# Optimization and (Under/over)fitting

EECS 442 – David Fouhey

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http://web.eecs.umich.edu/~fouhey/teaching/EECS442_W23/

# Regularized Least Squares

Add **regularization** to objective that prefers some solutions:

Before: $\arg\min_{\boldsymbol{w}} \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}\|_2^2$ → Loss

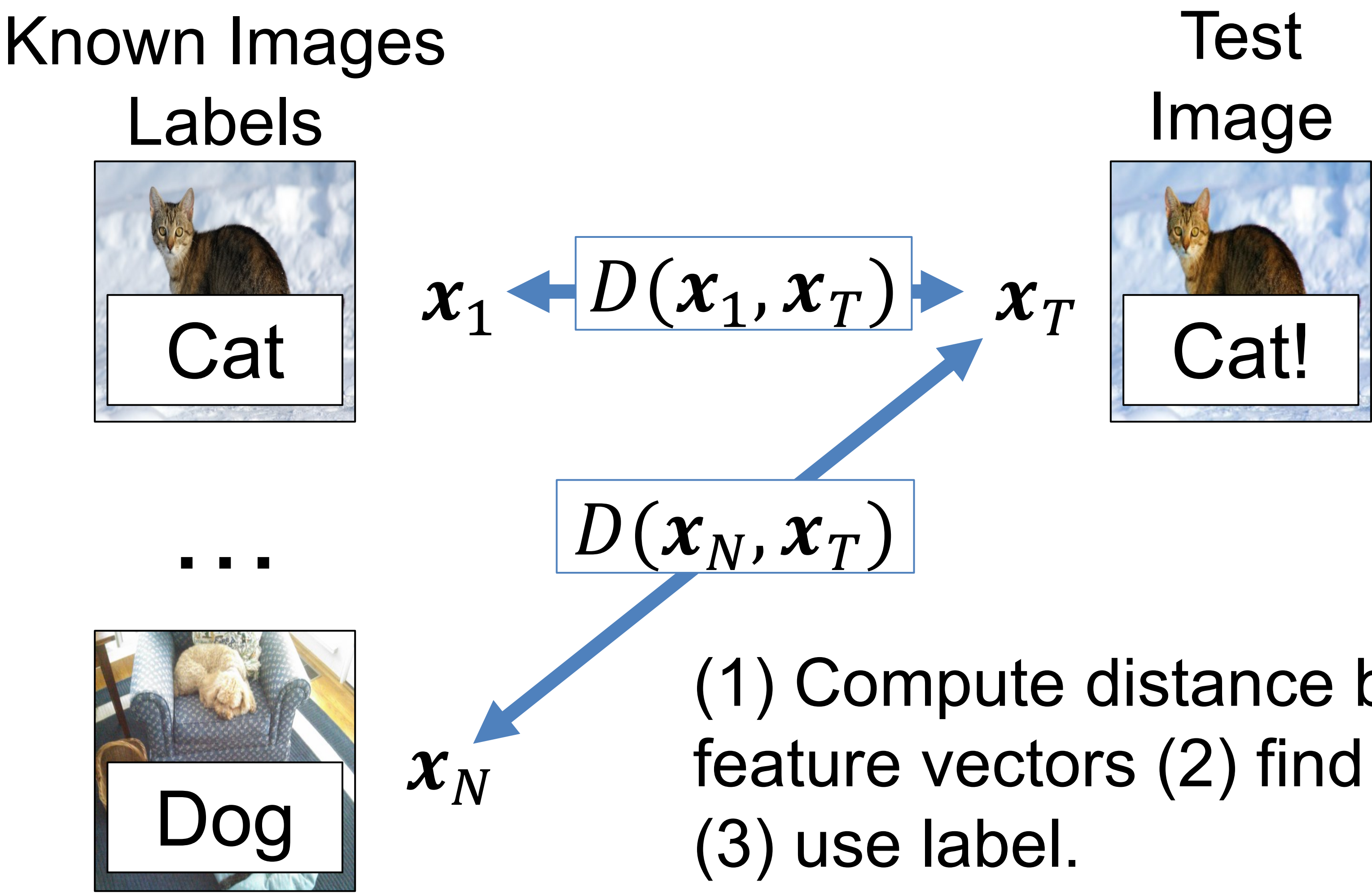
After: $\arg\min_{\boldsymbol{w}} \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}\|_2^2 + \lambda \|\boldsymbol{w}\|_2^2$

Loss Trade-off Regularization

Want model "smaller": pay a penalty for **w** with big norm

Intuitive Objective: accurate model (low loss) but not too complex (low regularization). λ controls how much of each.

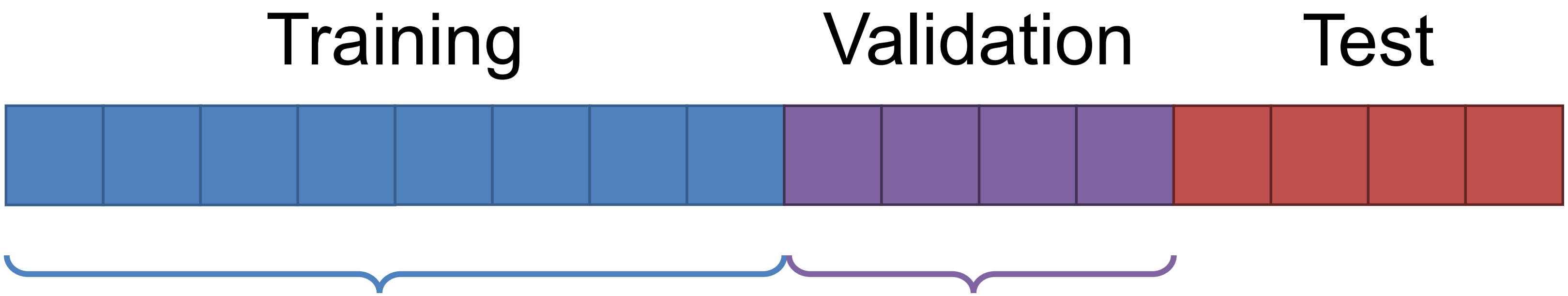
# Nearest Neighbors




(1) Compute distance between feature vectors (2) find nearest (3) use label.

# Picking Parameters

What distance? What value for k / λ?




Use these data points for lookup

Evaluate on these points for different k, λ, distances

# Linear Models

## Example Setup: 3 classes



Model – one weight per class: $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2$

**Want:**

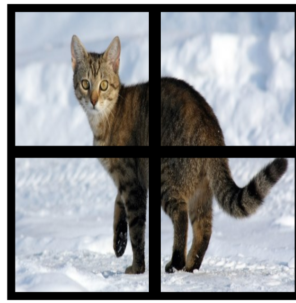
$\boldsymbol{w}_0^T \boldsymbol{x}$ big if cat

$\boldsymbol{w}_1^T \boldsymbol{x}$ big if dog

$\boldsymbol{w}_2^T \boldsymbol{x}$ big if hippo

Stack together: $\boldsymbol{W}_{3xF}$ where **x** is in $R^F$

# Linear Models



| | | | | | |
|---|---|---|---|---|---|
| Cat weight vector | 0.2 | -0.5 | 0.1 | 2.0 | 1.1 |
| Dog weight vector | 1.5 | 1.3 | 2.1 | 0.0 | 3.2 |
| Hippo weight vector | 0.0 | 0.3 | 0.2 | -0.3 | -1.2 |

$\boldsymbol{W}$

| |
|---|
| 56 |
| 231 |
| 24 |
| 2 |
| 1 |

$\boldsymbol{x}$

| | |
|---|---|
| -96.8 | Cat score |
| 437.9 | Dog score |
| 61.95 | Hippo score |

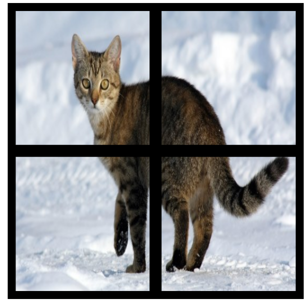
$\boldsymbol{Wx}$

Weight matrix a collection of scoring functions, one per class

Prediction is vector where jth component is “score” for jth class.

Diagram by: Karpathy, Fei-Fei

# How Badly Are We Doing?



Loss: dog score – cat score

How much higher-scored is "dog" vs "cat"?

$$(\boldsymbol{Wx})_2 - (\boldsymbol{Wx})_1$$

Don't give negative penalties

$$\max(0, (\boldsymbol{Wx})_2 - (\boldsymbol{Wx})_1)$$

| | | |
|---|---|---|
| $(\boldsymbol{Wx})_1$ | -96.8 | Cat score |
| $(\boldsymbol{Wx})_2$ | 437.9 | Dog score |
| $(\boldsymbol{Wx})_3$ | 61.95 | Hippo score |

$\boldsymbol{Wx}$

Prediction is vector where jth component is "score" for jth class.

Diagram by: Karpathy, Fei-Fei

# Objective 1: Multiclass SVM*

Inference (x): $\arg\max_{k} \, (\boldsymbol{W}\boldsymbol{x})_k$

(Take the class whose weight vector gives the highest score)

Training ($\mathbf{x}_i$,$y_i$):

$$\arg\min_{\boldsymbol{W}} \lambda\|\boldsymbol{W}\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \max(0, (\boldsymbol{W}\boldsymbol{x}_i)_j - (\boldsymbol{W}\boldsymbol{x}_i)_{y_i})$$

Regularization

Over all data points

For every class j that's NOT the correct one ($y_i$)

Pay no penalty if prediction for class $y_i$ is bigger than j. Otherwise, pay proportional to the score of the wrong class.

# Objective 1: Multiclass SVM

Inference (x): $\arg\max_k \; (\boldsymbol{W}\boldsymbol{x})_k$

(Take the class whose weight vector gives the highest score)

Training ($\mathbf{x}_i$,$y_i$):

$$\arg\min_{\boldsymbol{W}} \lambda \|\boldsymbol{W}\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \max(0, (\boldsymbol{W}\boldsymbol{x}_i)_j - (\boldsymbol{W}\boldsymbol{x}_i)_{y_i} + m)$$

Regularization

Over all data points

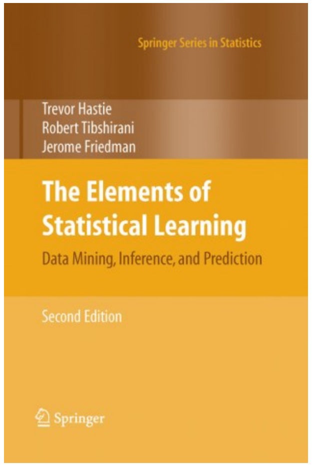
For every class j that's NOT the correct one ($y_i$)

Pay no penalty if prediction for class $y_i$ is bigger than j by m ("margin"). Otherwise, pay proportional to the score of the wrong class.

# Objective 1:

## Called: Support Vector Machine

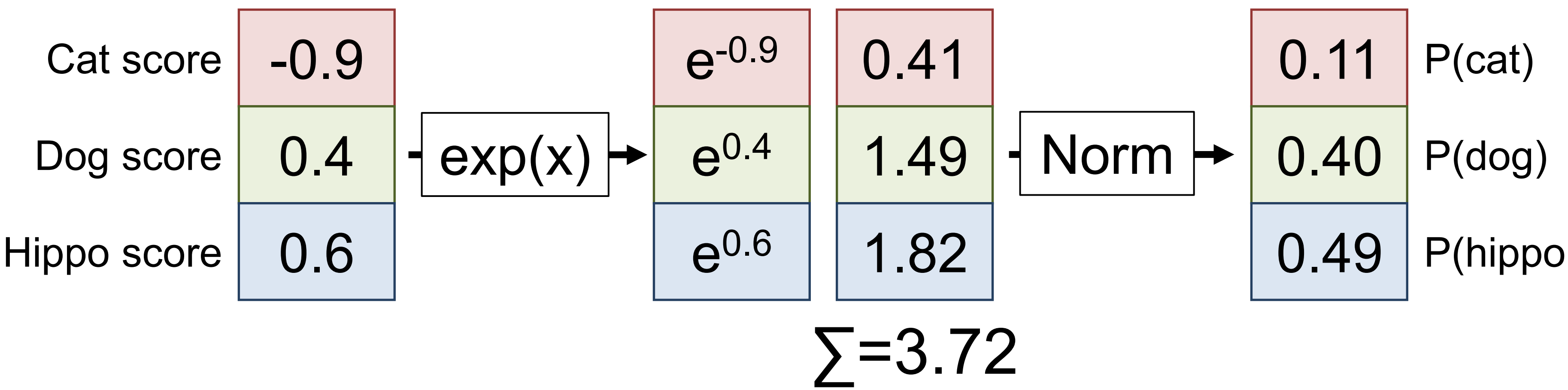
Lots of great theory as to why this is a sensible thing to do (but kernel SVMs are just secret nearest neighbor machines). See:




Useful book (Free too!):
The Elements of Statistical Learning
Hastie, Tibshirani, Friedman
https://web.stanford.edu/~hastie/ElemStatLearn/

# Objective 2: Making Probabilities

## Converting Scores to “Probability Distribution”




Generally P(class j): $$\frac{\exp((Wx)_j)}{\sum_k \exp((Wx)_k)}$$

*Called softmax function*

Inspired by: Karpathy, Fei-Fei

# Objective 2: Softmax

Inference (x): $\arg\max_{k} \ (\boldsymbol{Wx})_k$

(Take the class whose weight vector gives the highest score)

P(class j): $\dfrac{\exp((Wx)_j)}{\sum_k \exp((Wx)_k)}$

**Why can we skip the exp/sum exp thing to make a decision?**

# Objective 2: Softmax

Inference (x): $\arg\max_{k} \; (\boldsymbol{W}\boldsymbol{x})_k$

(Take the class whose weight vector gives the highest score)

Training ($\mathbf{x}_i$,$y_i$):

$$\arg\min_{\boldsymbol{W}} \lambda\|W\|_2^2 + \sum_{i=1}^{n} -\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k)}\right)$$

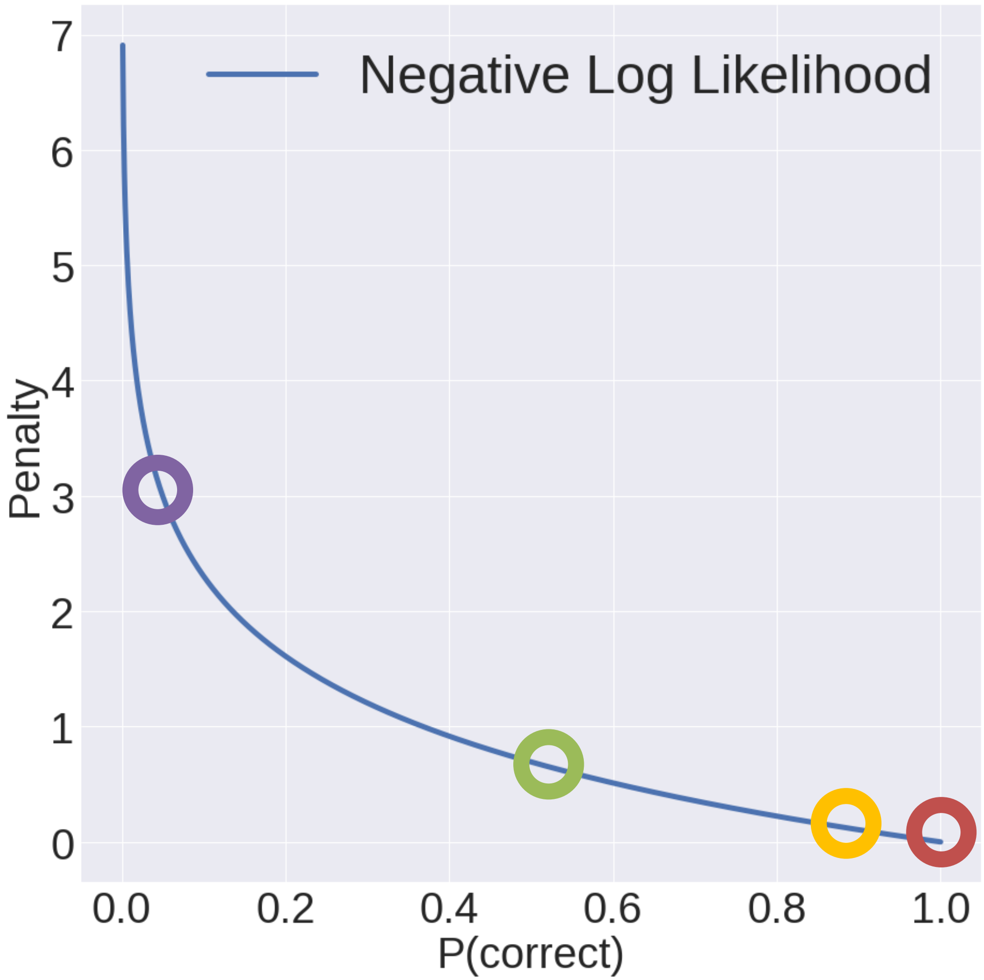
P(correct class)

Regularization

Over all data points

Pay penalty for not making correct class likely.
"Negative log-likelihood"

# Objective 2: Softmax




**P(correct) = 0.05: 3.0 penalty**

**P(correct) = 0.5: 0.11 penalty**

**P(correct) = 0.9: 0.11 penalty**

**P(correct) = 1: No penalty!**

# How Do We Optimize Things?

Goal: find the **w** minimizing some loss function L.

$$\arg\min_{\boldsymbol{w} \in R^N} L(\boldsymbol{w})$$

Works for lots of different Ls:

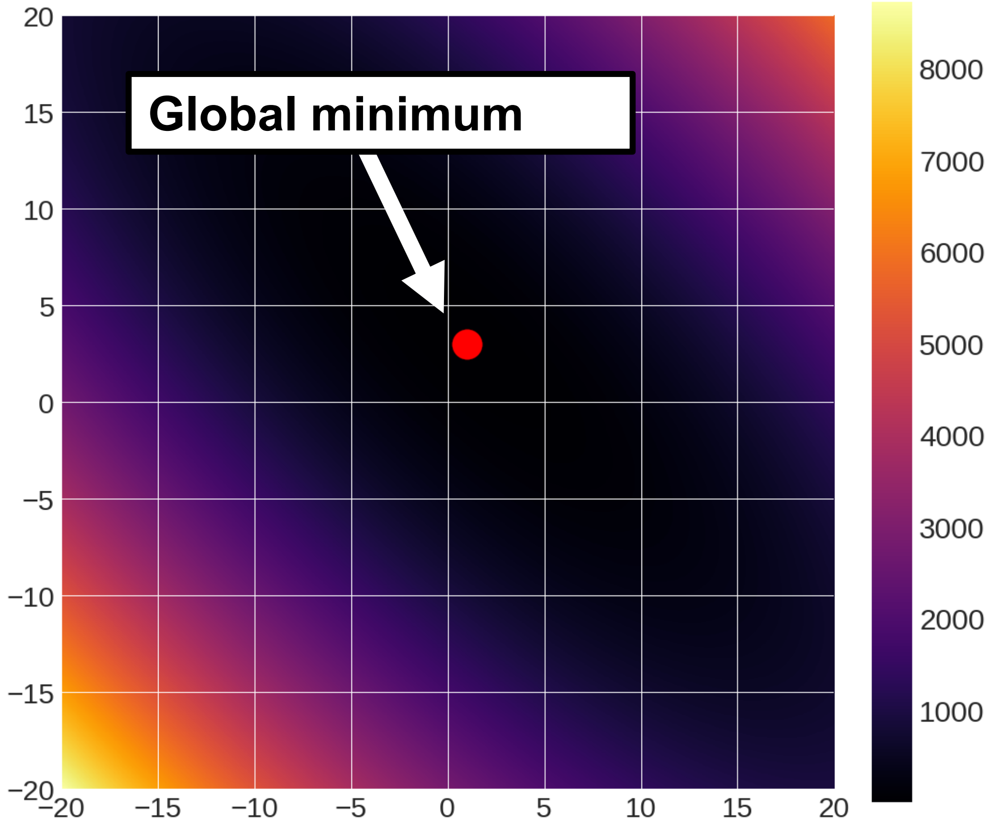
$$L(\boldsymbol{W}) = \boldsymbol{\lambda} \|\boldsymbol{W}\|_{\boldsymbol{2}}^{\boldsymbol{2}} + \sum_{i=1}^{n} -\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k))}\right)$$

$$L(\boldsymbol{w}) = \lambda \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} \left(y_i - \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{x}_{\boldsymbol{i}}\right)^2$$

$$L(\boldsymbol{w}) = C \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} \max\left(0,1 - y_i \boldsymbol{w}^T \boldsymbol{x}_{\boldsymbol{i}}\right)$$

# Sample Function to Optimize

$$f(x,y) = (x+2y-7)^2 + (2x+y-5)^2$$

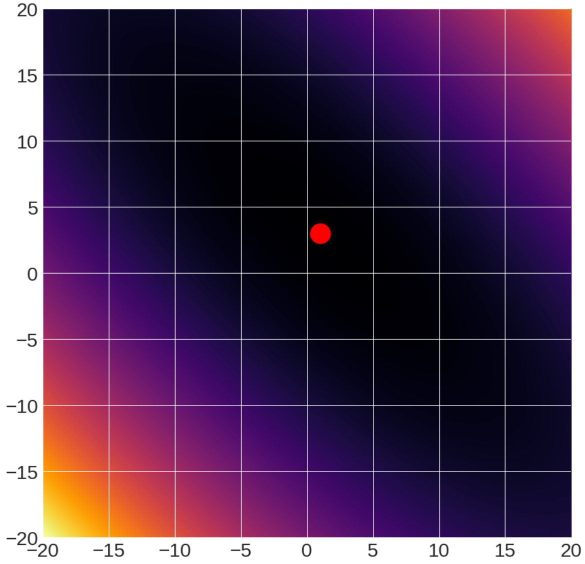


# Sample Function to Optimize

- I'll switch back and forth between this 2D function (called the *Booth Function*) and other more-learning-focused functions
- Beauty of optimization is that it's all the same in principle
- But don't draw too many conclusions: 2D space has qualitative differences from 1000D space

See intro of: Dauphin et al. *Identifying and attacking the saddle point problem in high-dimensional non-convex optimization* NIPS 2014
https://ganguli-gang.stanford.edu/pdf/14.SaddlePoint.NIPS.pdf

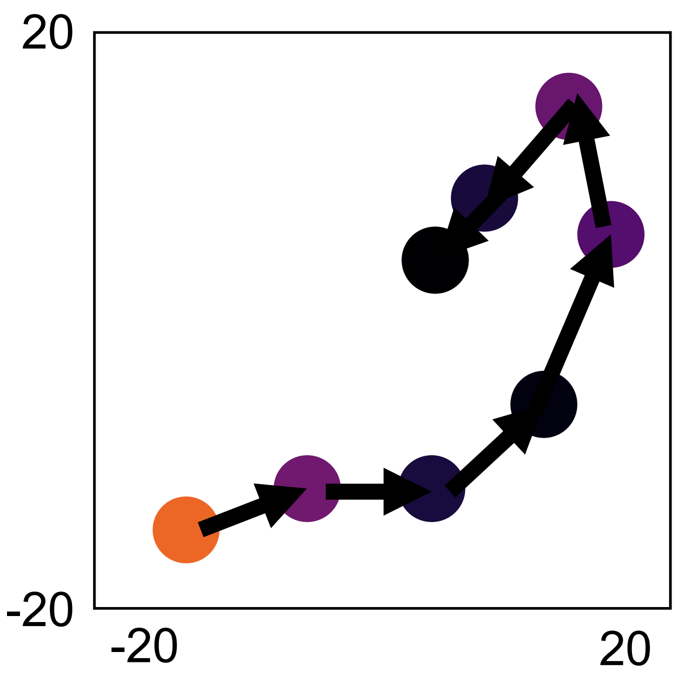
# A Caveat

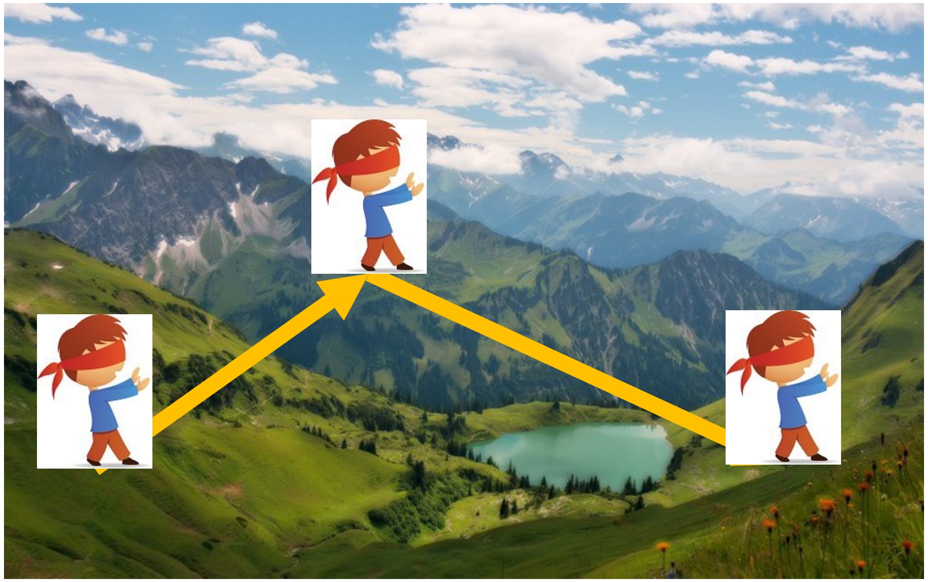



- Each point in the picture is a function evaluation
- Here it takes microseconds – so we can easily see the answer
- Functions we want to optimize may take hours to evaluate

# A Caveat

Model in your head: moving around a landscape with a teleportation device



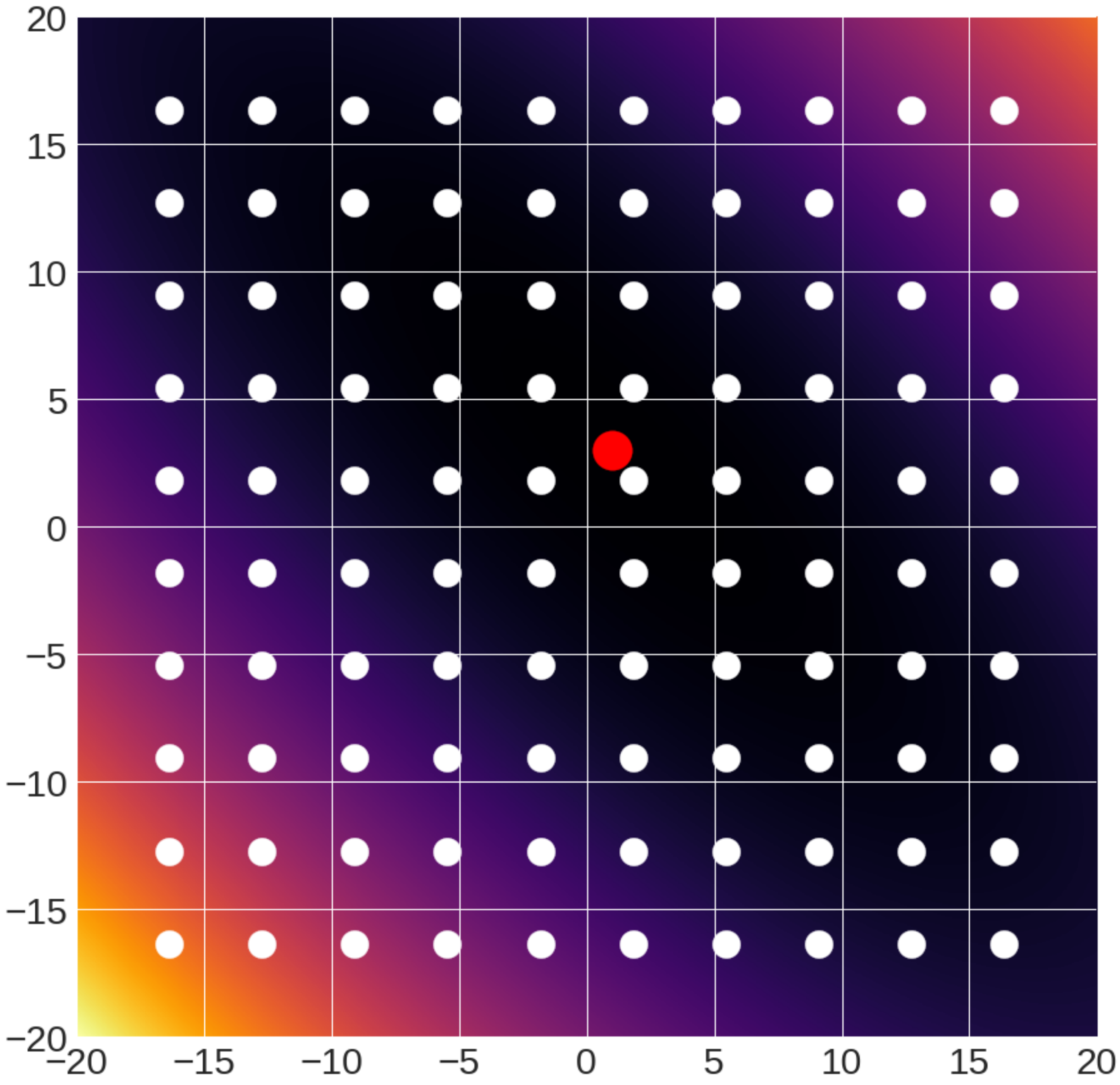



Landscape diagram: Karpathy and Fei-Fei

# Option #1A – Grid Search

```
#systematically try things
best, bestScore = None, Inf
for dim1Value in dim1Values:
        ….
        for dimNValue in dimNValues:
                w = [dim1Value, …, dimNValue]
                if L(w) < bestScore:
                        best, bestScore = w, L(w)
return best
```

# Option #1A – Grid Search



# Option #1A – Grid Search

**Pros:**

1. Super simple
2. Only requires being able to evaluate model

**Cons:**

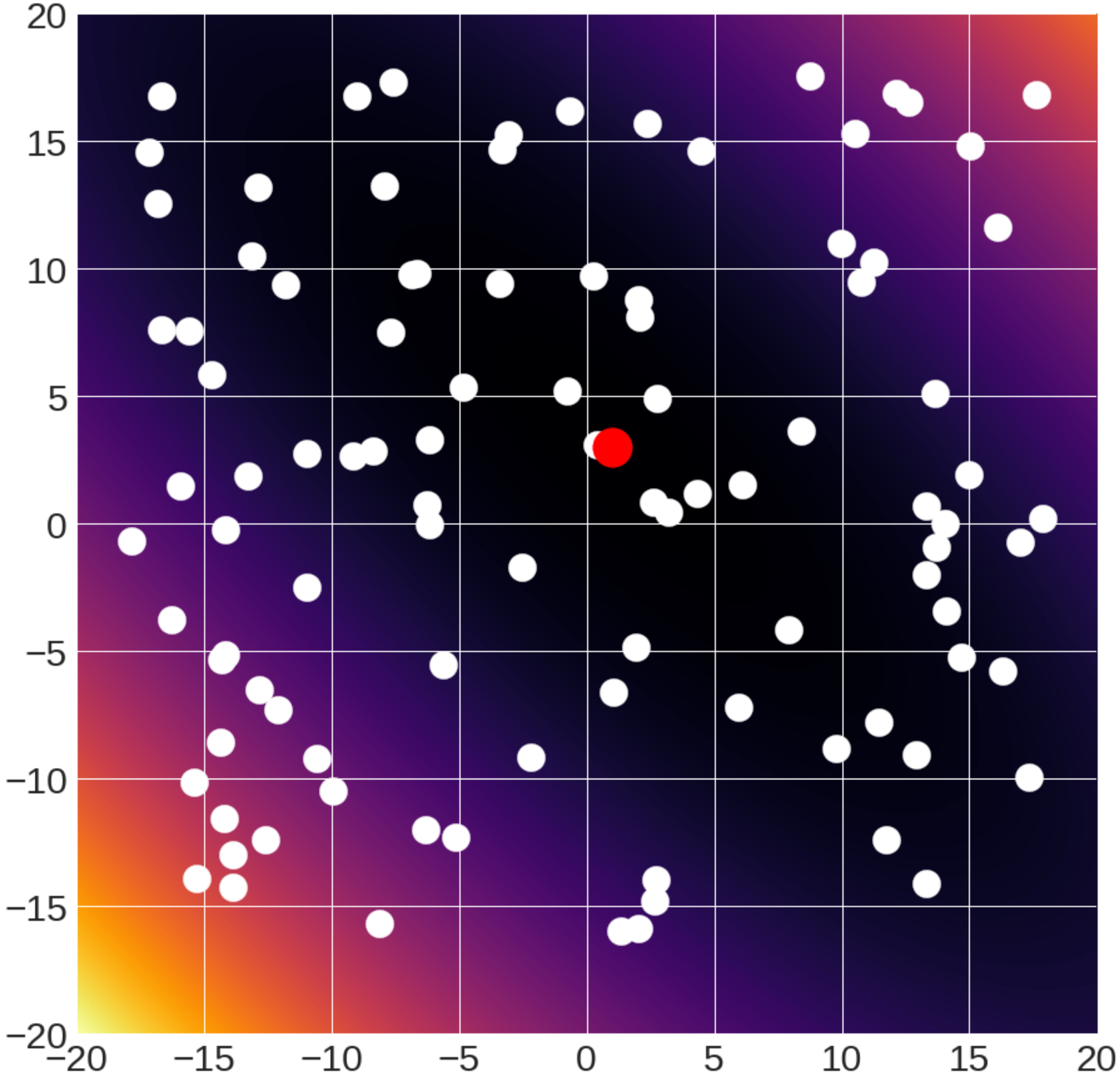
1. Scales horribly to high dimensional spaces

Complexity: $\text{samplesPerDim}^{\text{numberOfDims}}$

# Option #1B – Random Search

```
#Do random stuff RANSAC Style
best, bestScore = None, Inf
for iter in range(numIters):
        w = random(N,1) #sample
        score = L(w) #evaluate
        if score < bestScore:
                best, bestScore = w, score
return best
```

# Option #1B – Random Search



# Option #1B – Random Search

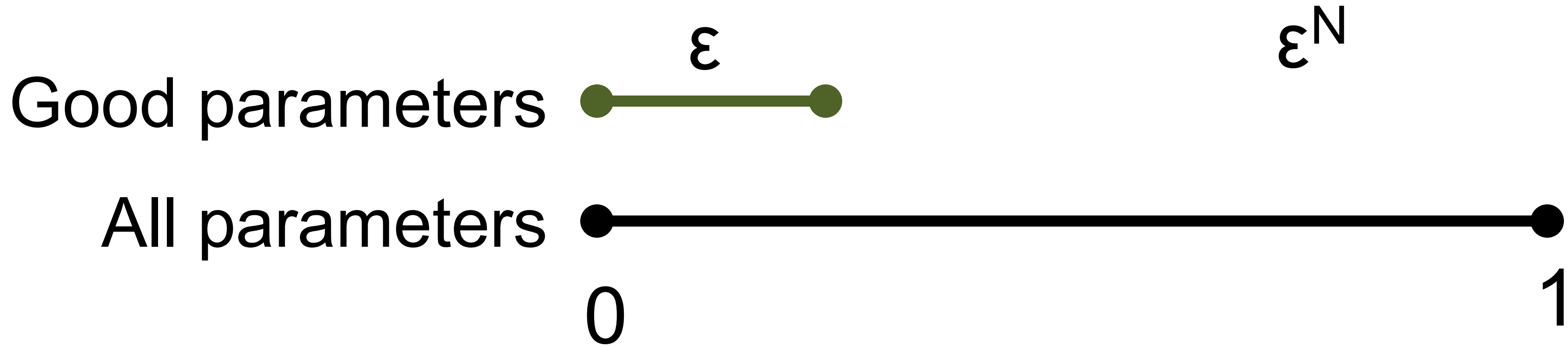
**Pros:**

1. Super simple
2. Only requires being able to sample model and evaluate it

**Cons:**

1. Slow –throwing darts at high dimensional dart board
2. Might miss something

P(all correct) = $\varepsilon^N$



# When Do You Use Options 1A/1B?

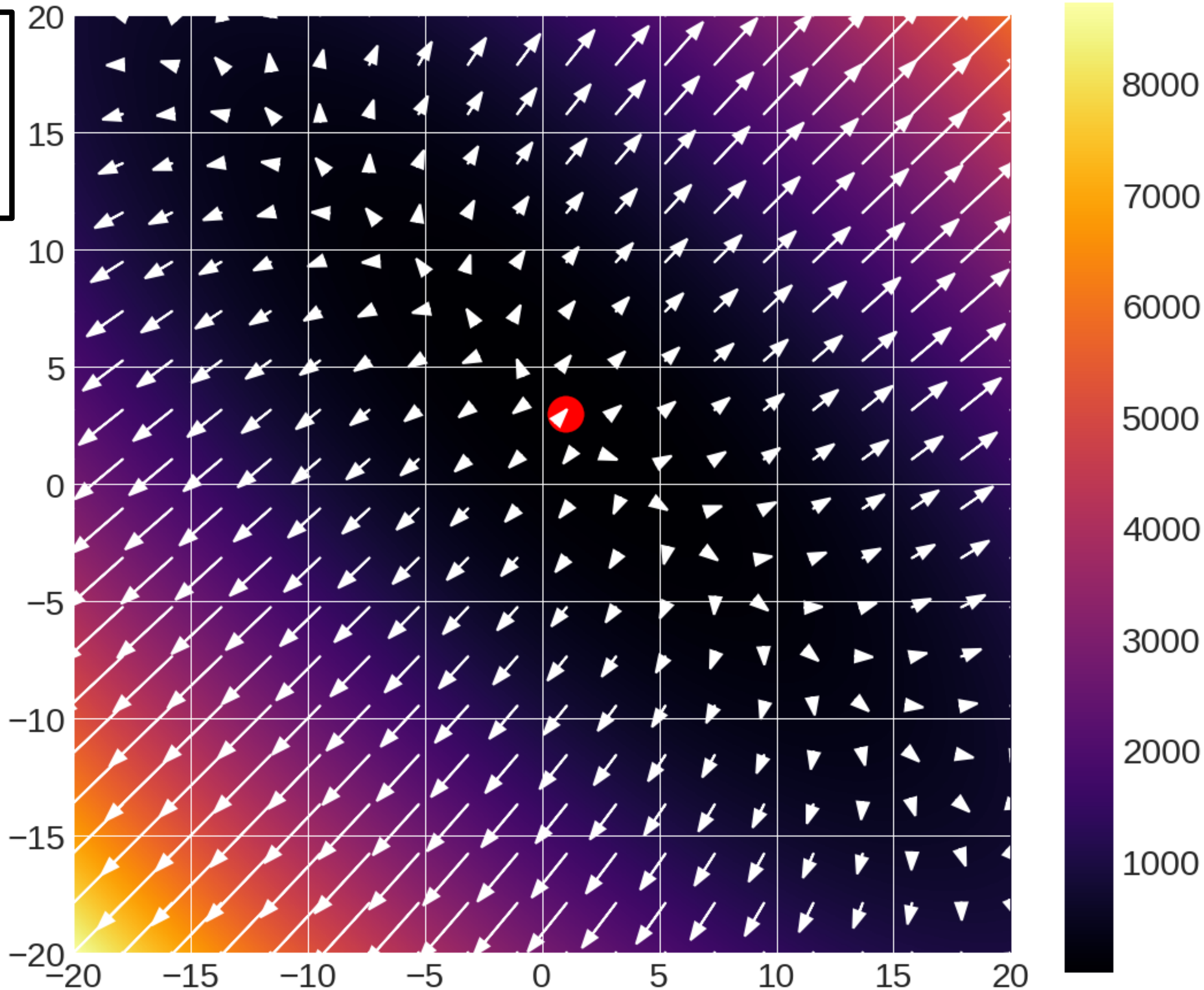
Use these when

- Number of dimensions small, space bounded
- Objective is impossible to analyze (e.g., validation accuracy if one uses a distance function)

Random search is more effective; grid search makes it easy to systematically test something (people love certainty)

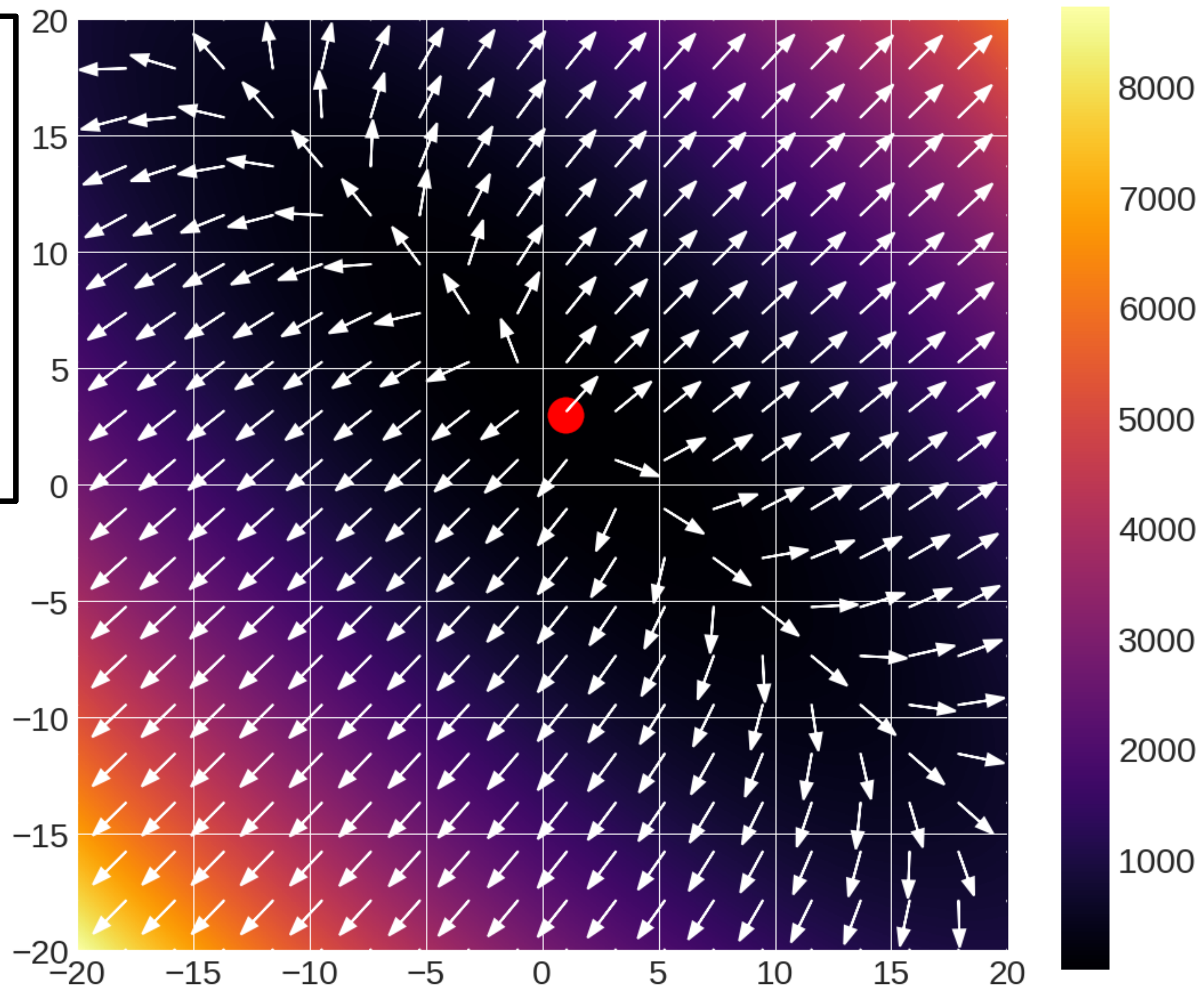
# Option 2 – Use The Gradient

Arrows:
**gradient**



# Option 2 – Use The Gradient

Arrows:
**gradient direction**
(scaled to unit length)



# Option 2 – Use The Gradient

Want: $\arg\min_{\boldsymbol{w}} L(\boldsymbol{w})$

**What's the geometric interpretation of:**

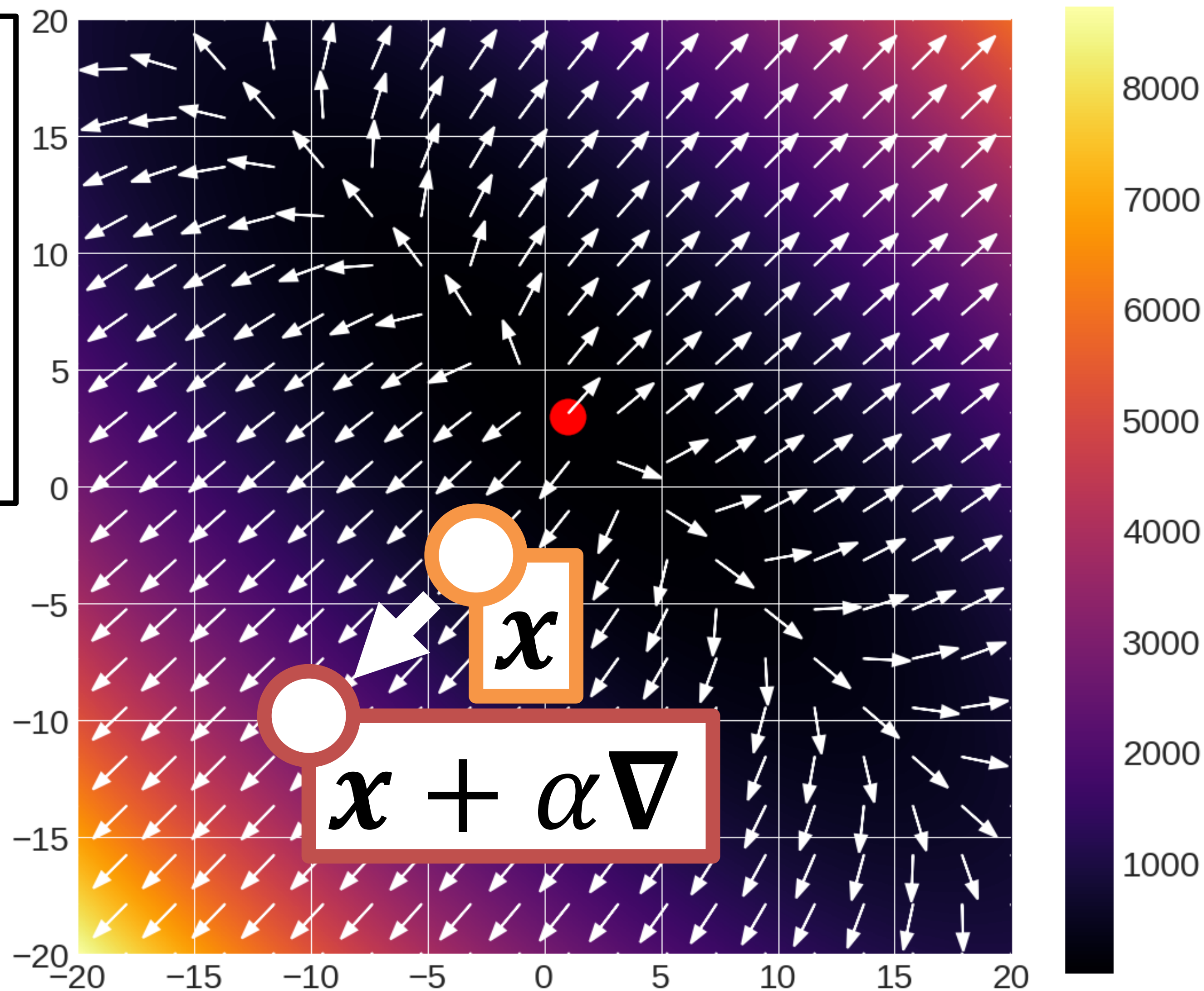
$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \begin{bmatrix} \partial L / \partial \boldsymbol{x}_1 \\ \vdots \\ \partial L / \partial \boldsymbol{x}_N \end{bmatrix}$$

**Which is bigger (for small α)?**

$$L(\boldsymbol{w}) \quad \overset{\leq ?}{\underset{> ?}{}} \quad L(\boldsymbol{w} + \alpha \nabla_{\boldsymbol{w}} L(\boldsymbol{w}))$$

# Option 2 – Use The Gradient

Arrows: gradient direction (scaled to unit length)



# Option 2 – Use The Gradient

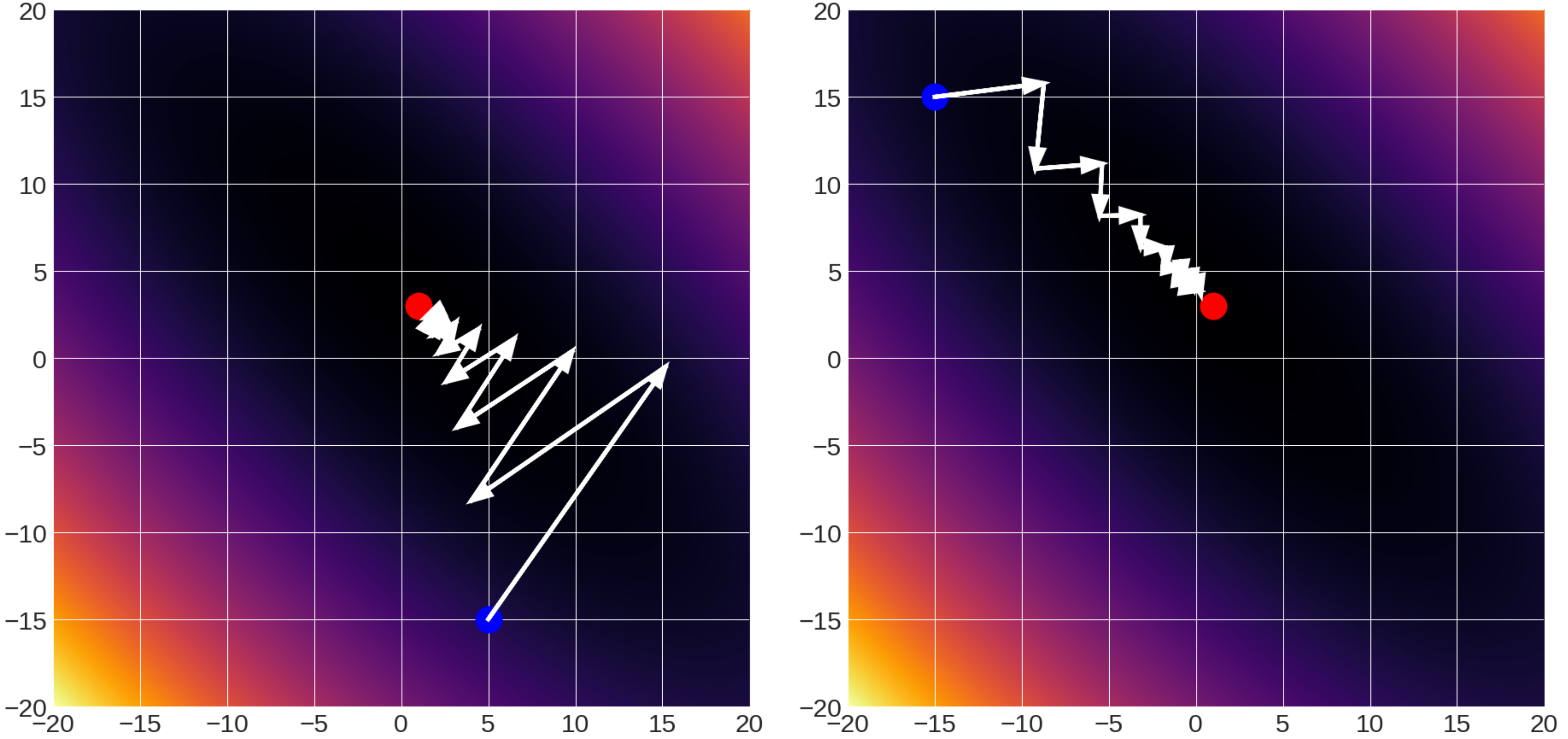
Method: at each step, move in direction of negative gradient

**w0** = *initialize*() #initialize
for iter in range(numIters):
  **g** = $\nabla_{\boldsymbol{w}} L(\boldsymbol{w})$ #eval gradient
  **w** = **w** + -*stepsize*(iter)****g** #update w
return **w**

# Gradient Descent

Given starting point (blue)

$w_{i+1} = w_i + -9.8x10^{-2}$ x gradient



# Computing the Gradient

## How Do You Compute The Gradient?
## Numerical Method:

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \begin{bmatrix} \frac{\partial L(w)}{\partial x_1} \\ \vdots \\ \frac{\partial L(w)}{\partial x_n} \end{bmatrix}$$

**How do you compute this?**

$$\frac{\partial f(x)}{\partial x} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

In practice, use:

$$\frac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon}$$

# Computing the Gradient

## How Do You Compute The Gradient?

## Numerical Method:

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \begin{bmatrix} \frac{\partial L(w)}{\partial x_1} \\ \vdots \\ \frac{\partial L(w)}{\partial x_n} \end{bmatrix}$$

Use: $\frac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon}$

**How many function evaluations per dimension?**

# Computing the Gradient

## How Do You Compute The Gradient?

## Analytical Method:

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \begin{bmatrix} \frac{\partial L(w)}{\partial x_1} \\ \vdots \\ \frac{\partial L(w)}{\partial x_n} \end{bmatrix}$$

Use calculus!

# Computing the Gradient

$$L(\boldsymbol{w}) = \lambda \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} \left(y_i - \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{x}_{\boldsymbol{i}}\right)^2$$

$$\downarrow \quad \frac{\partial}{\partial \boldsymbol{w}} \quad \downarrow$$

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = 2\lambda \boldsymbol{w} + \sum_{i=1}^{n} -\left(2\left(y_i - \boldsymbol{w}^T \boldsymbol{x}_{\boldsymbol{i}}\right)\boldsymbol{x}_i\right)$$

Note: if you look at other derivations, things are written either ($y-w^Tx$) or ($w^Tx - y$); the gradients will differ by a minus.

# Interpreting Gradients (1 Sample)

Recall:

$\mathbf{w} = \mathbf{w} + -\nabla_{\boldsymbol{w}} L(\boldsymbol{w})$ #update w

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = 2\lambda\boldsymbol{w} + -\left(2\left(y - \boldsymbol{w}^T\boldsymbol{x}\right)\boldsymbol{x}\right)$$

Push **w** towards 0 ... α

$$-\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = \overbrace{-2\lambda\boldsymbol{w}}^{\text{Push } \mathbf{w} \text{ towards } 0} + \underbrace{\overbrace{\left(2\left(y - \boldsymbol{w}^T\boldsymbol{x}\right)\right.}^{\alpha}\left.\boldsymbol{x}\right)}$$

If $y > w^T x$ (too *low*): then $w = w + \alpha x$ for some $\alpha$

**Before**: $w^T x$

**After**: $(w + \alpha x)^T x = w^T x + \alpha x^T x$

# Quick annoying detail: subgradients

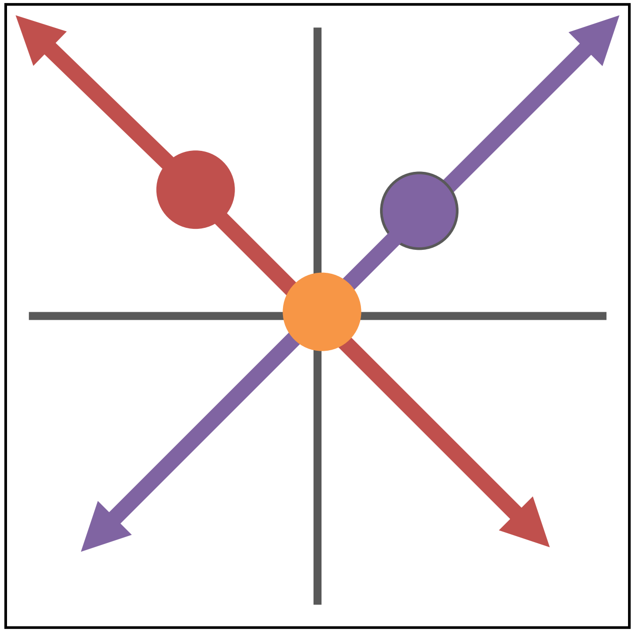
What is the derivative of |x|?

<u>Derivatives/Gradients</u>

Defined everywhere but 0

$$\frac{\partial}{\partial x} f(x) = \text{sign}(x) \quad x \neq 0$$

$$\text{undefined} \quad x = 0$$




Oh no! A discontinuity!

# Quick annoying detail: subgradients

## Subgradient: any underestimate of function

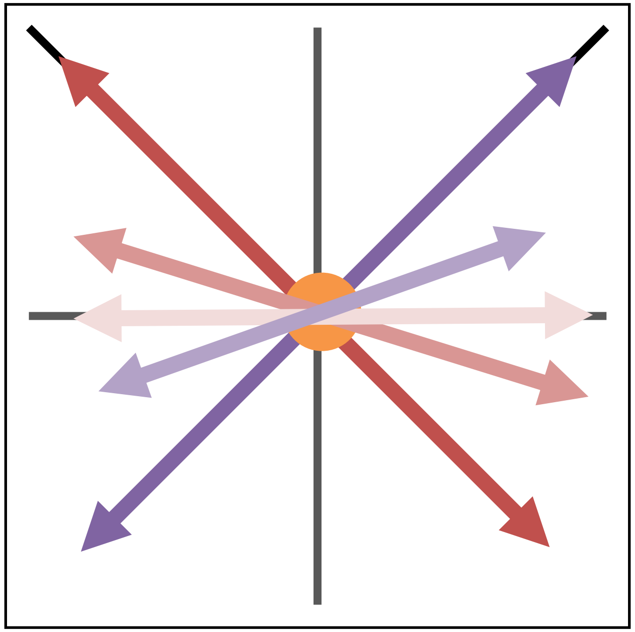
<u>Subderivatives/subgradients</u>

Defined everywhere

$$\frac{\partial}{\partial x} f(x) = \text{sign}(x) \quad x \neq 0$$

$$\frac{\partial}{\partial x} f(x) \in [-1,1] \quad x = 0$$

|x|



*In practice*: at discontinuity, pick value on either side.

# Computing The Gradient

- Numerical: foolproof but slow
- Analytical: can mess things up ☺
- In practice: do analytical, but check with numerical (called a gradient check)

Slide: Karpathy and Fei-Fei

# Implementing Gradient Descent

Loss is a function that we can evaluate over data

All Data

$$-\nabla_{\boldsymbol{w}} L(\boldsymbol{w}) = -2\lambda \boldsymbol{w} + \sum_{i=1}^{n} \left(2\left(y_i - \boldsymbol{w}^T \boldsymbol{x}_{\boldsymbol{i}}\right)\boldsymbol{x}_i\right)$$

Subset B

$$-\nabla_{\boldsymbol{w}} L_B(\boldsymbol{w}) = -2\lambda \boldsymbol{w} + \sum_{i \in B} \left(2\left(y_i - \boldsymbol{w}^T \boldsymbol{x}_{\boldsymbol{i}}\right)\boldsymbol{x}_i\right)$$

# Implementing Gradient Descent

Option 1: Vanilla Gradient Descent
Compute gradient of L over all data points

for iter in range(numIters):
    **g** = gradient(data,L)
    **w** = **w** + -*stepsize*(iter)*__g__ #update w

# Implementing Gradient Descent

Option 2: *Stochastic* Gradient Descent
Compute gradient of L over 1 random sample

for iter in range(numIters):
    index = randint(0,#data)
    **g** = gradient(data[index],L)
    **w** = **w** + -*stepsize*(iter)***g** #update w

# Implementing Gradient Descent

Option 3: *Minibatch* Gradient Descent
Compute gradient of L over subset of B samples

for iter in range(numIters):
    subset = choose_samples(#data,B)
    **g** = gradient(data[subset],L)
    **w** = **w** + -*stepsize*(iter)***g** #update w

Typical batch sizes: ~100 (although there's lots of great research on huge batch sizes)

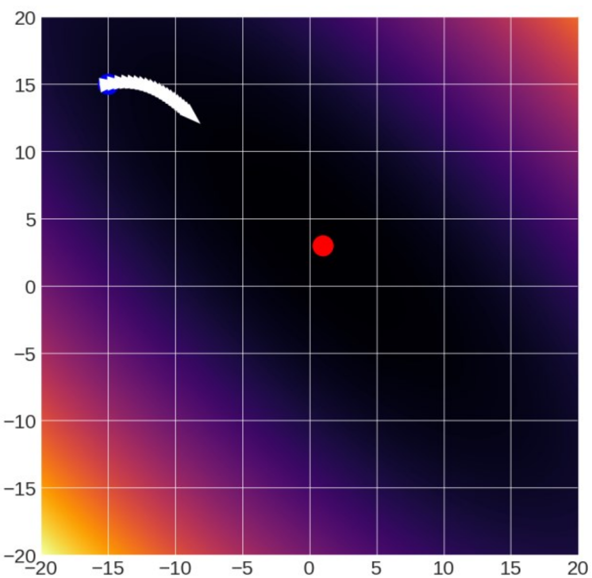
# Gradient Descent Details

## Step size (also called **learning rate / lr**)

*critical parameter*

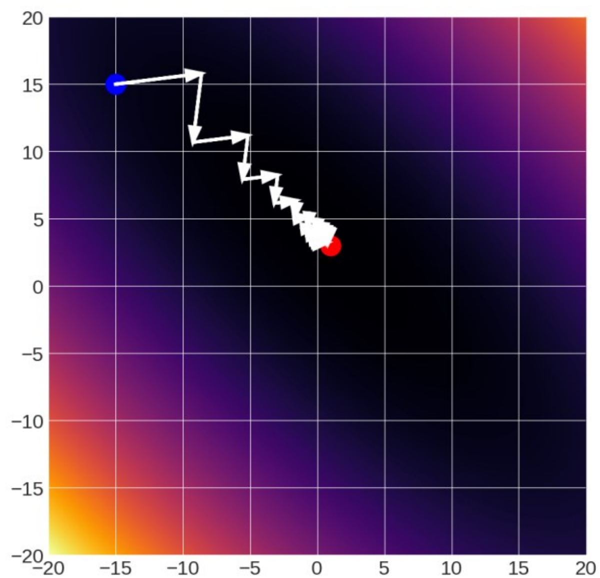
$1x10^{-2}$

falls short



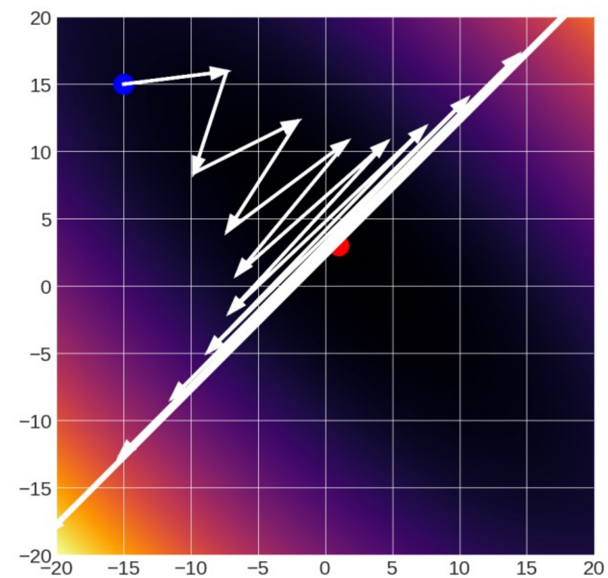

$10x10^{-2}$

converges



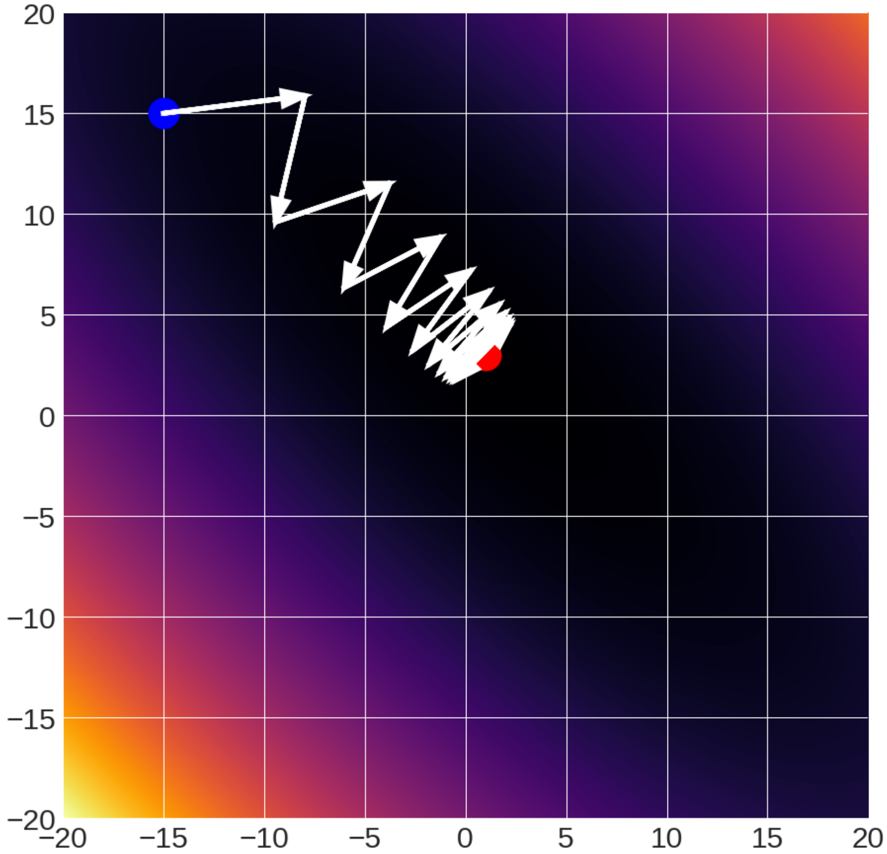

$12x10^{-2}$

diverges



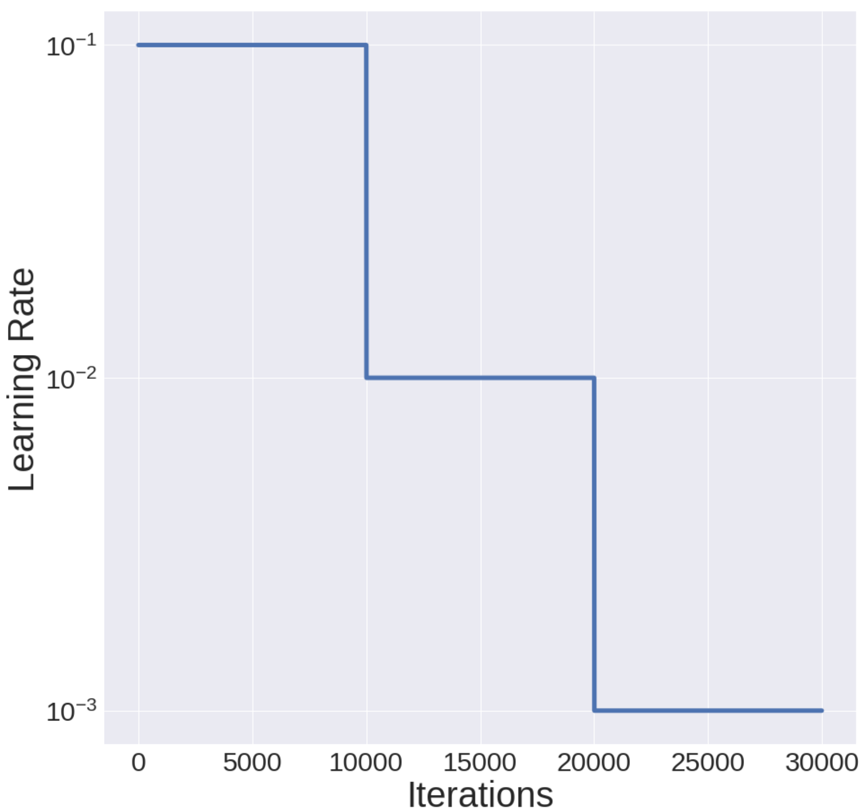
# Gradient Descent Details

$11\text{x}10^{-2}$ :oscillates
(Raw gradients)



# Gradient Descent Details

One solution: start with initial rate lr, multiply by f every N interations



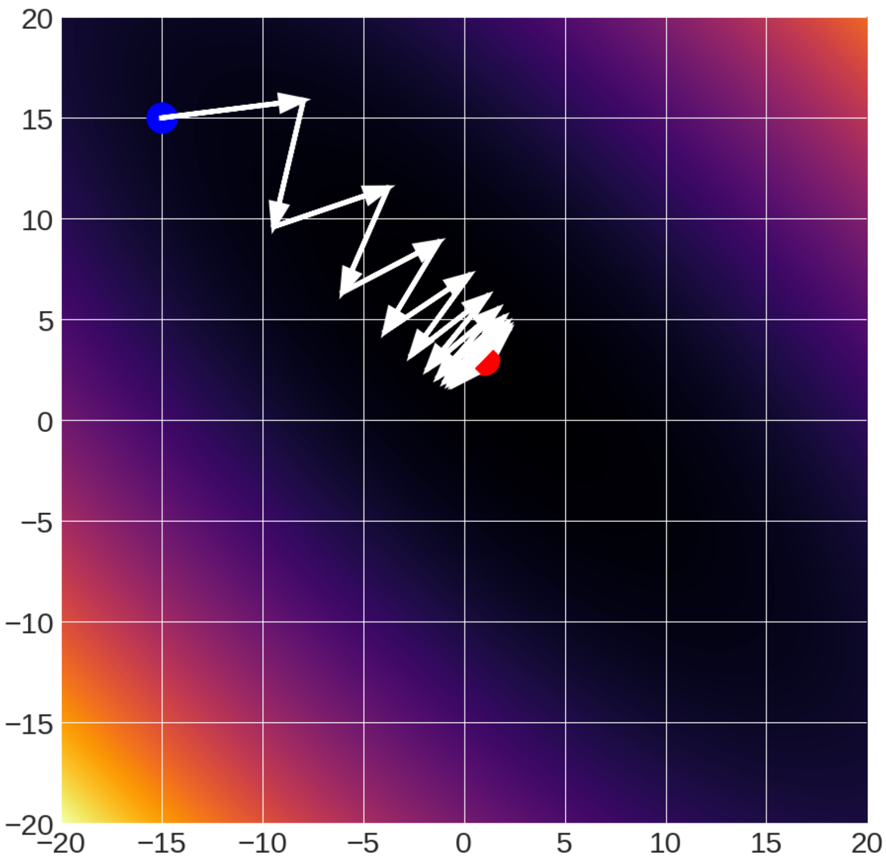

init_lr = $10^{-1}$

f = 0.1

N = 10K

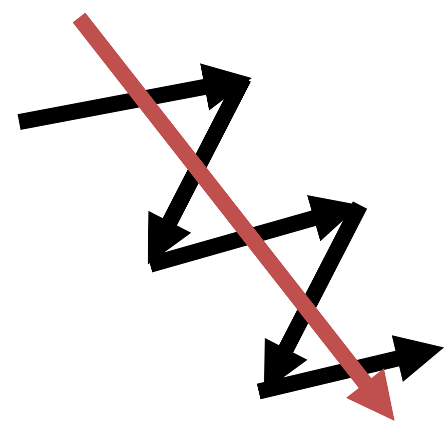
# Gradient Descent Details

11x10$^{-2}$ :oscillates
(Raw gradients)



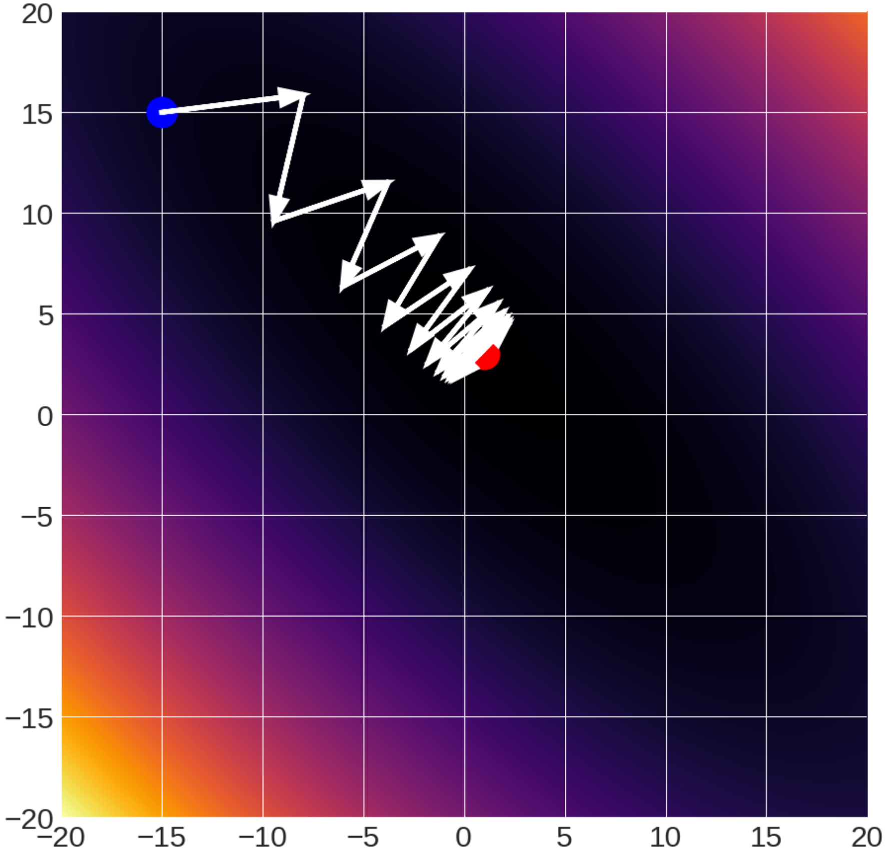

Solution:
Average gradients

With exponentially decaying weights, called “momentum”

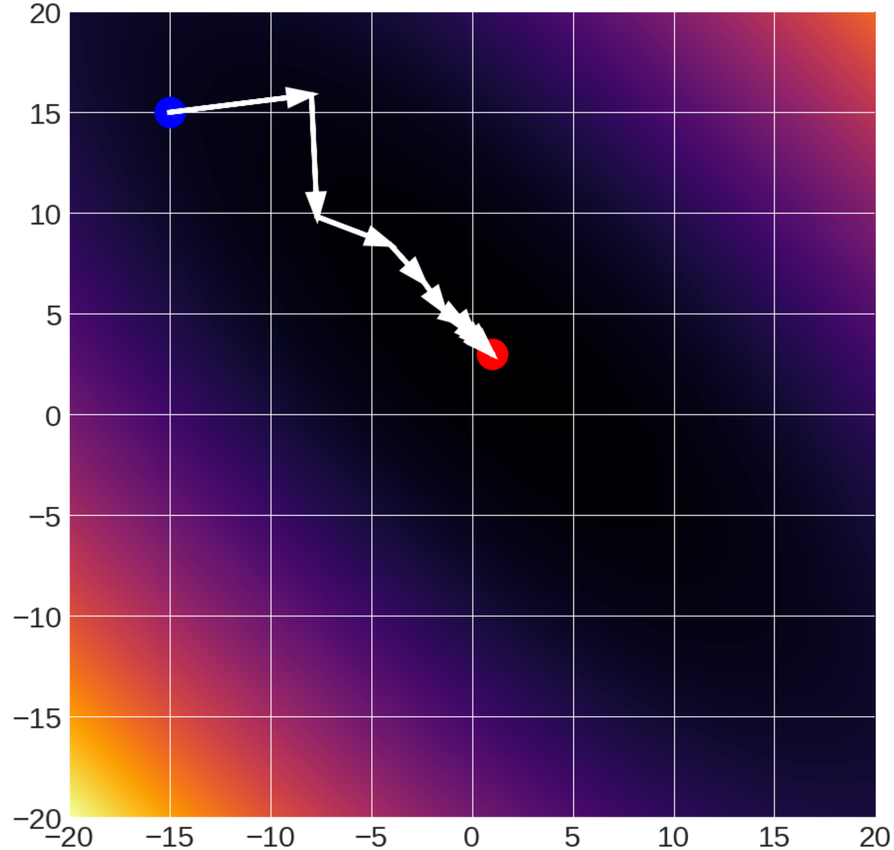


# Gradient Descent Details

$11\text{x}10^{-2}$ :oscillates
(Raw gradients)

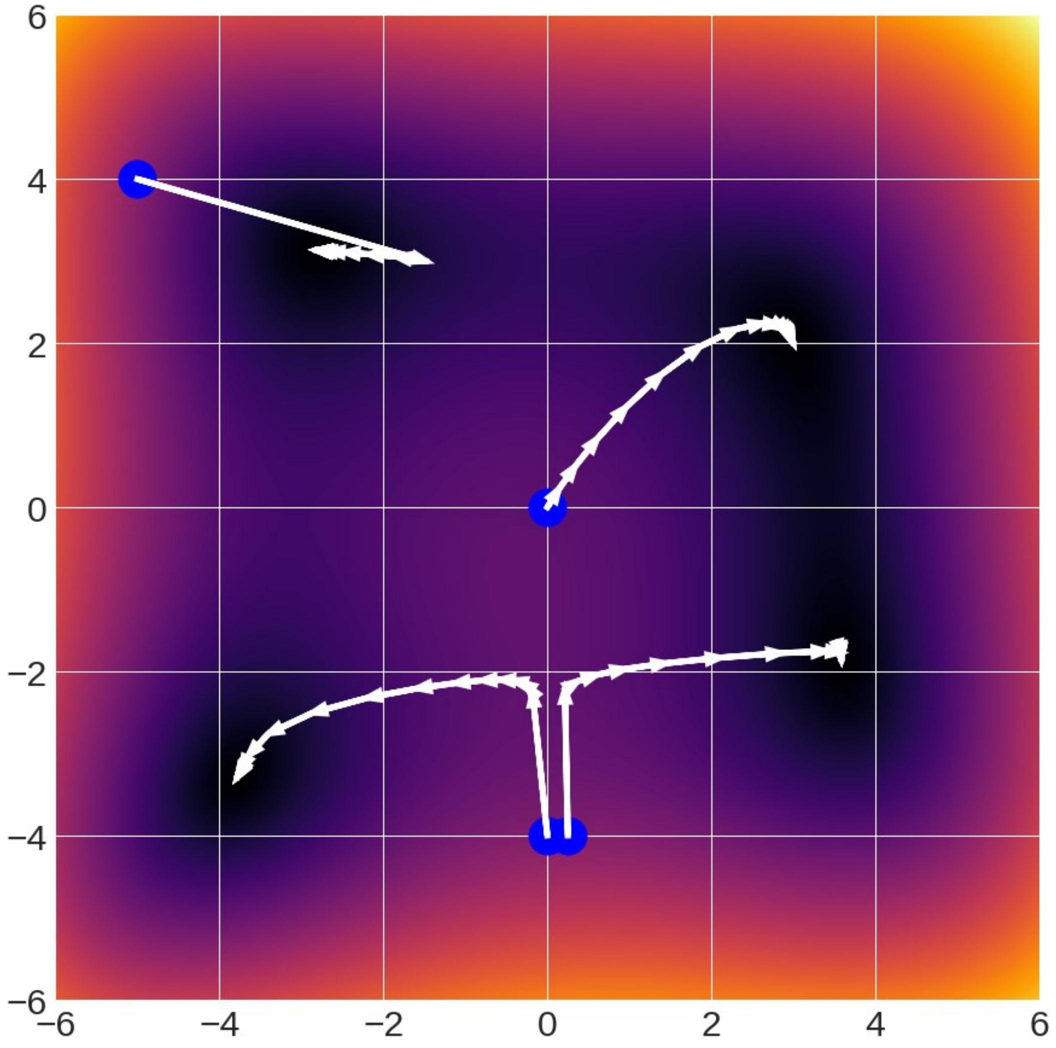



$11\text{x}10^{-2}$
(0.25 momentum)



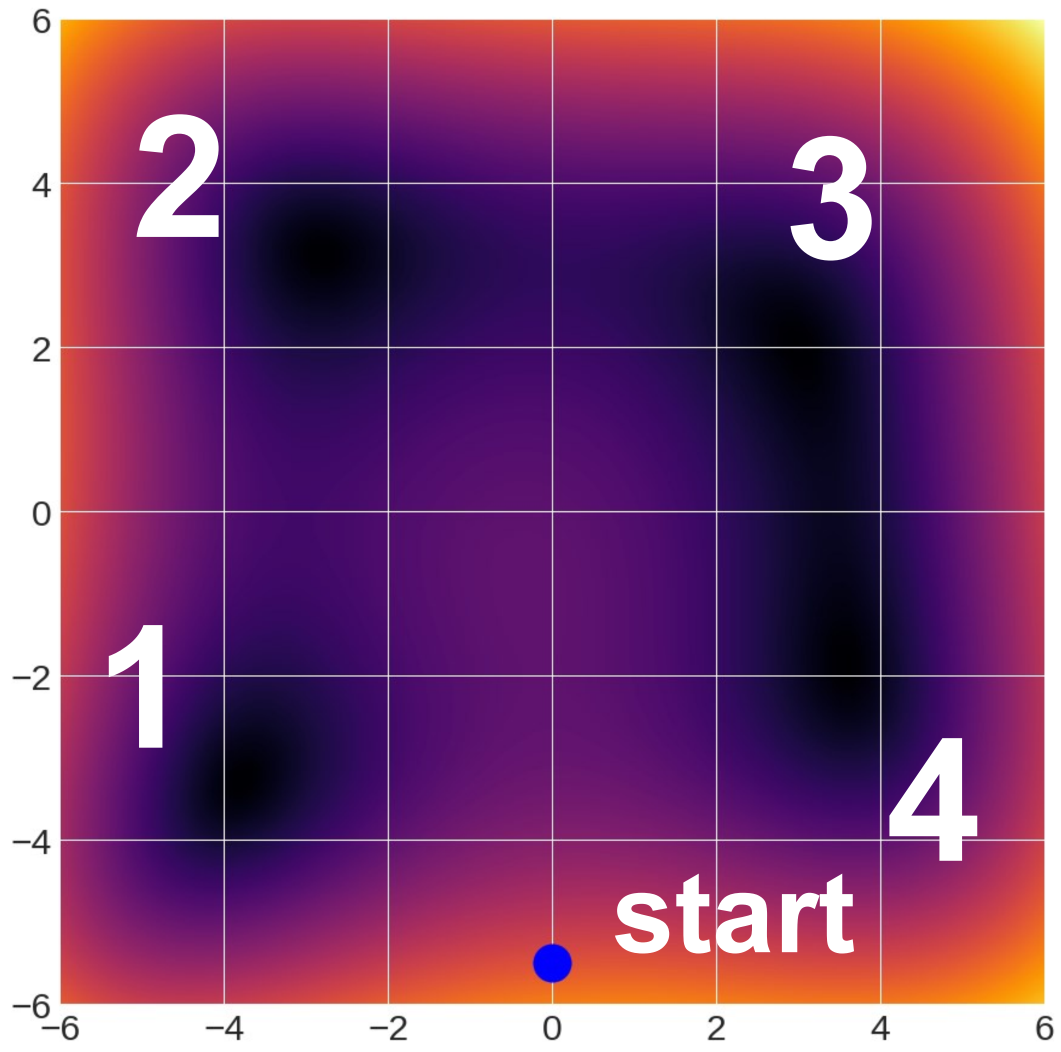
# Gradient Descent Details

Multiple Minima
→
Gradient Descent
Finds **local minimum**

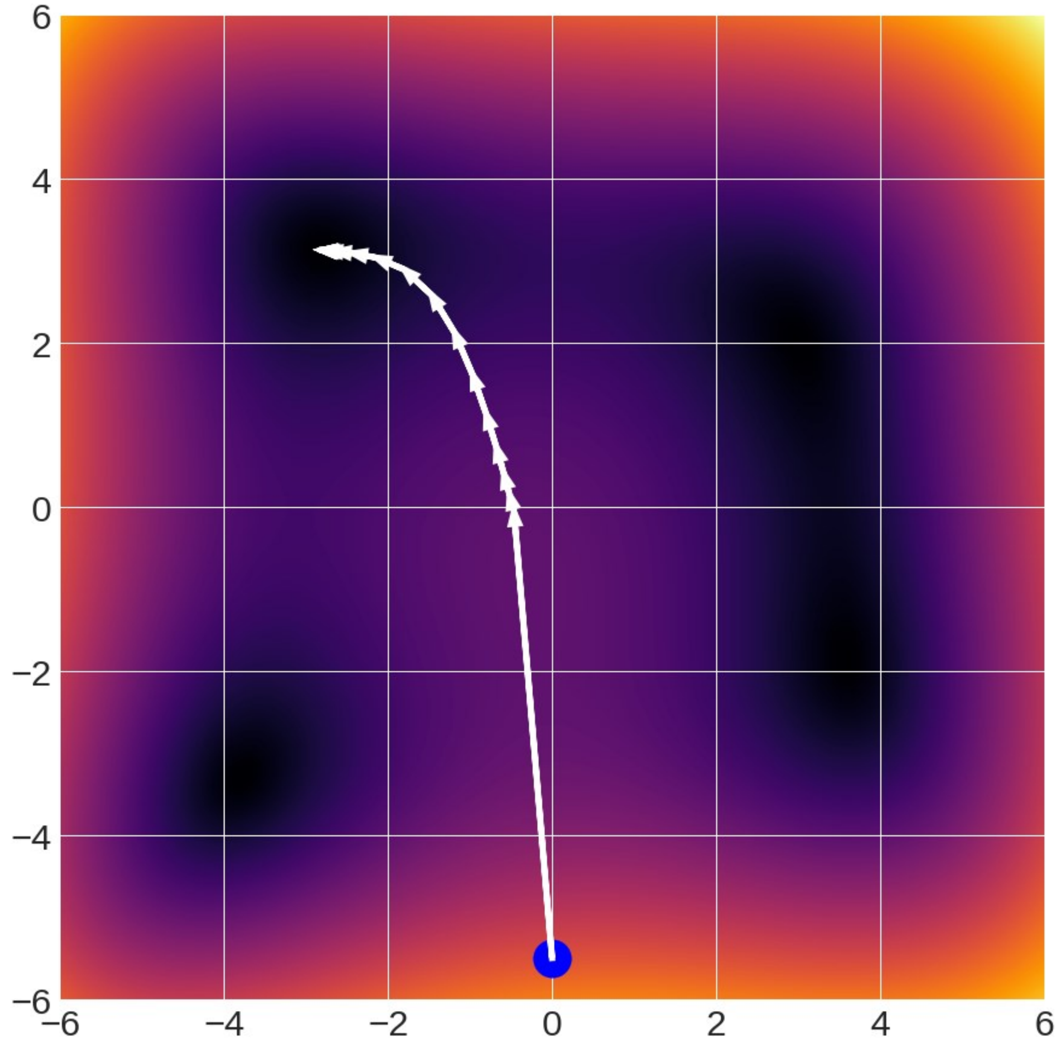


# Gradient Descent Details

Guess the minimum!

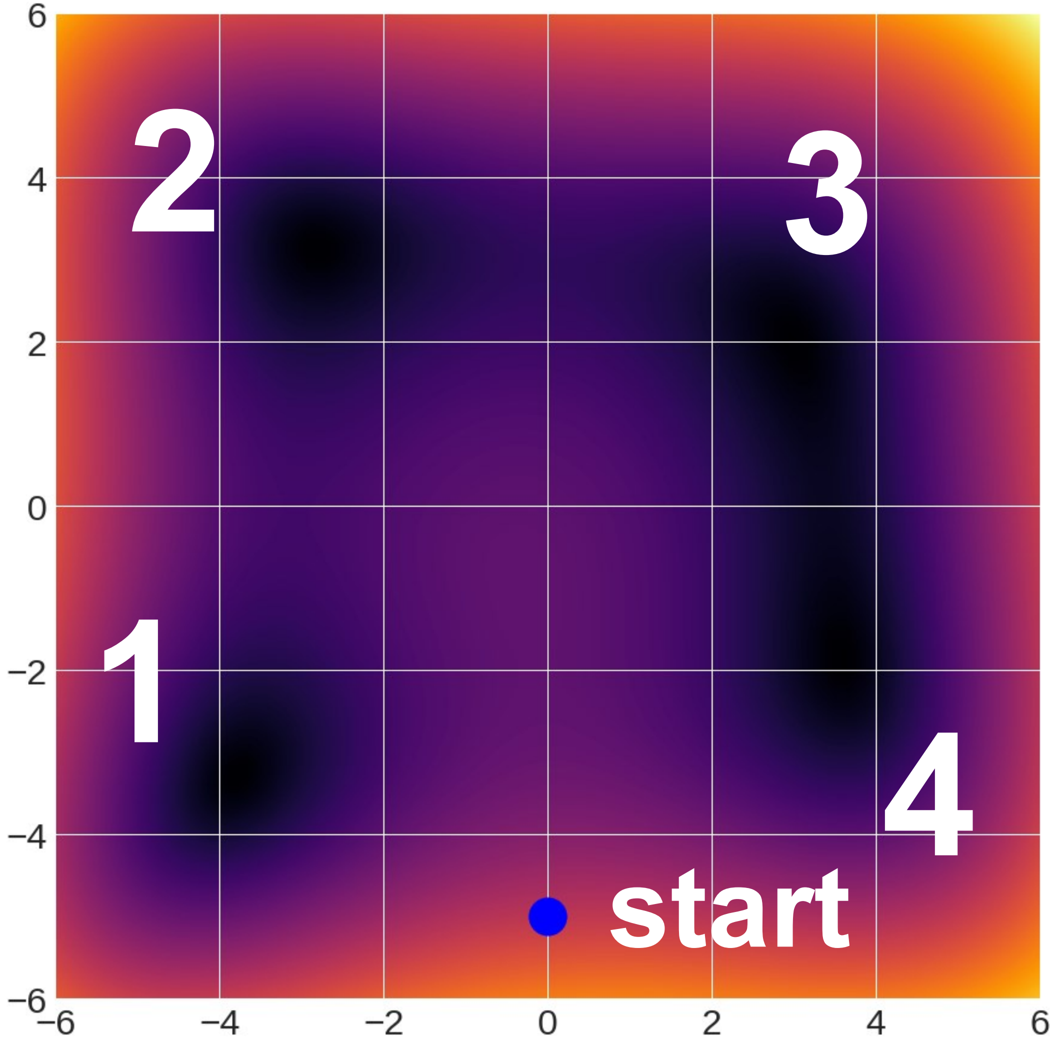


# Gradient Descent Details



# Gradient Descent Details

Guess the minimum!

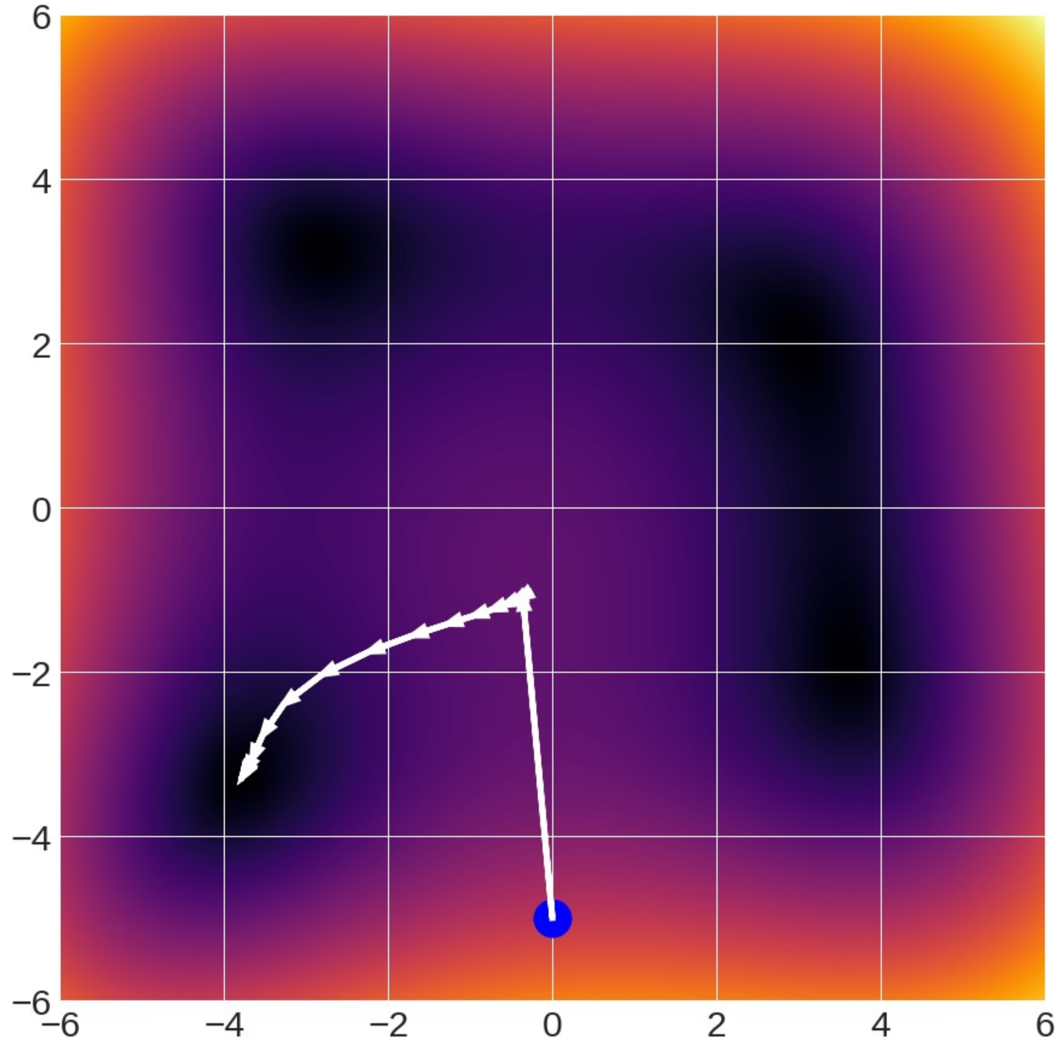


# Gradient Descent Details

Dynamics are fairly complex

Many important functions are **convex**: any local minimum is a global minimum

Many important functions are not.



# In practice

- **Conventional wisdom**: minibatch stochastic gradient descent (SGD) + momentum (package implements it for you) + some sensibly changing learning rate
- The above is typically what is meant by “SGD”
- Other update rules exist (e.g., AdamW). Can often work better on some problems.

# Optimizing Everything

$$L(\boldsymbol{W}) = \boldsymbol{\lambda}\|\boldsymbol{W}\|_{\boldsymbol{2}}^{\boldsymbol{2}} + \sum_{i=1}^{n} -\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k))}\right)$$

$$L(\boldsymbol{w}) = \lambda\|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n}\left(y_i - \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{x}_{\boldsymbol{i}}\right)^2$$

- Optimize **w** on training set with SGD to maximize training accuracy
- Optimize λ with random/grid search to maximize validation accuracy
- Note: Optimizing λ on training sets it to 0

# (Over/Under)fitting and Complexity

Let's fit a polynomial: given x, predict y

Note: can do non-linear regression with copies of x

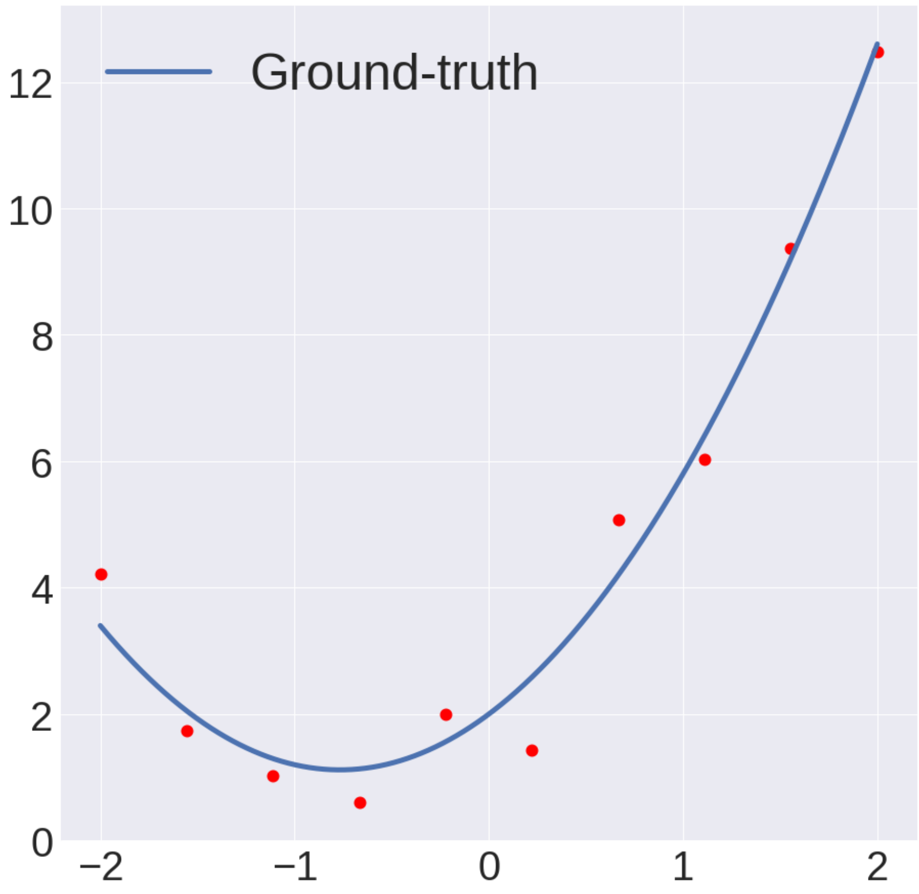
$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_1^F & \cdots & x_1^2 & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ x_N^F & \cdots & x_N^2 & x_N & 1 \end{bmatrix} \begin{bmatrix} w_F \\ \vdots \\ w_2 \\ w_1 \\ w_0 \end{bmatrix}$$

**Matrix of all polynomial degrees**

**Weights: one per polynomial degree**

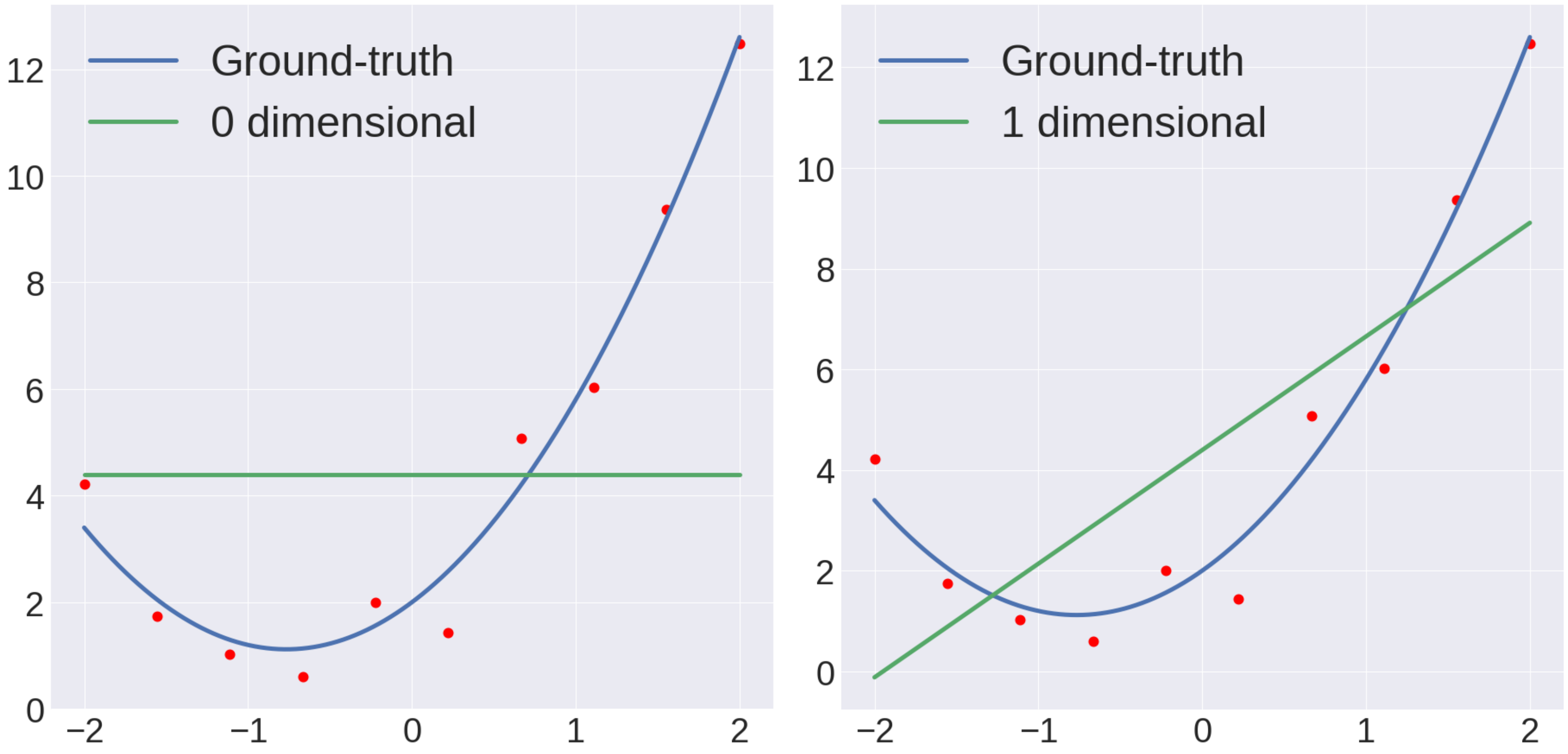
# (Over/Under)fitting and Complexity

Model: $1.5x^2 + 2.3x+2 + N(0,0.5)$

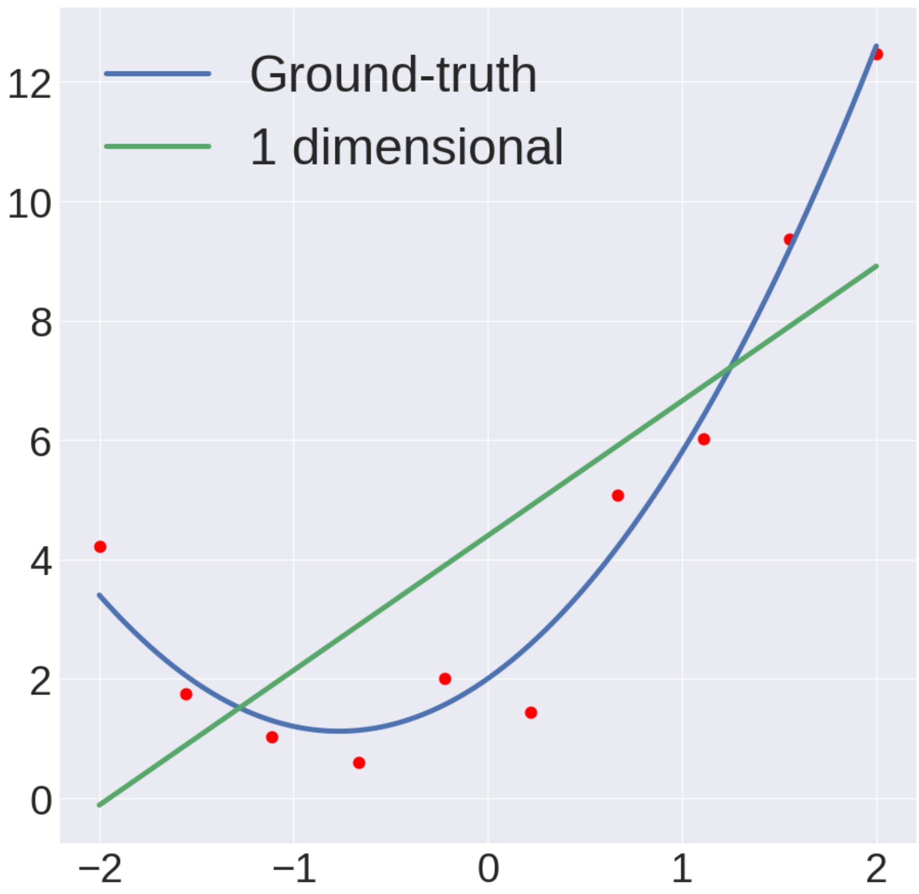


# Underfitting

Model: $1.5x^2 + 2.3x+2 + N(0,0.5)$



# Underfitting

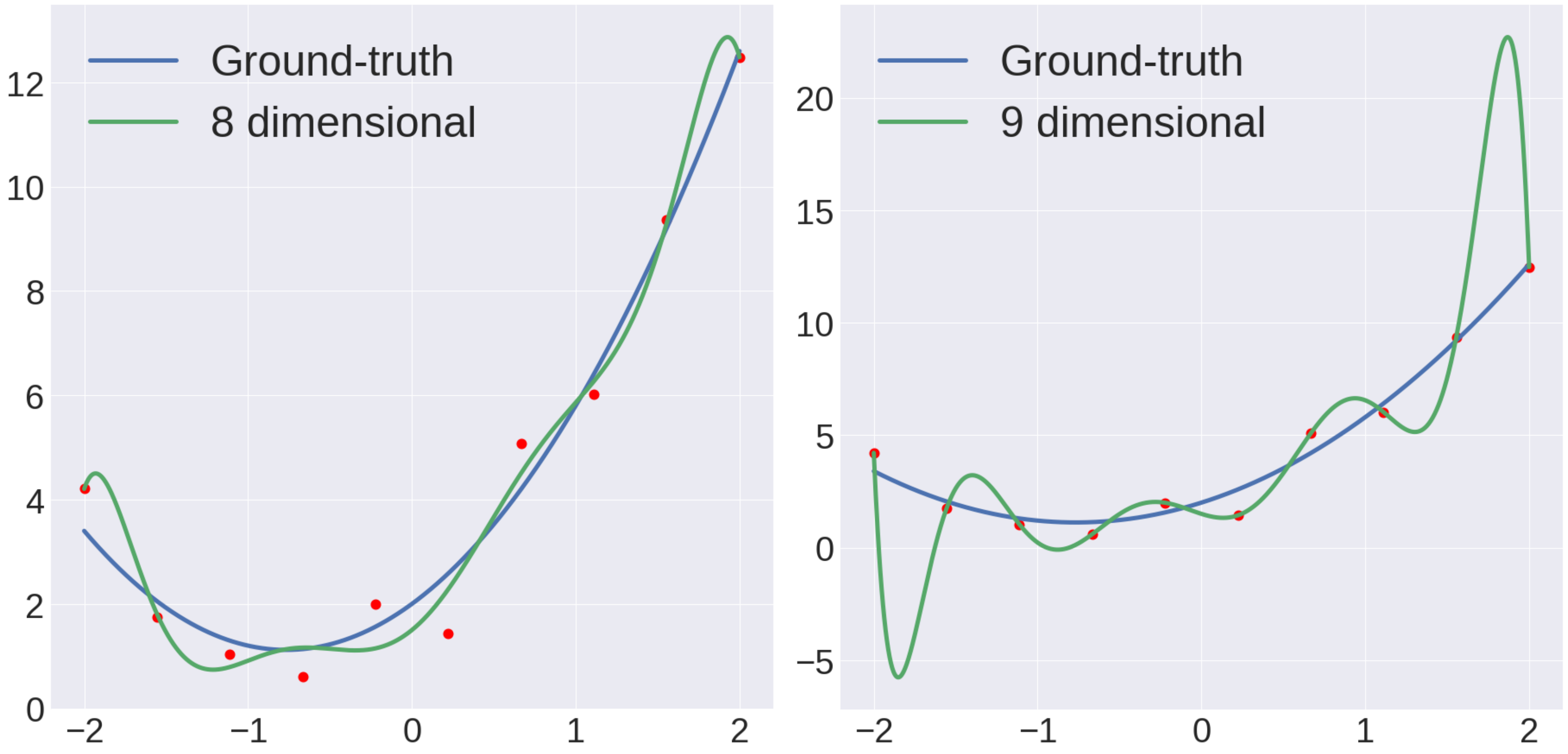



Model doesn’t have the parameters to fit the data.

*Bias* (statistics): Error intrinsic to the model.

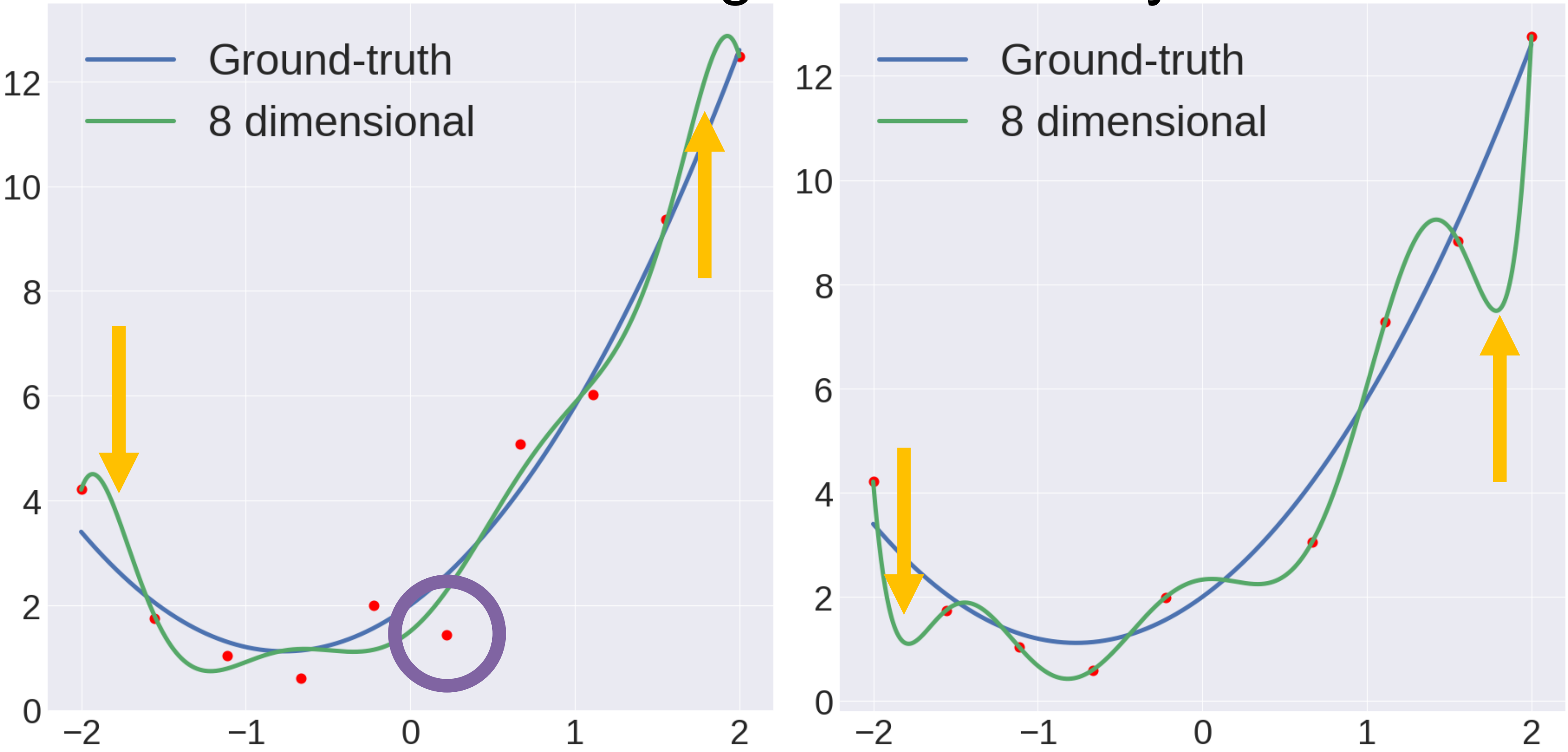
# Overfitting

Model: $1.5x^2 + 2.3x+2 + N(0,0.5)$



# Overfitting

Model has high ***variance***: remove **one point**, and model changes dramatically



# (Continuous) Model Complexity

$$\arg\min_{\boldsymbol{W}} \lambda \|\boldsymbol{W}\|_2^2 + \sum_{i=1}^{n} -\log\left(\frac{\exp((Wx)_{y_i})}{\sum_k \exp((Wx)_k))}\right)$$

Regularization: penalty for complex model

Pay penalty for negative log-likelihood of correct class

Intuitively: big weights = more complex model

Model 1: $0.01*x_1 + 1.3*x_2 + -0.02*x_3 + -2.1x_4 + 10$

Model 2: $37.2*x_1 + 13.4*x_2 + 5.6*x_3 + -6.1x_4 + 30$

# Fitting Model

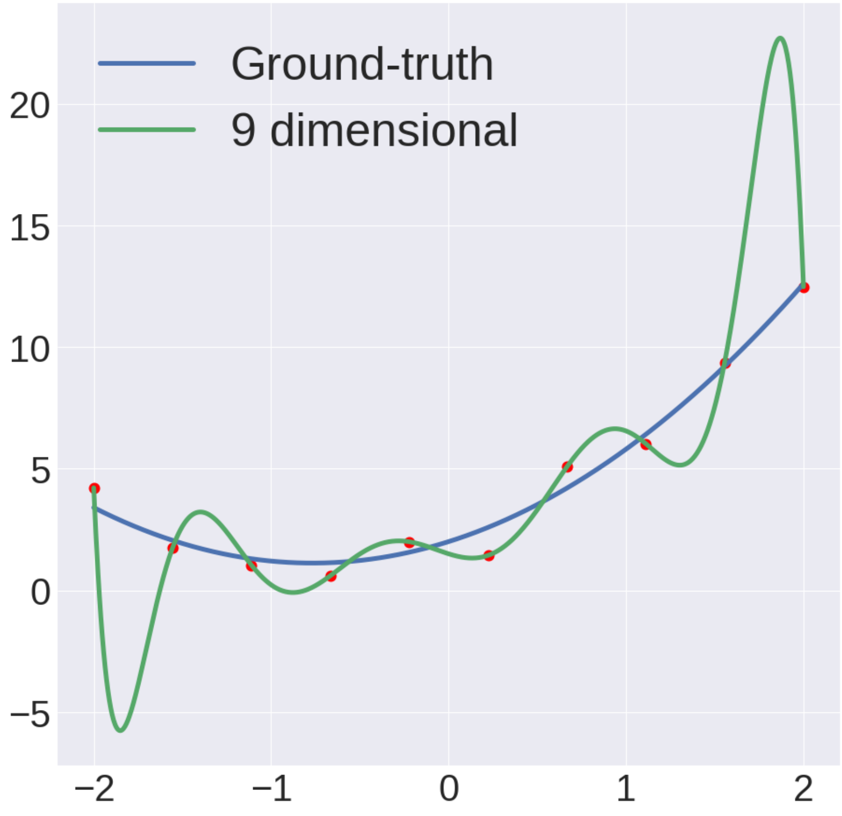
Again, fitting polynomial, but with regularization

$$\arg\min_{\mathrm{w}} ||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}|| + \lambda ||\boldsymbol{w}||$$

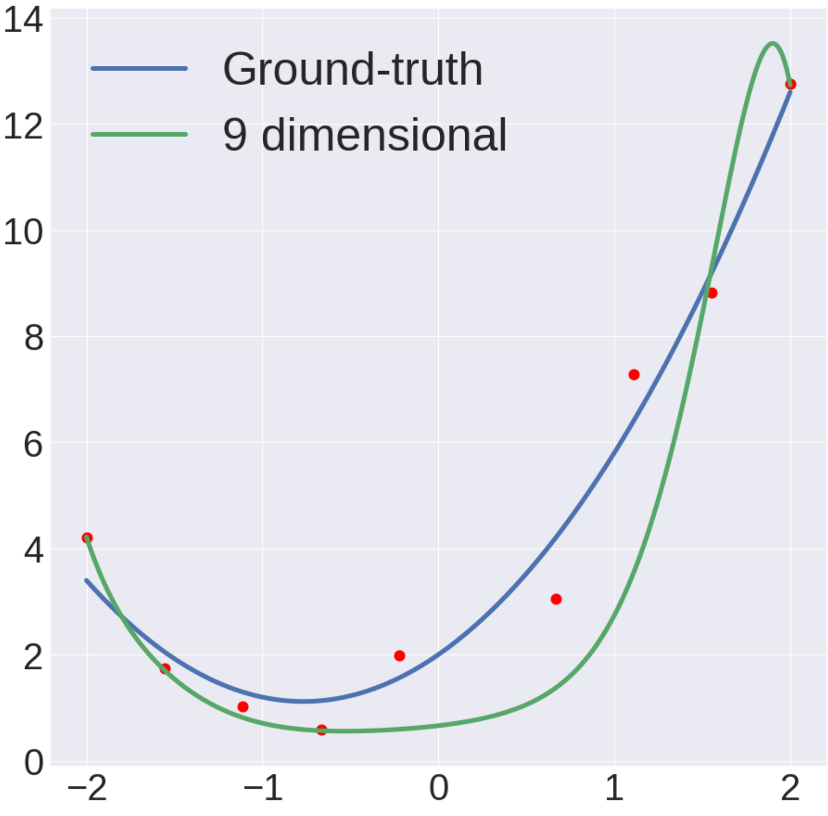
$$\begin{bmatrix} x_1^F & \cdots & x_1^2 & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ x_N^F & \cdots & x_N^2 & x_N & 1 \end{bmatrix} \quad \begin{bmatrix} w_F \\ \vdots \\ w_0 \end{bmatrix}$$

# Adding Regularization

## No regularization: fits all data points




## Regularization: can't fit all data points



# In General

Error on new data comes from combination of:

1. Bias: model is oversimplified and can’t fit the underlying data
2. Variance: you don’t have the ability to estimate your model from limited data
3. Inherent: the data is intrinsically difficult

Bias and variance trade-off. Fixing one hurts the other.

# Underfitting and Overfitting

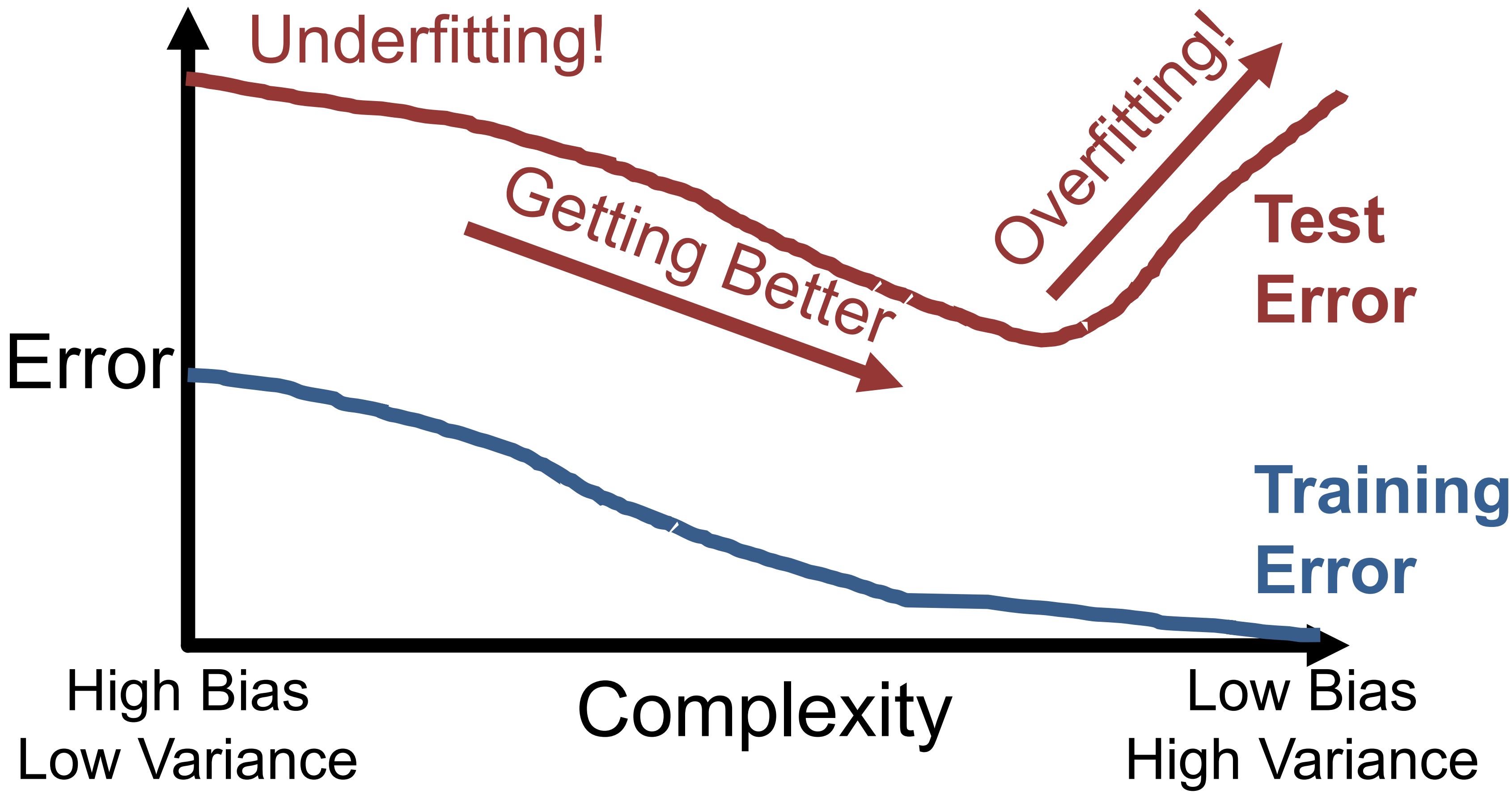



Diagram adapted from: D. Hoiem

# Underfitting and Overfitting

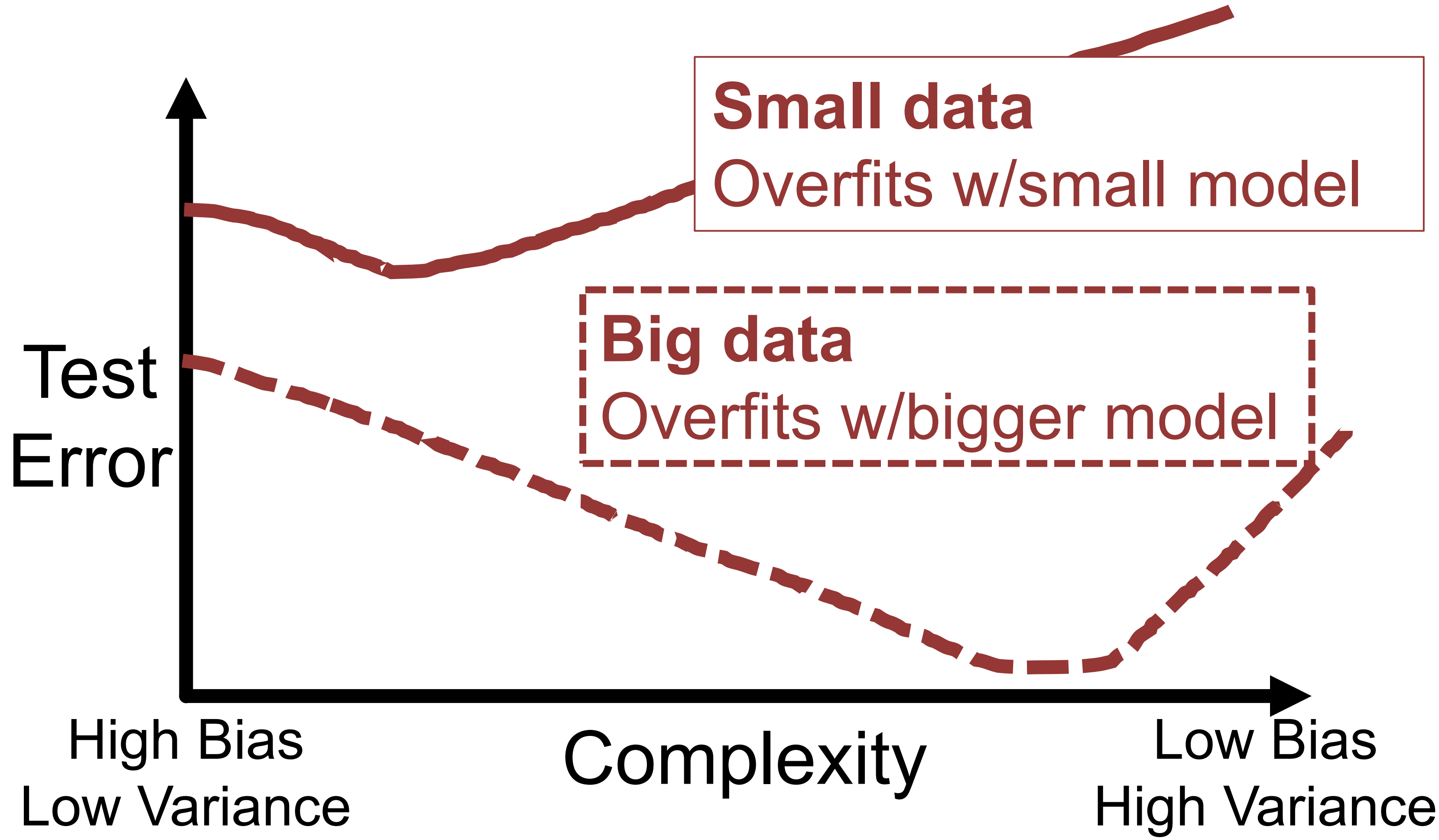



Diagram adapted from: D. Hoiem

# Underfitting and Overfitting

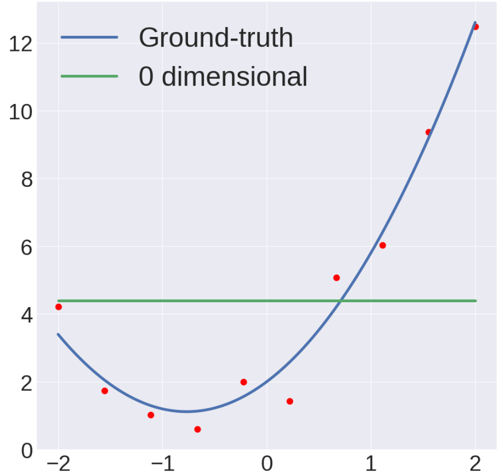
Small data

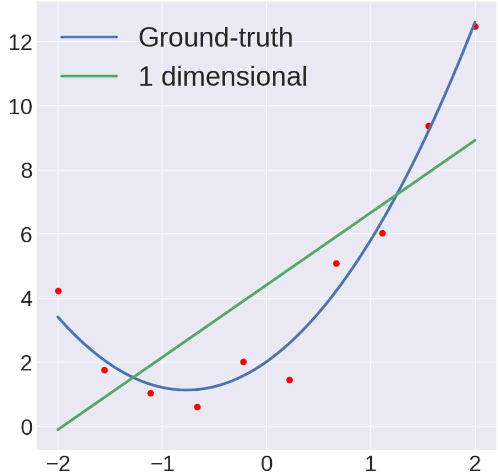

Lots of this behavior doesn't seem to replicate in models with high learning capacity.

See below for details.

Diagram adapted from: D. Hoiem; comic: https://webcomicname.com/
*Reconciling modern machine learning practice and the bias-variance trade-off*, Belkin et al. https://arxiv.org/pdf/1812.11118.pdf

# Underfitting



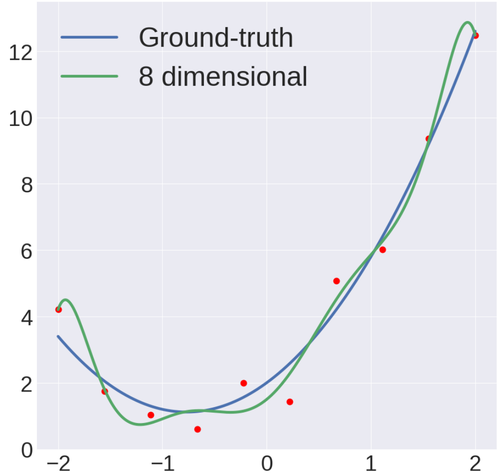



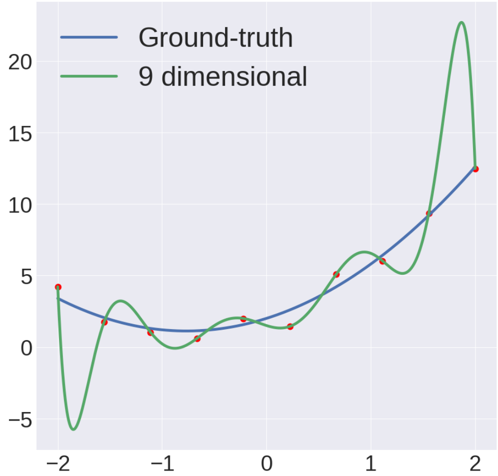

Do poorly on both training and validation data due to bias.

Solution:

1. More features
2. More powerful model
3. Reduce regularization

# Overfitting




Do well on training data, but poorly on validation data due to variance

Solution:

1. More data
2. Less powerful model
3. Regularize your model more

*Cris Dima rule*: first make sure you *can* overfit, then stop overfitting.

# Next Class

- Non-linear models (neural nets)

# Let’s Compute Another Gradient

- Below is another derivation that’s worth looking at on your own time if you’re curious

# Computing The Gradient

## Multiclass Support Vector Machine

$$\arg\min_{\boldsymbol{W}} \lambda \|\boldsymbol{W}\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \max(0, (\boldsymbol{W}\boldsymbol{x}_i)_j - (\boldsymbol{W}\boldsymbol{x}_i)_{y_i} + m)$$

Notation:

$W \rightarrow$ rows $w_i$ (i.e., per-class scorer)

$(Wx_i)_j \rightarrow w_j^T x_i$

$$\arg\min_{\boldsymbol{W}} \lambda \sum_{j=1}^{K} \|\boldsymbol{w}_j\|_2^2 + \sum_{i=1}^{n} \sum_{j \neq y_i} \max(0, \boldsymbol{w}_j^T \boldsymbol{x}_i - \boldsymbol{w}_{y_i}^T \boldsymbol{x}_i + m)$$

Derivation setup: Karpathy and Fei-Fei

# Computing The Gradient

$$\arg\min_{\boldsymbol{W}} \boldsymbol{\lambda} \sum_{\boldsymbol{j=1}}^{\boldsymbol{K}} \|\boldsymbol{w_j}\|_2^2 + \sum_{i=1}^{n} \underbrace{\sum_{j \neq y_i} \max(0, \boldsymbol{w}_j^T \boldsymbol{x}_i - \boldsymbol{w}_{y_i}^T \boldsymbol{x}_i + m)}$$

$$\frac{\partial}{\partial w_j}: \quad \begin{array}{ll} w_j^T x_i - w_{y_i}^T x_i + m \leq 0: & 0 \\ w_j^T x_i - w_{y_i}^T x_i + m > 0: & x_i \end{array}$$

$$\rightarrow 1(w_j^T\ x_i - w_{y_i}^T x_i + m > 0) x_i$$

Derivation setup: Karpathy and Fei-Fei

# Computing The Gradient

$$\arg\min_{\boldsymbol{W}} \boldsymbol{\lambda} \sum_{\boldsymbol{j=1}}^{\boldsymbol{K}} \left\|\boldsymbol{w_j}\right\|_{\boldsymbol{2}}^{\boldsymbol{2}} + \sum_{i=1}^{n} \underbrace{\sum_{j \neq y_i} \max(0, \boldsymbol{w}_j^T \boldsymbol{x}_i - \boldsymbol{w}_{y_i}^T \boldsymbol{x}_i + m)}$$

$$\frac{\partial}{\partial w_{y_i}}: \quad \sum_{j \neq y_i} 1\left(w_j^T x_i - w_{y_i}^T x_i + m > 0\right)(-x_i)$$

Derivation setup: Karpathy and Fei-Fei

# Interpreting The Gradient

$$-\frac{\partial}{\partial w_j}: \underbrace{1(w_j^T x_i - w_{y_i}^T x_i + m > 0)}\ \underbrace{-x_i}$$

If we do not predict the correct class by at least a score difference of m …

Want incorrect class's scoring vector to score that point lower.

**Recall:**
**Before**: $w^T x$;
**After**: $(w-\alpha x)^T x = w^T x - a x^T x$

Derivation setup: Karpathy and Fei-Fei