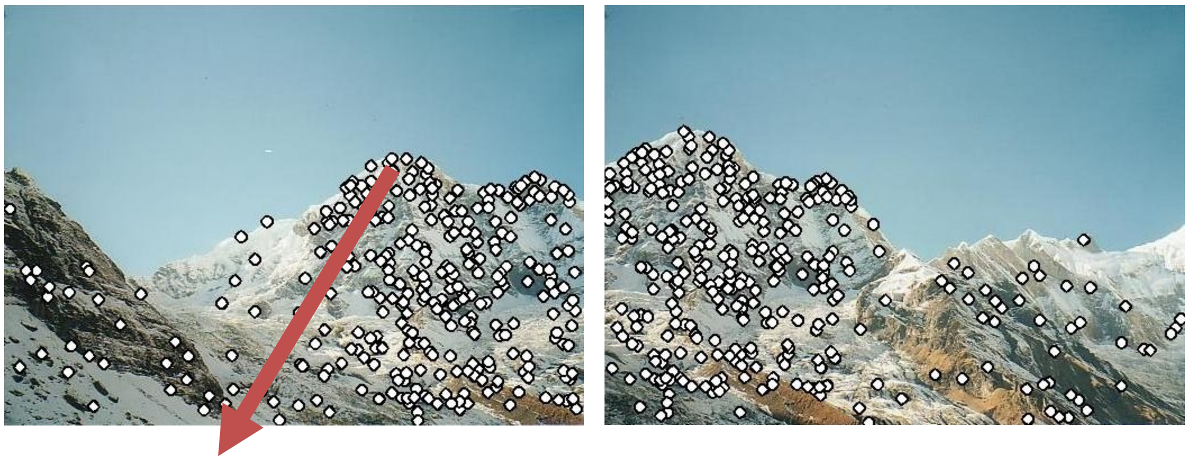




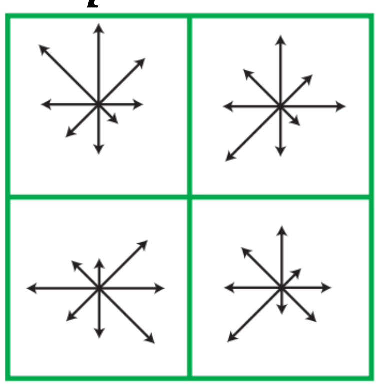
- (Multi-scale) Harris; or
- Laplacian of Gaussian

Putting It Together 2

Describe Regions Near Features



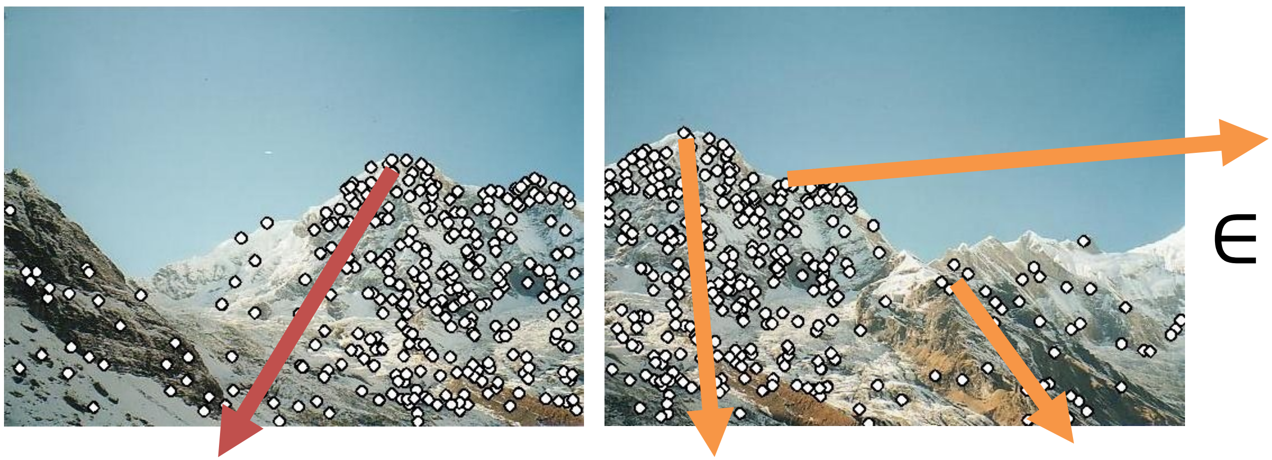
$x_q \in R^{128}$



Build histogram of gradient orientations (SIFT)
(But in practice use opencv)

Putting It Together 3

Match Features Based On Region



$x_3 \in R^{128}$

$x_q \in R^{128}$ $x_1 \in R^{128}$ $x_2 \in R^{128}$

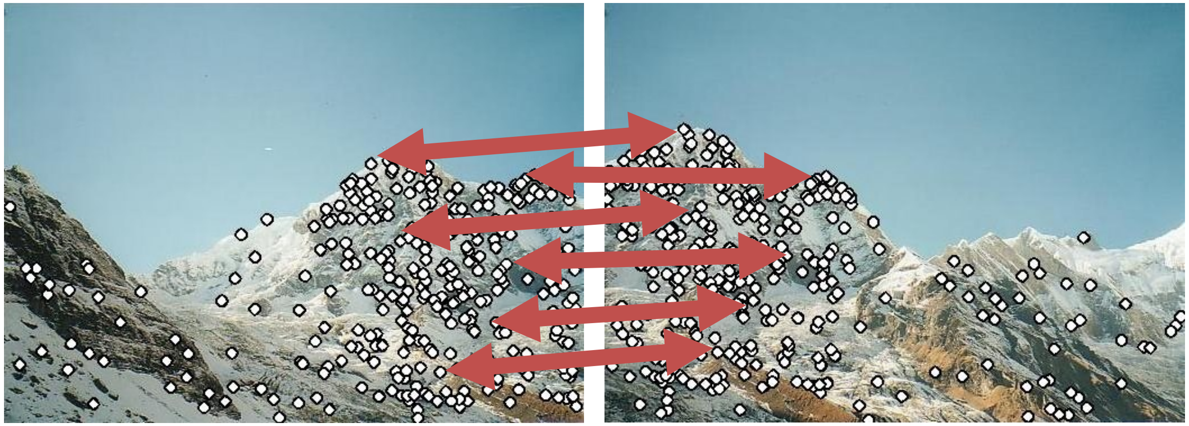
Sort by distance to: x_q $\|x_q - x_1\| < \|x_q - x_2\| < \|x_q - x_3\|$

Accept match if: $\|x_q - x_1\| / \|x_q - x_2\|$

Nearest neighbor is far closer than 2nd nearest neighbor

Putting It Together 4

Fit transformation H via RANSAC



for trial in range(Ntrials):
    Pick sample
    Fit model
    Check if more inliers
Re-fit model with most inliers

$$\arg\min_{\|\boldsymbol{h}\|=1} \|\boldsymbol{A}\boldsymbol{h}\|^2$$

Putting It Together 5

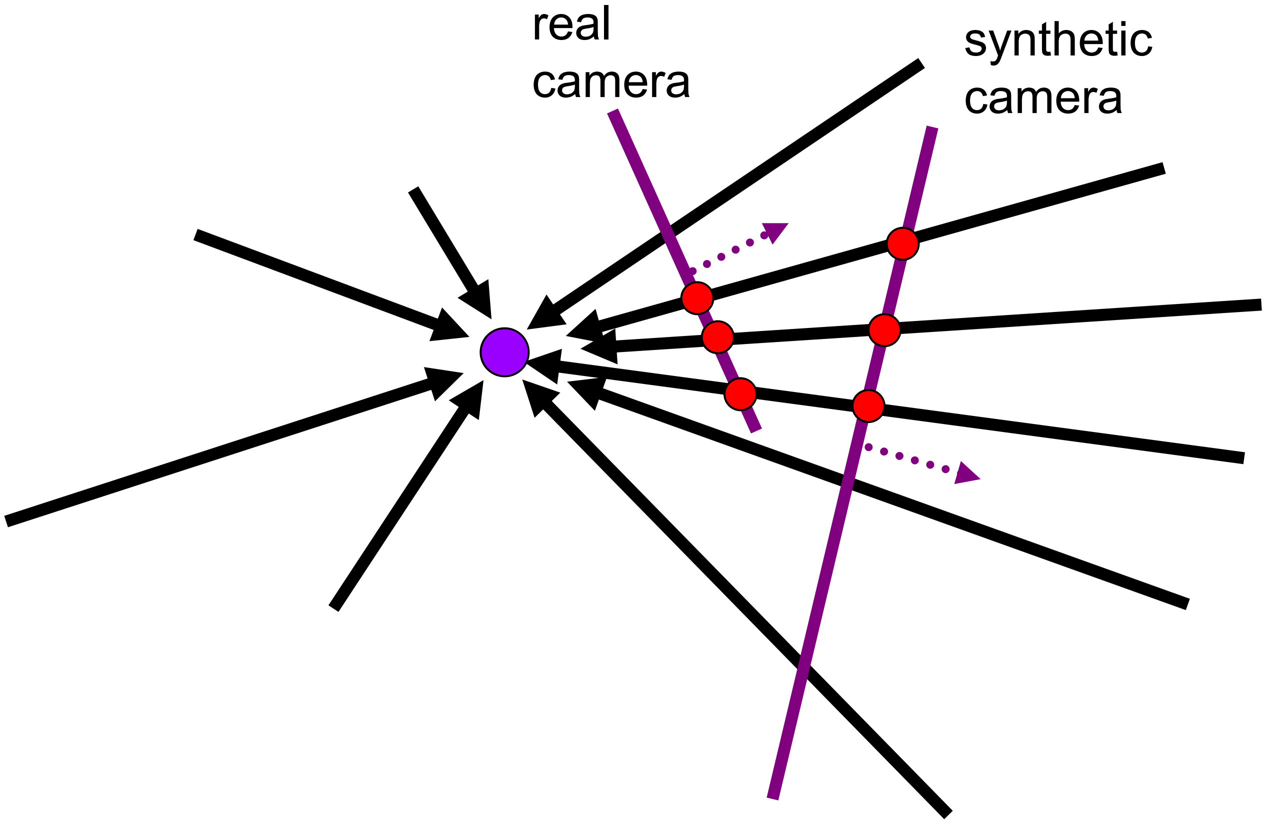
Warp images together

Resample images with inverse warping and blend

(but in practice, just call opencv for inverse warping)

Backup

A pencil of rays contains all views



Can generate any synthetic camera view
as long as it has **the same center of projection**!

Slide Credit: A. Efros

Bonus Art

Analyzing Patterns

St. Lucy Altarpiece, **D. Veneziano**

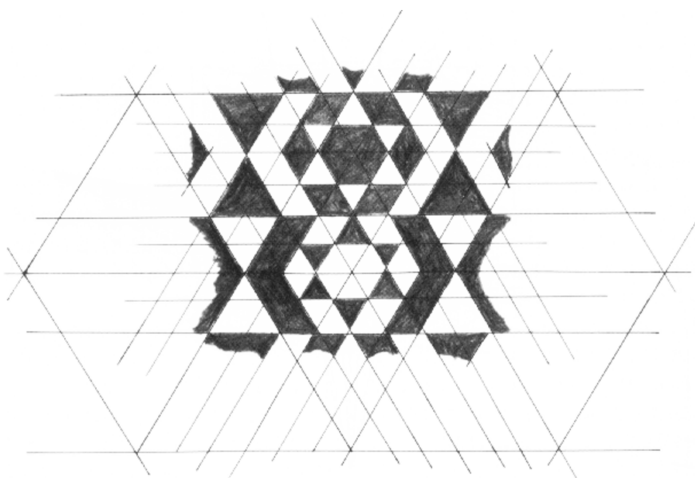
Slide from A. Criminisi

What is the (complicated) shape of the floor pattern?

Automatically rectified floor

Analyzing Patterns

Automatic rectification



From Martin Kemp, *The Science of Art* *(manual reconstruction)*

Slide from A. Criminisi

Homography Derivation

- This has gotten cut in favor of showing more of the setup.
- The key to the set-up is to try to move towards a setup where you can pull [h1,h2,h3] out, or where each row is a linear equation in [h1,h2,h3]

Want: $$\begin{bmatrix} x_i' \\ y_i' \\ w_i' \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix}$$

$$\boldsymbol{p_i} = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix} \equiv \boldsymbol{H}\boldsymbol{p_i} \equiv \begin{bmatrix} \boldsymbol{h}_1^T \\ \boldsymbol{h}_2^T \\ \boldsymbol{h}_3^T \end{bmatrix} \boldsymbol{p_i} \equiv \begin{bmatrix} \boldsymbol{h}_1^T \boldsymbol{p_i} \\ \boldsymbol{h}_2^T \boldsymbol{p_i} \\ \boldsymbol{h}_3^T \boldsymbol{p_i} \end{bmatrix}$$

Recall: $\boldsymbol{a} \equiv \boldsymbol{b} \rightarrow \boldsymbol{a} = \lambda \boldsymbol{b}$ **In turn** $\rightarrow \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$

In the end want: $$\begin{bmatrix} x_i' \\ y_i' \\ w_i' \end{bmatrix} \times \begin{bmatrix} \boldsymbol{h}_1^T \boldsymbol{p_i} \\ \boldsymbol{h}_2^T \boldsymbol{p_i} \\ \boldsymbol{h}_3^T \boldsymbol{p_i} \end{bmatrix} = \boldsymbol{0}$$

Why Cross products? Cross products have explicit forms

Fitting Transformation

Want:

$$\begin{bmatrix} x_i' \\ y_i' \\ w_i' \end{bmatrix} \times \begin{bmatrix} \boldsymbol{h}_1^T \boldsymbol{p}_i \\ \boldsymbol{h}_2^T \boldsymbol{p}_i \\ \boldsymbol{h}_3^T \boldsymbol{p}_i \end{bmatrix} = \boldsymbol{0}$$

Note: calculate this explicitly. It looks ugly, but do it by doing [a,b,c] x [a',b',c'] then re-substituting.

Cross-product

$$\begin{bmatrix} y_i' \boldsymbol{h}_3^T \boldsymbol{p}_i - w_i' \boldsymbol{h}_2^T \boldsymbol{p}_i \\ w_i' \boldsymbol{h}_1^T \boldsymbol{p}_i - x_i' \boldsymbol{h}_3^T \boldsymbol{p}_i \\ x_i' \boldsymbol{h}_2^T \boldsymbol{p}_i - y_i' \boldsymbol{h}_1^T \boldsymbol{p}_i \end{bmatrix} = \boldsymbol{0}$$

You want to be able to right-multiply by [h1,h2,h3]

Re-arrange and put 0s in

$$\begin{bmatrix} \boldsymbol{h}_1^T \boldsymbol{0} - w_i' \boldsymbol{h}_2^T \boldsymbol{p}_i + y_i' \boldsymbol{h}_3^T \boldsymbol{p}_i \\ w_i' \boldsymbol{h}_1^T \boldsymbol{p}_i + \boldsymbol{h}_2^T \boldsymbol{0} - x_i' \boldsymbol{h}_3^T \boldsymbol{p}_i \\ -y_i' \boldsymbol{h}_1^T \boldsymbol{p}_i + x_i' \boldsymbol{h}_2^T \boldsymbol{p}_i + \boldsymbol{h}_3^T \boldsymbol{0} \end{bmatrix} = \boldsymbol{0}$$

Fitting Transformation

Equation
$$\begin{bmatrix} \boldsymbol{h}_1^T\boldsymbol{0} - w_i'\boldsymbol{h}_2^T\boldsymbol{p_i} + y_i'\boldsymbol{h}_3^T\boldsymbol{p_i} \\ w_i'\boldsymbol{h}_1^T\boldsymbol{p_i} + \boldsymbol{h}_2^T\boldsymbol{0} - x_i'\boldsymbol{h}_3^T\boldsymbol{p_i} \\ -y_i'\boldsymbol{h}_1^T\boldsymbol{p_i} + x_i'\boldsymbol{h}_2^T\boldsymbol{p_i} + \boldsymbol{h}_3^T\boldsymbol{0} \end{bmatrix} = \boldsymbol{0}$$

Pull out h
$$\begin{bmatrix} \boldsymbol{0}^T & -w'{}_i\boldsymbol{p}_{\boldsymbol{i}}^T & y'{}_i\boldsymbol{p}_{\boldsymbol{i}}^T \\ w_i'\boldsymbol{p}_{\boldsymbol{i}}^T & \boldsymbol{0}^T & -x_i'\boldsymbol{p}_{\boldsymbol{i}}^T \\ -y_i'\boldsymbol{p}_{\boldsymbol{i}}^T & x_i'\boldsymbol{p}_{\boldsymbol{i}}^T & \boldsymbol{0}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \end{bmatrix} = \boldsymbol{0}$$

Only two linearly independent equations

Yank out **h** once you have all the coefficients.
If you're head-scratching about the two equations, it's not obvious to me at first glance that the three equations aren't linearly independent either.

Simplification: Two-band Blending

- Brown & Lowe, 2003
  - Only use two bands: high freq. and low freq.
  - Blend low freq. smoothly
  - Blend high freq. with no smoothing: binary alpha

Figure Credit: Brown & Lowe

2-band “Laplacian Stack” Blending

Low frequency (λ > 2 pixels)

High frequency (λ < 2 pixels)

Linear Blending

2-band Blending