Epipolar Geometry

EECS 442 – David Fouhey

Winter 2023, University of Michigan

https://web.eecs.umich.edu/~fouhey/teaching/EECS442_W23/

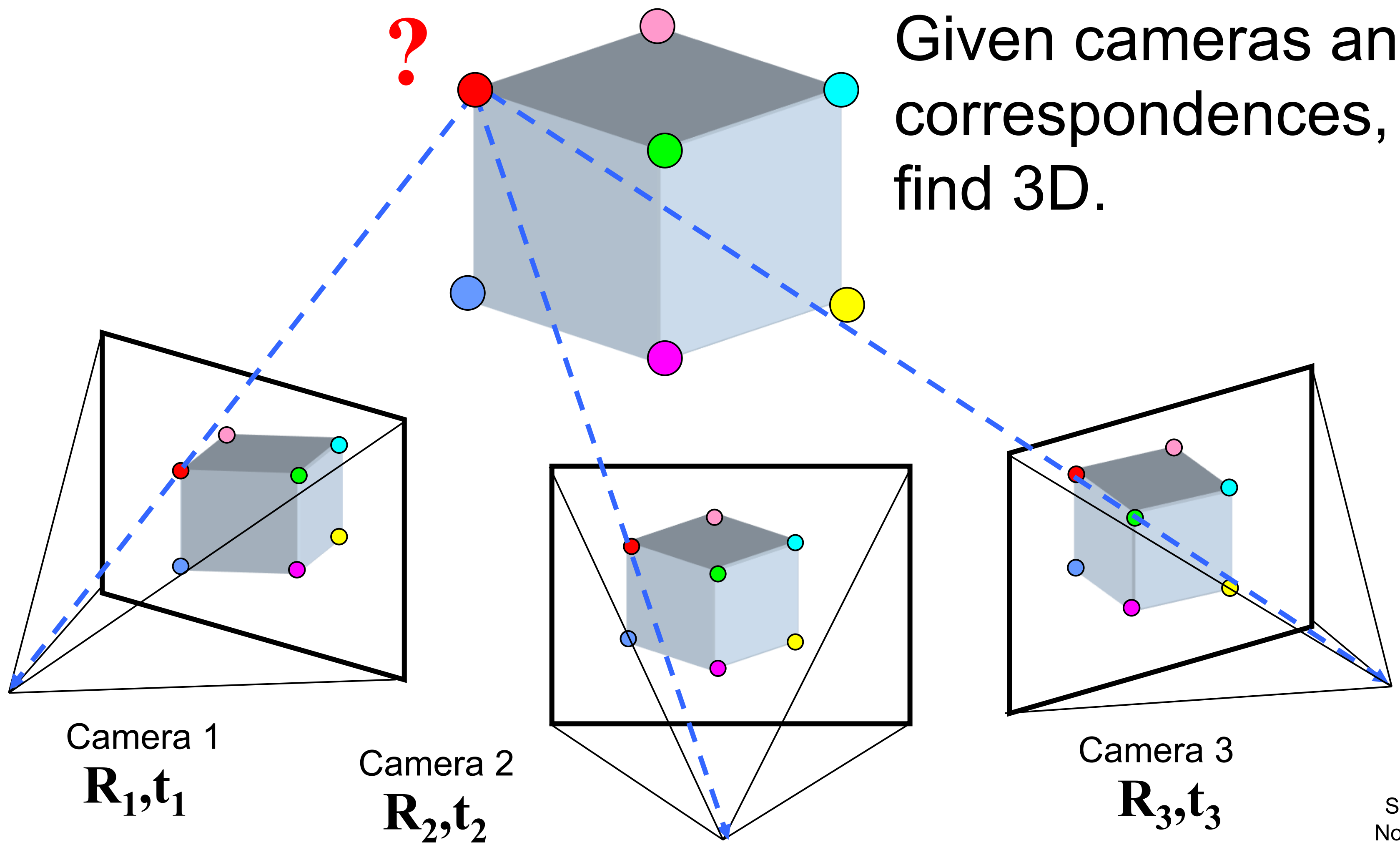
Slide Credit: These slides are largely a re-make of S. Lazebnik's diagrams and follow her general outline.

Multi-view geometry

Image Credit: S. Lazebnik

Multi-view geometry problems

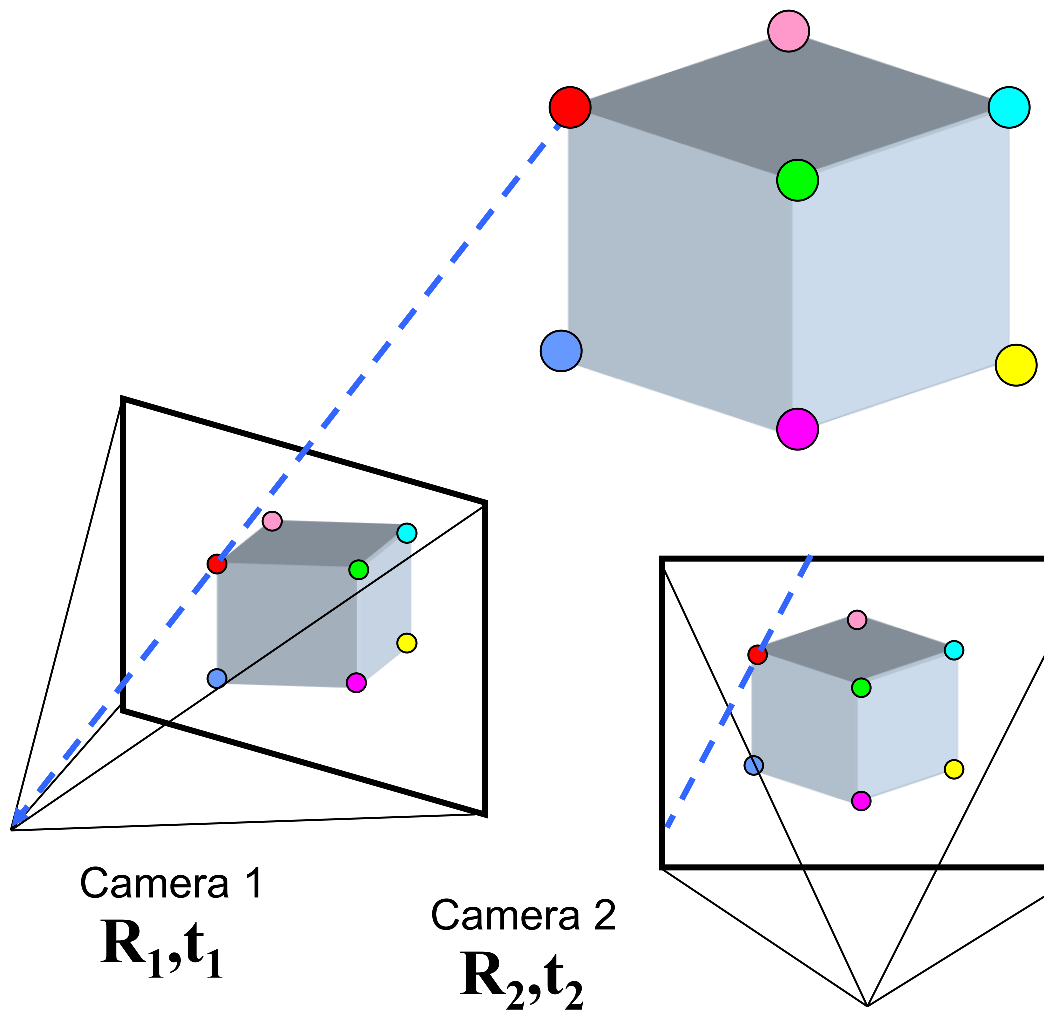
Recovering structure: Given cameras and correspondences, find 3D.



Slide credit: Noah Snavely

Multi-view geometry problems

Stereo/Epipolar Geomery:
Given 2 cameras and find where a point could be



Slide credit: Noah Snavely

Two-view geometry

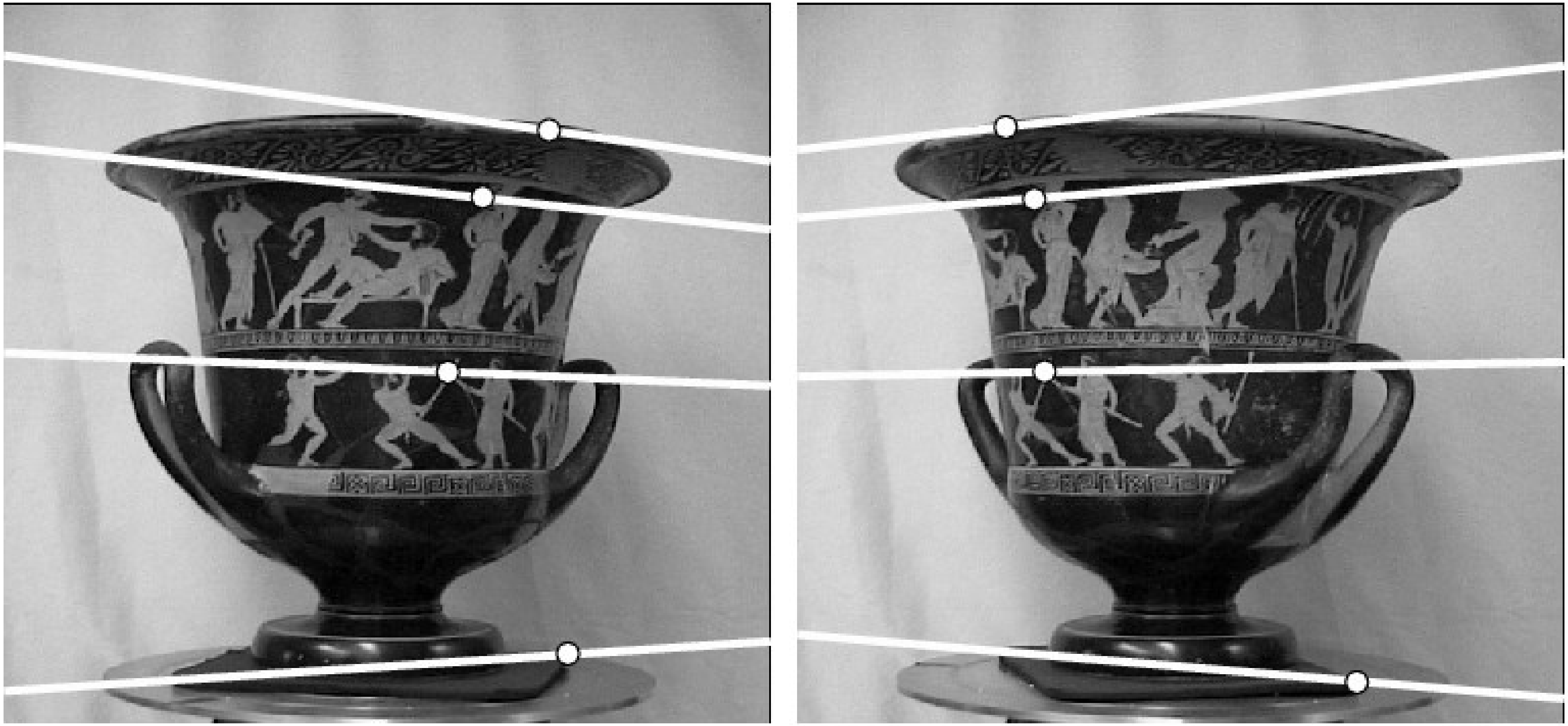
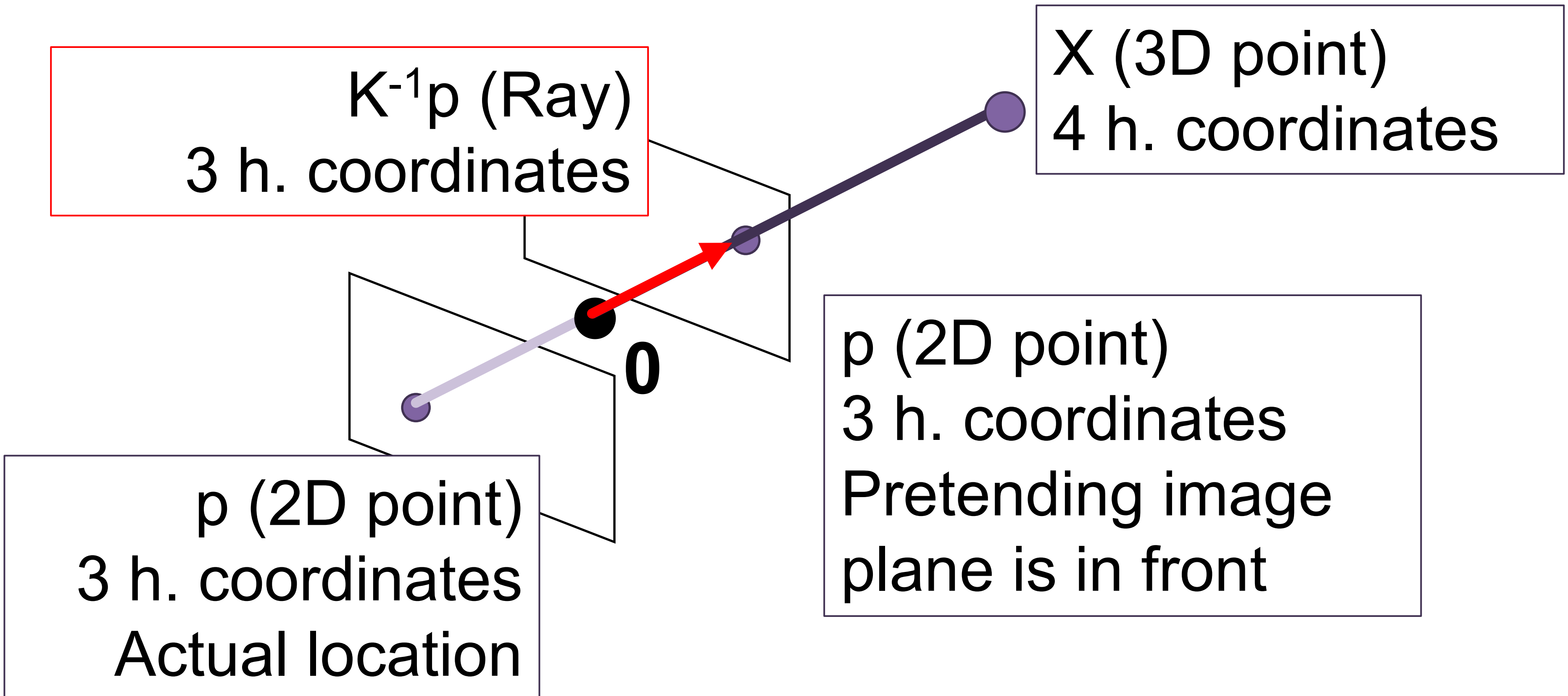


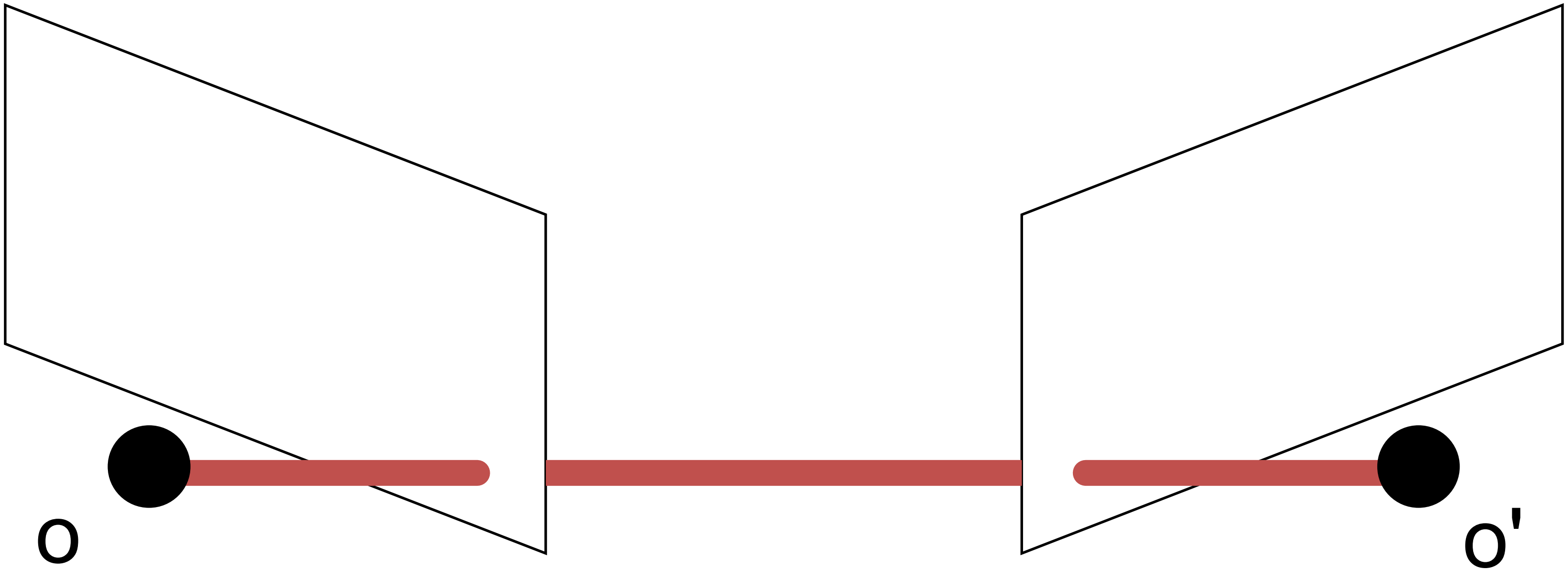
Image Credit: Hartley & Zisserman

Camera Geometry Reminder



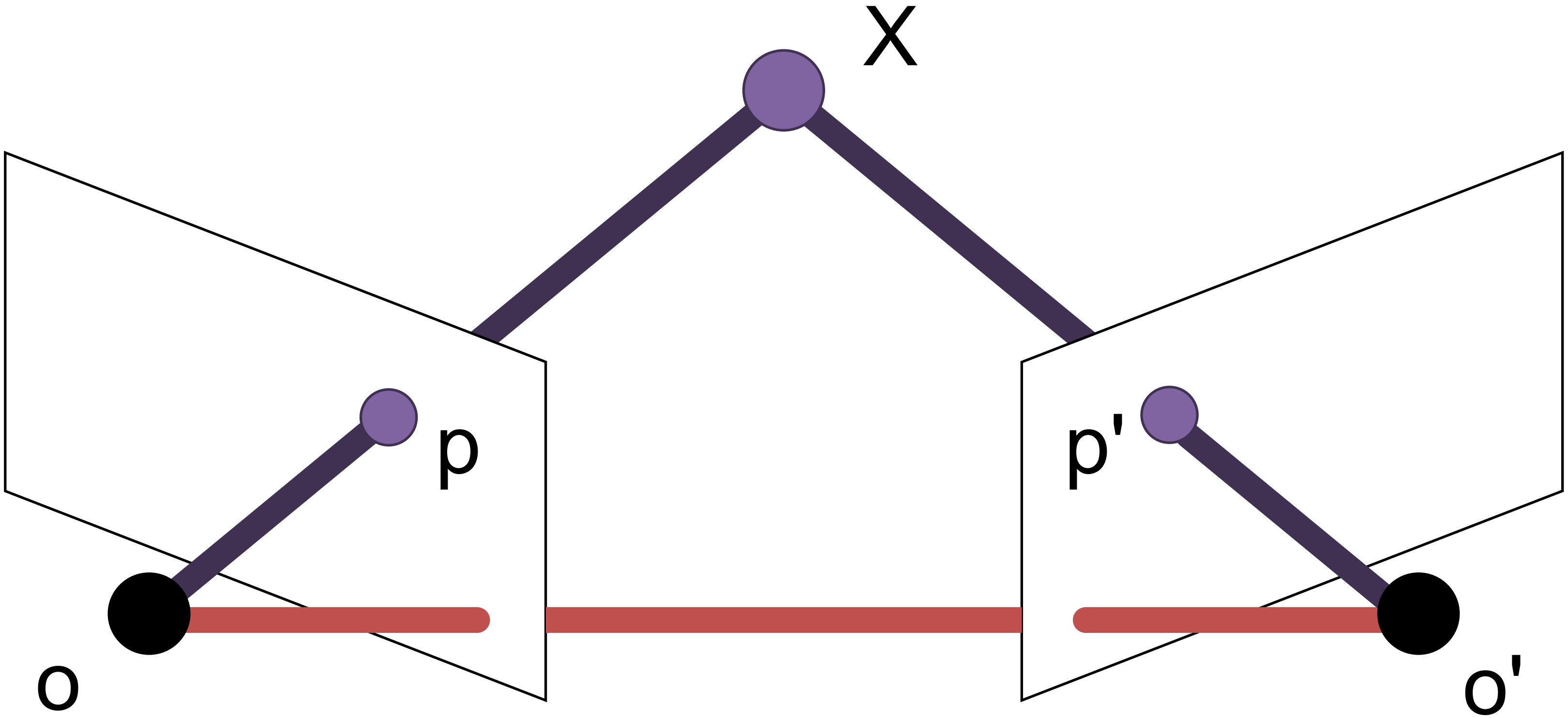
Have camera with pinhole at origin **0**

Epipolar Geometry



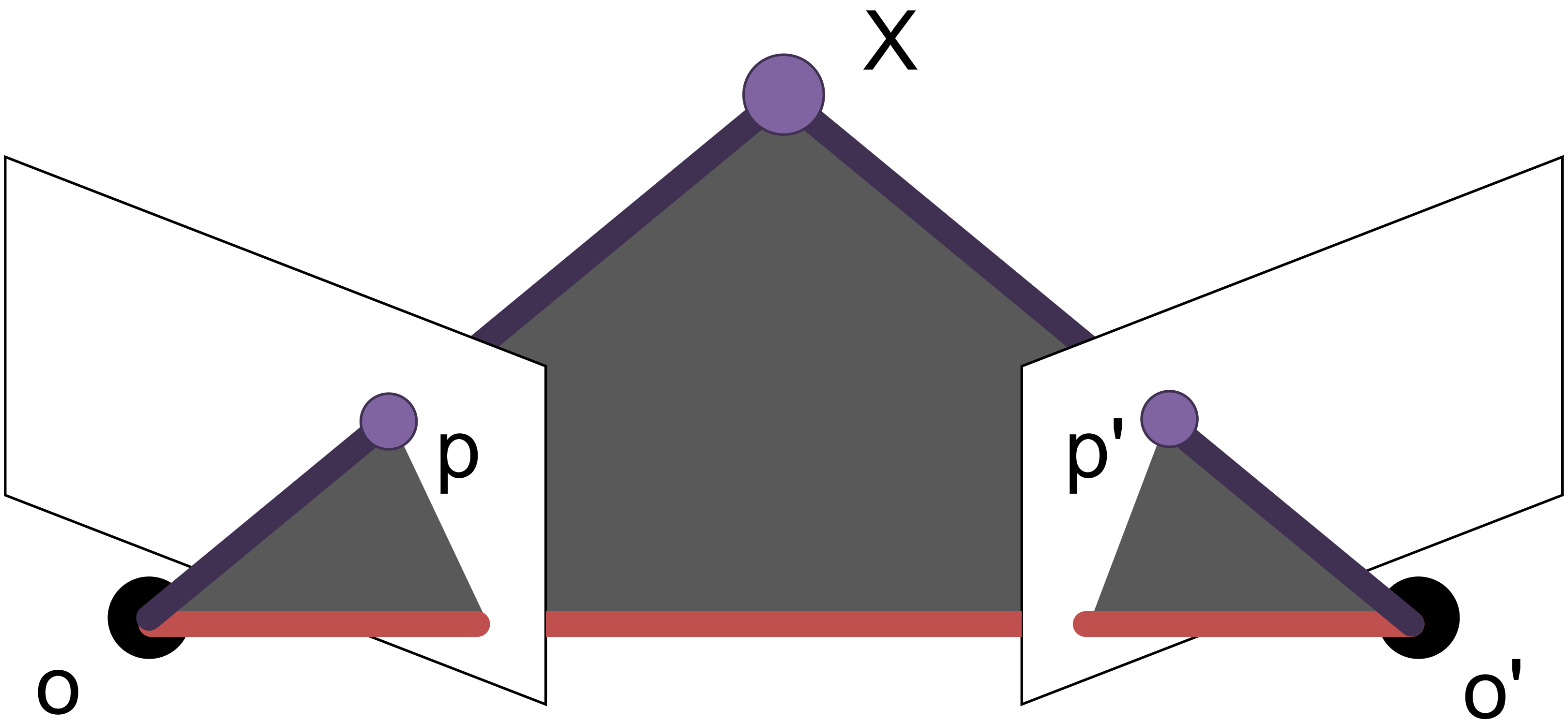
Suppose we have two cameras at origins o, o’
Baseline is the line connecting the origins

Epipolar Geometry



Now add a **point X**, which projects to p and p’

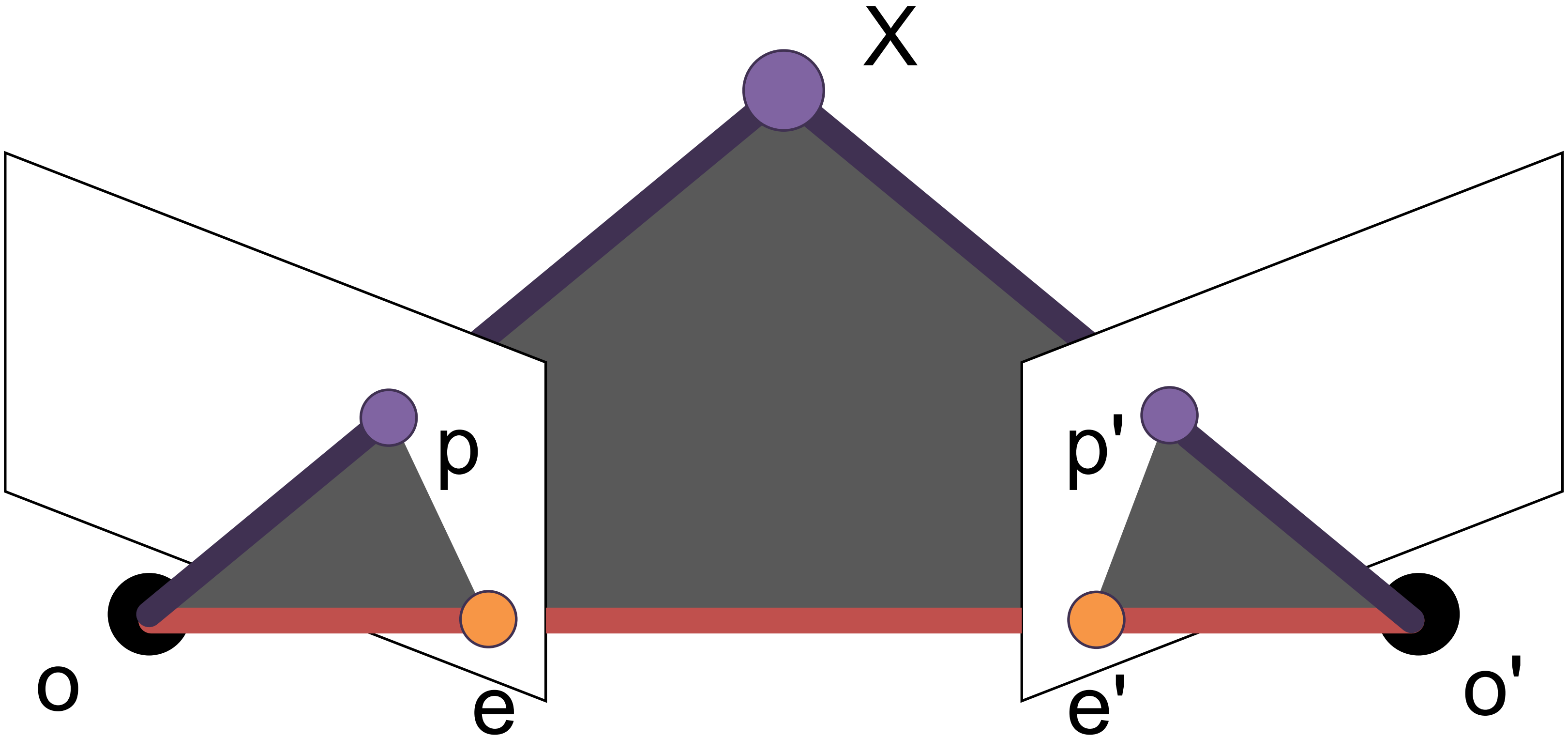
Epipolar Geometry



The plane formed by X, o, and o’ is called the epipolar plane

There is a family of planes per o, o’

Epipolar Geometry

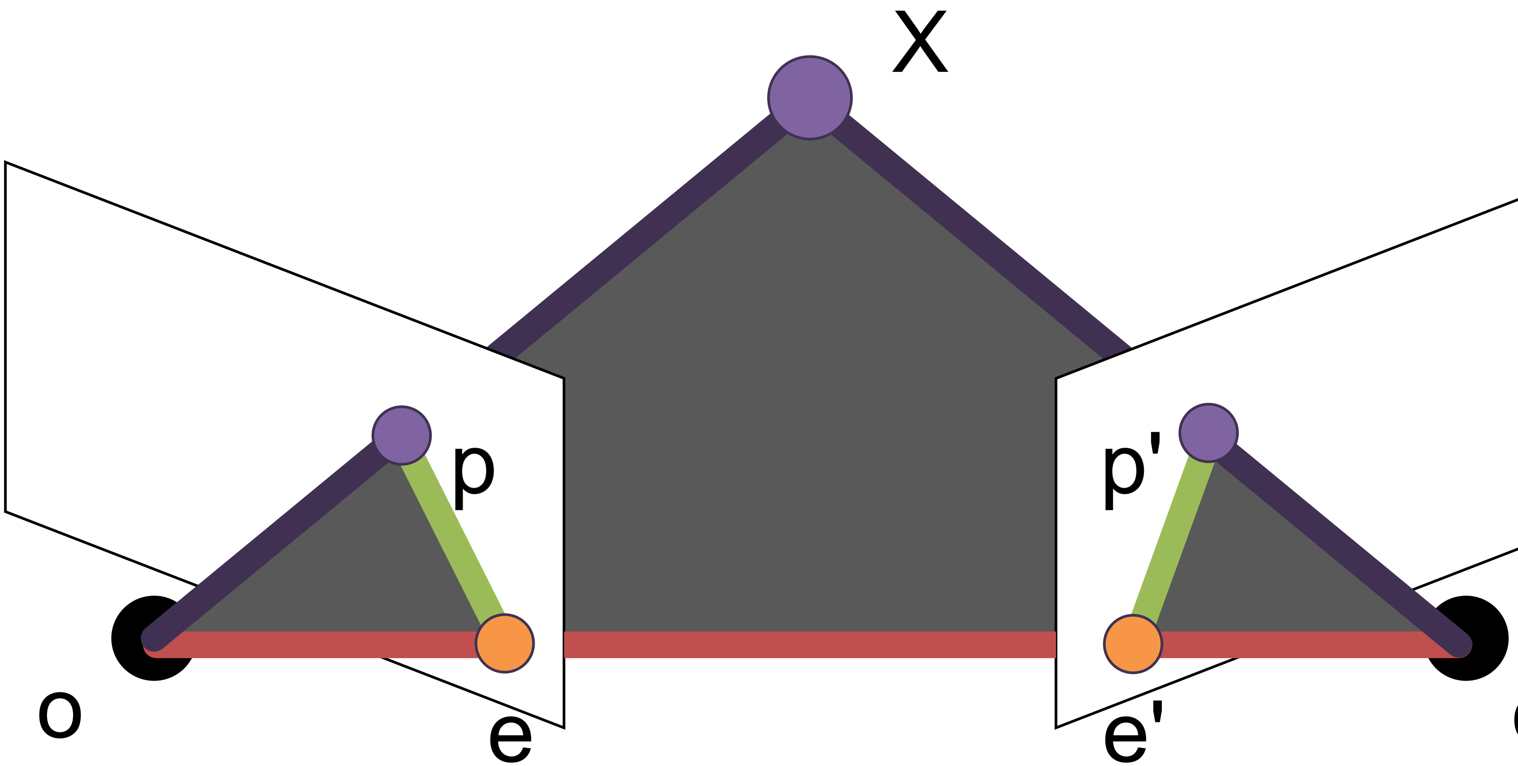


- Epipoles e, e’ are where the baseline intersects the image planes
- Projection of other camera in the image plane

The Epipole

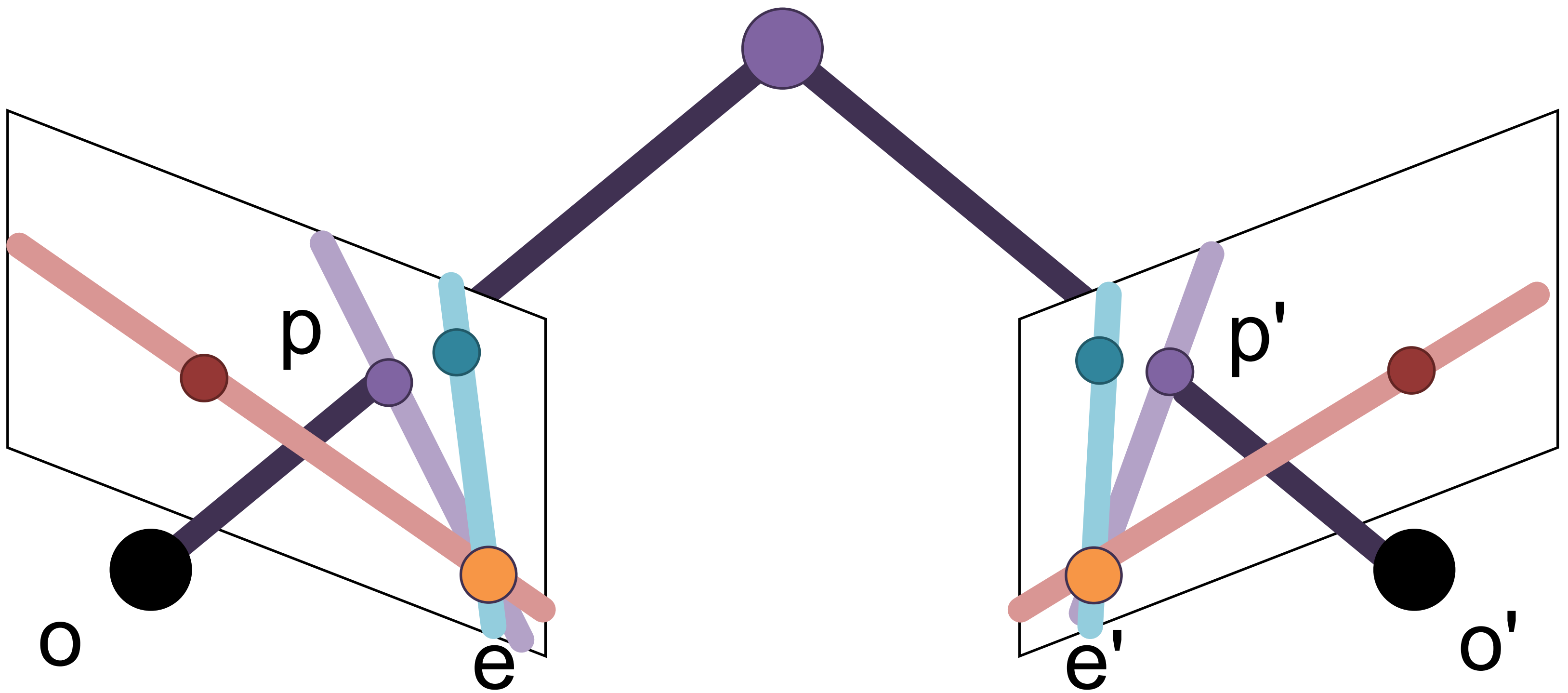
Photo by Frank Dellaert

Epipolar Geometry



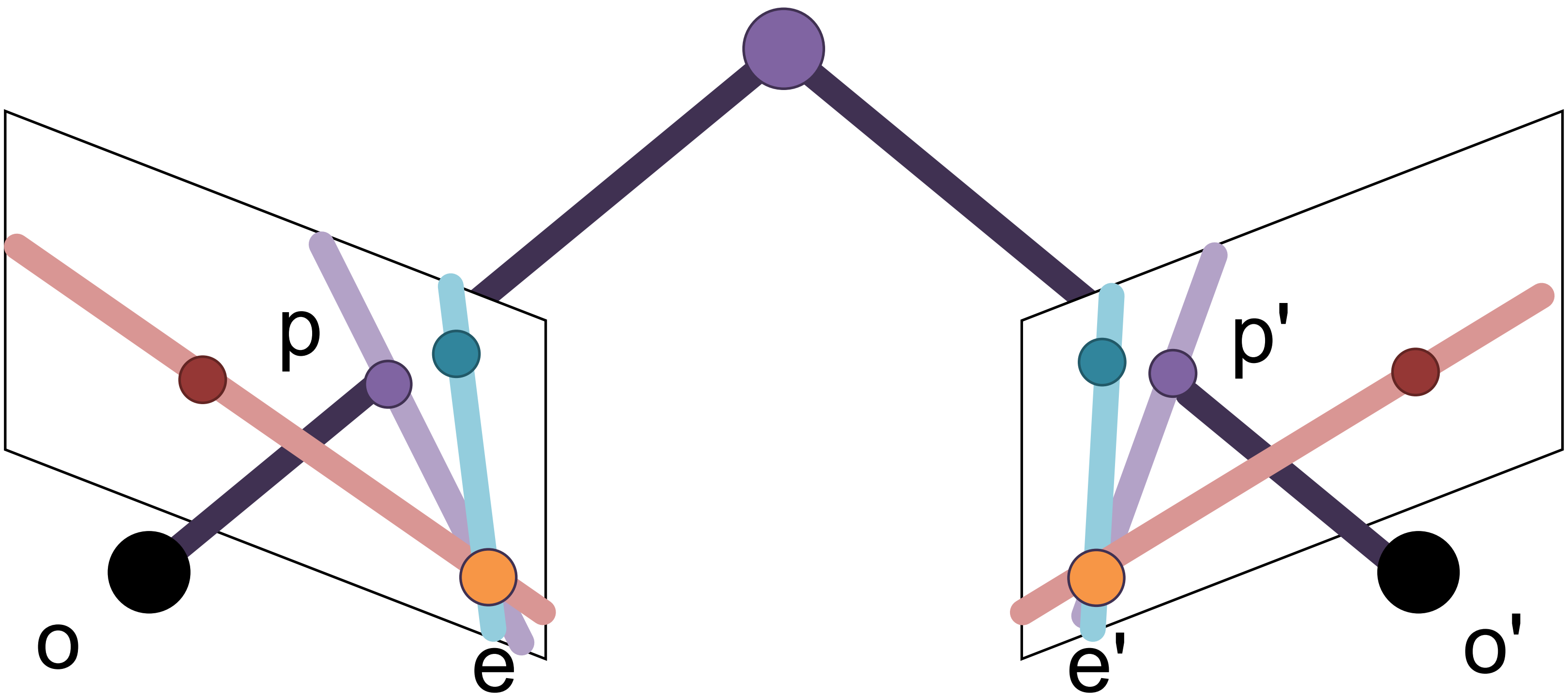
- Epipolar lines go between the epipoles and the projections of the points.
- Intersection of epipolar plane with image plane

Example: Converging Cameras



Epipoles finite, maybe in image; epipolar lines converge

Example: Converging Cameras



Epipolar lines come in pairs: given a point p, we can construct the epipolar line for p’.

Example 1: Converging Cameras

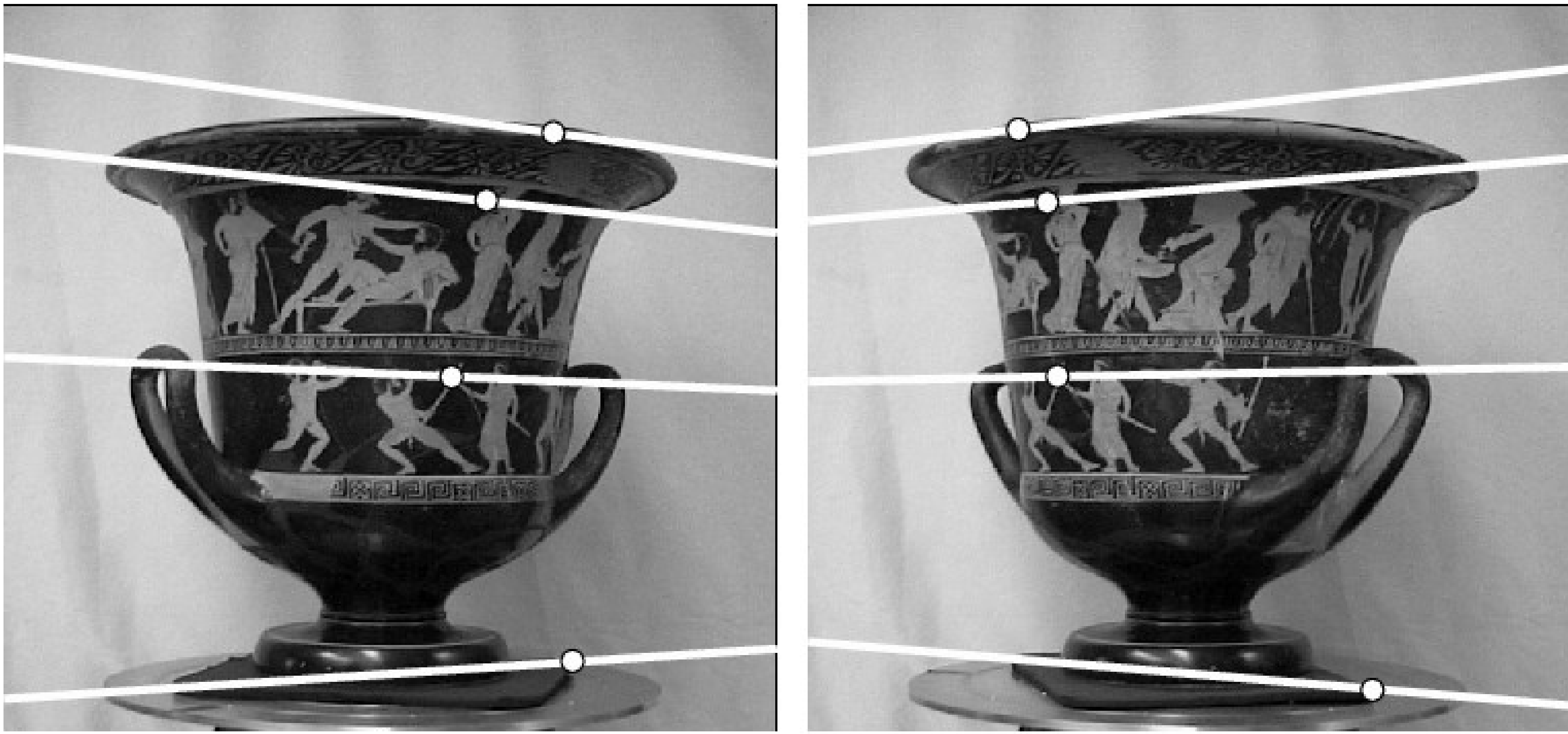
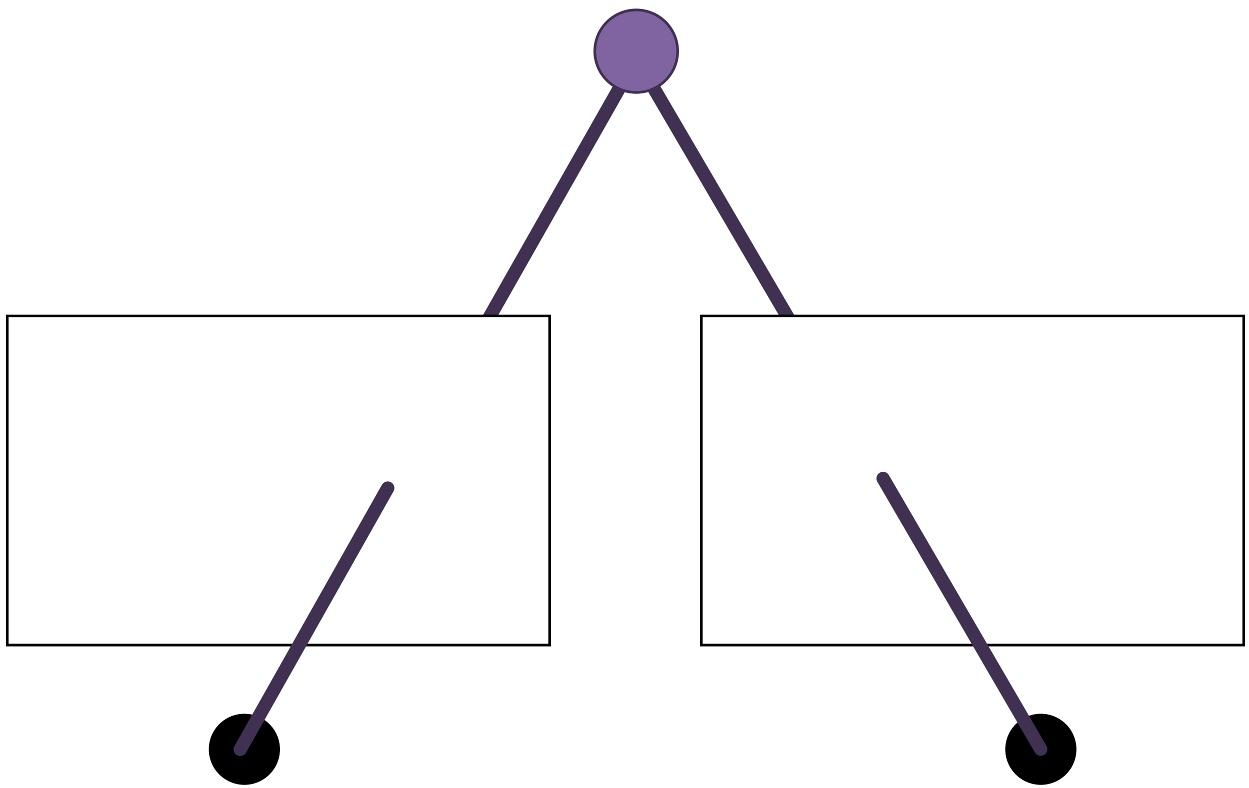


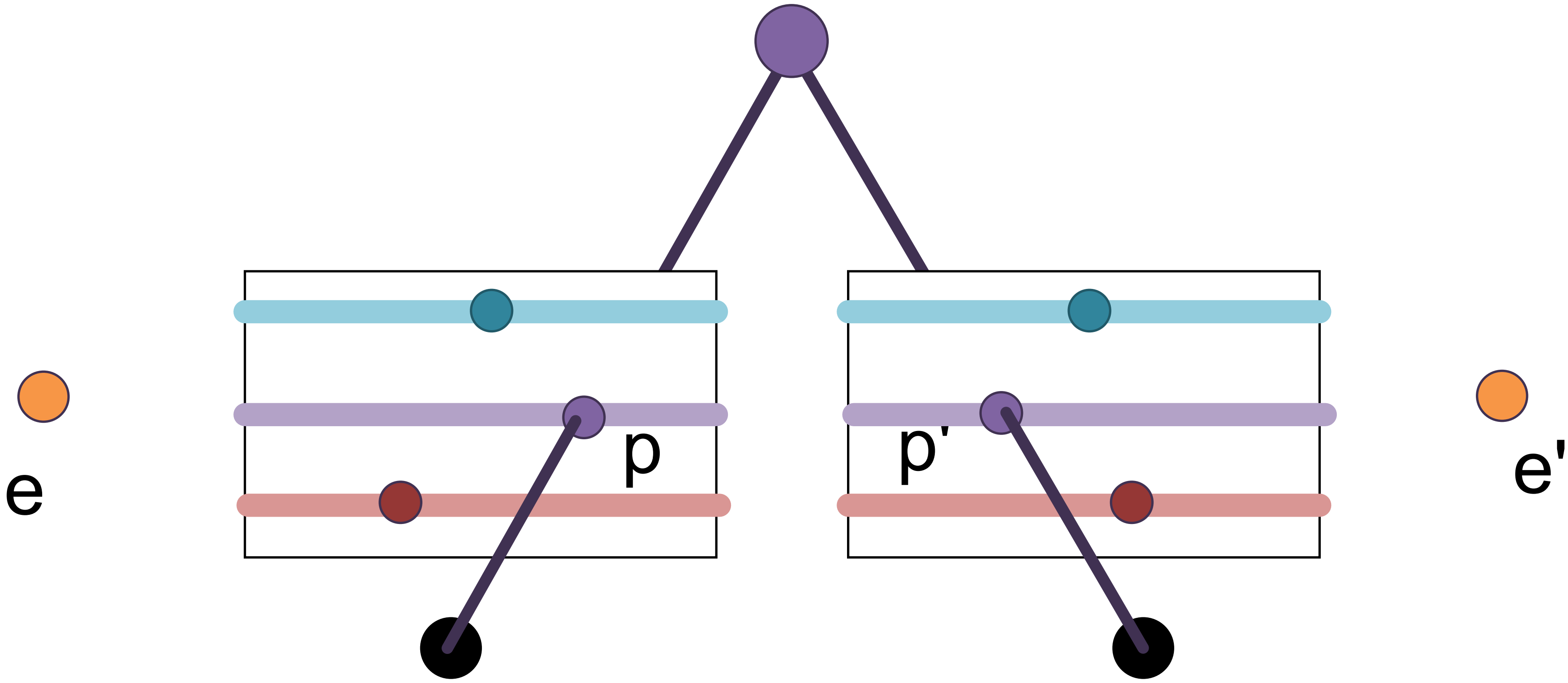
Image Credit: Hartley & Zisserman

Example: Parallel to Image Plane



Suppose the cameras are both facing outwards.
Where are the epipoles (proj. of other camera)?

Example: Parallel to Image Plane



Epipoles *infinitely* far away, epipolar lines parallel

Example: Parallel to Image Plane

Image Credit: Hartley & Zisserman

Example: Forward Motion

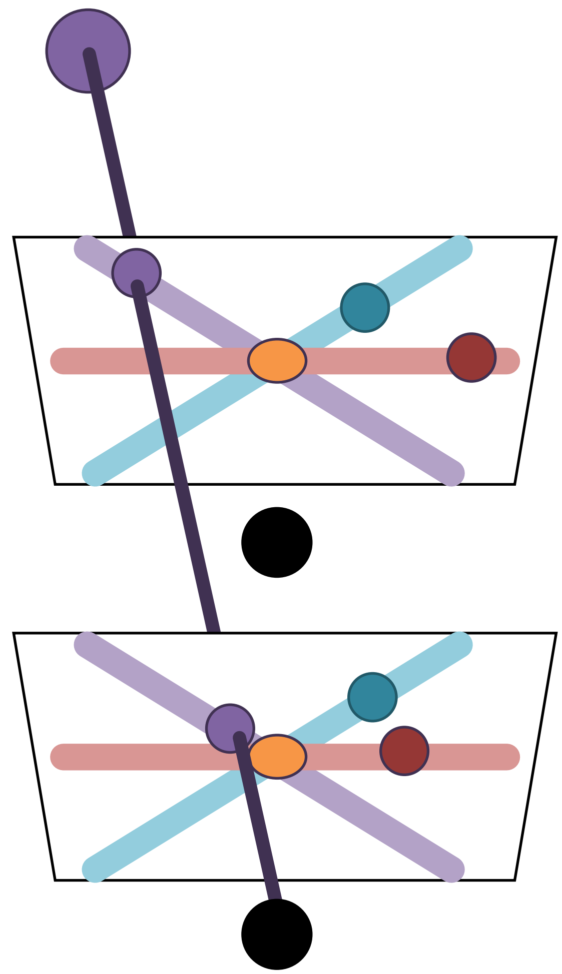
Image Credit: Hartley & Zisserman

Example: Forward Motion

Image Credit: Hartley & Zisserman

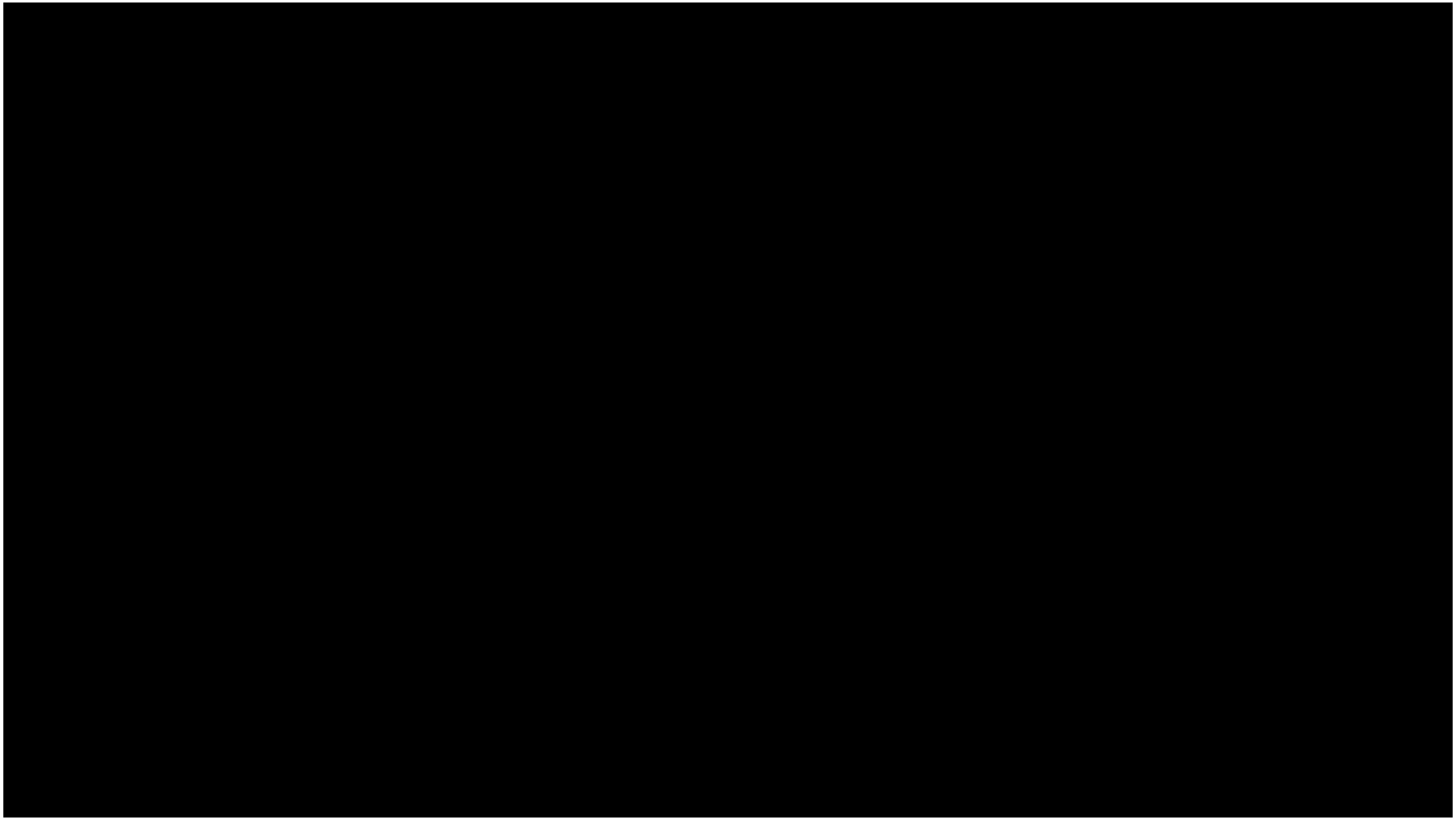
Example: Forward Motion

Epipole is focus of expansion / principal point of the camera.



Epipolar lines go out from principal point

Motion perpendicular to image plane



http://vimeo.com/48425421

HW6 Preview

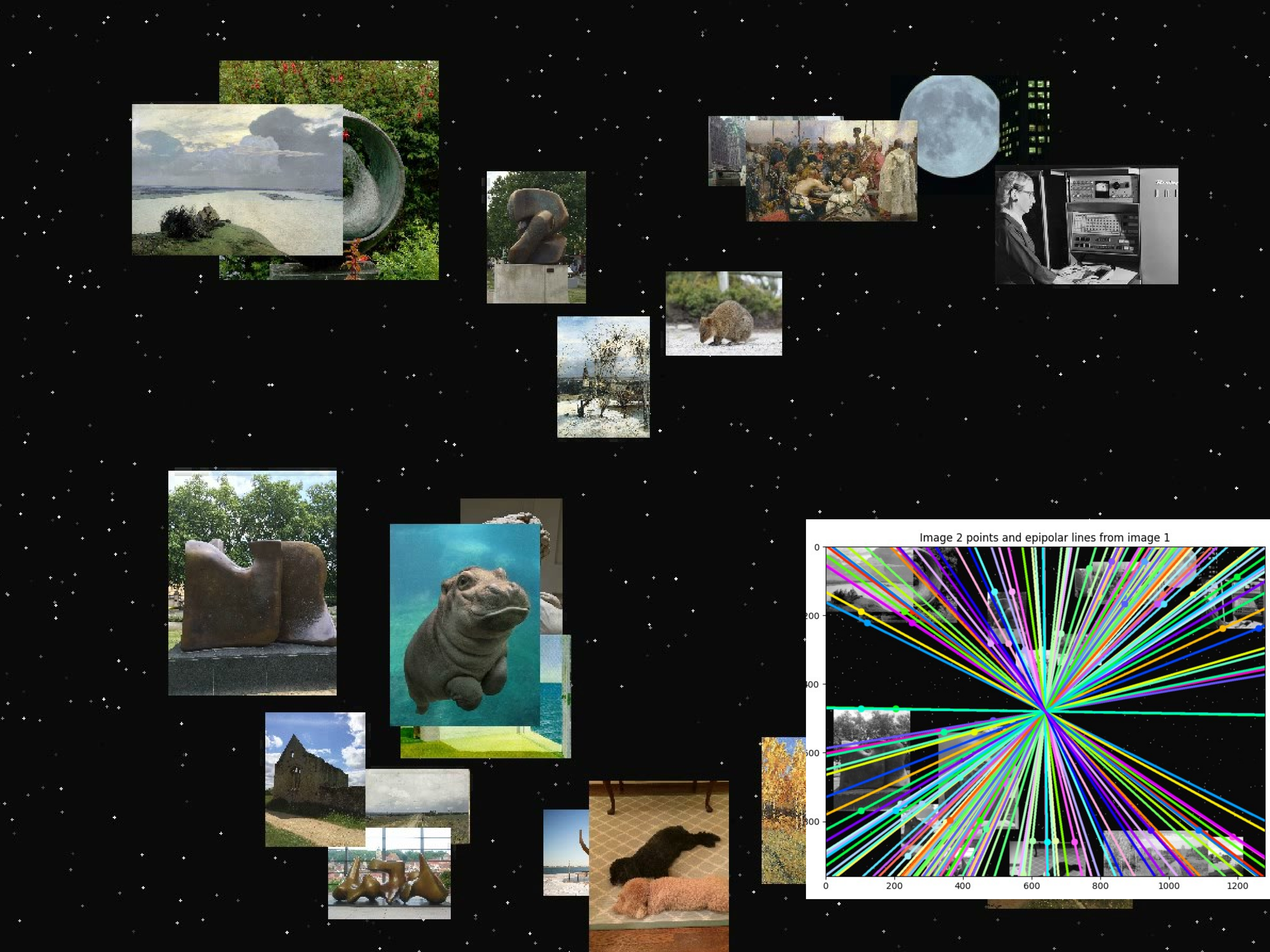


Image 2 points and epipolar lines from image 1
0
200
400
600
800
1000
1200


Image 2 points and epipolar lines from image 1
0
200
400
600
800
1000
1200

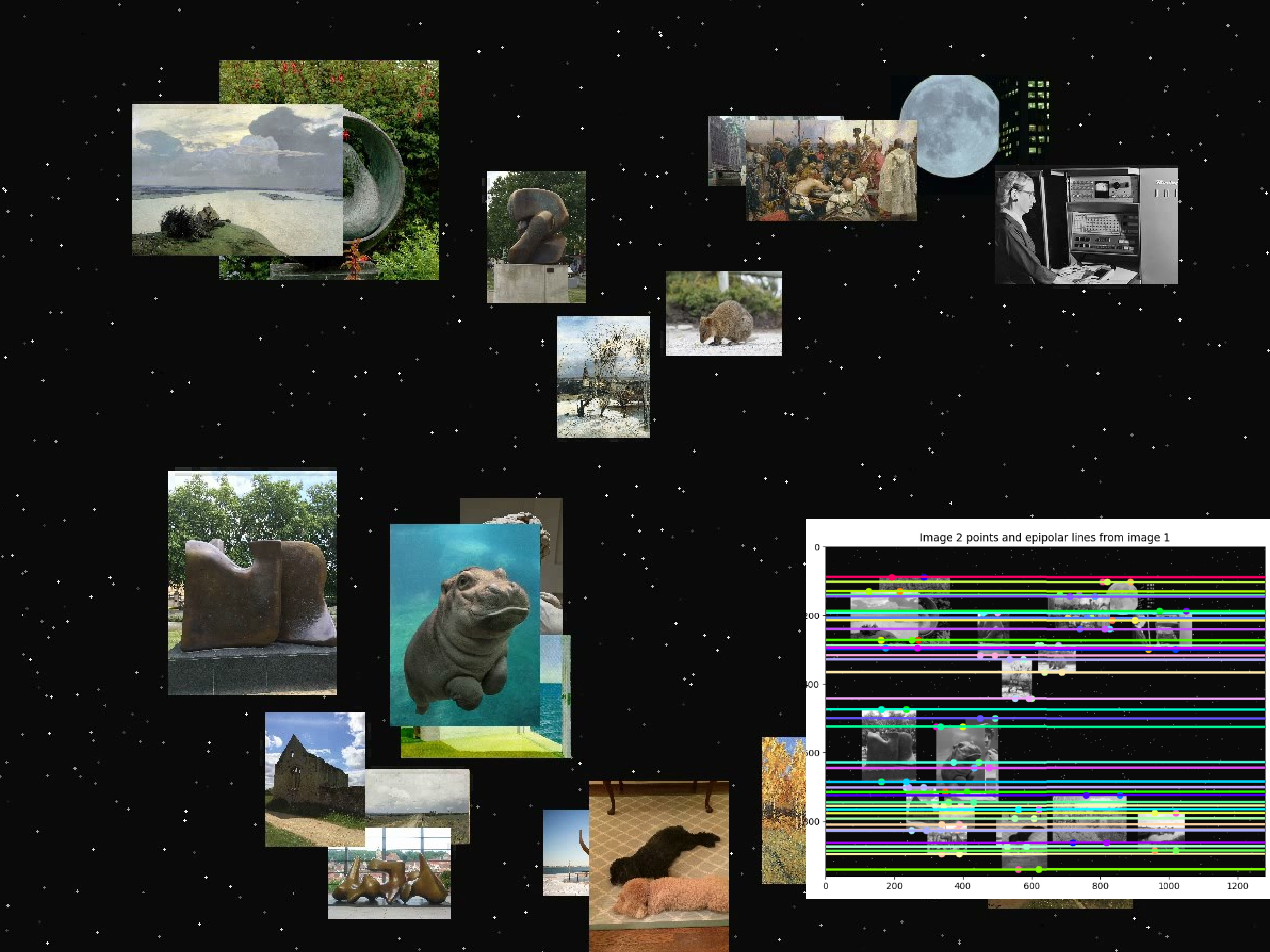
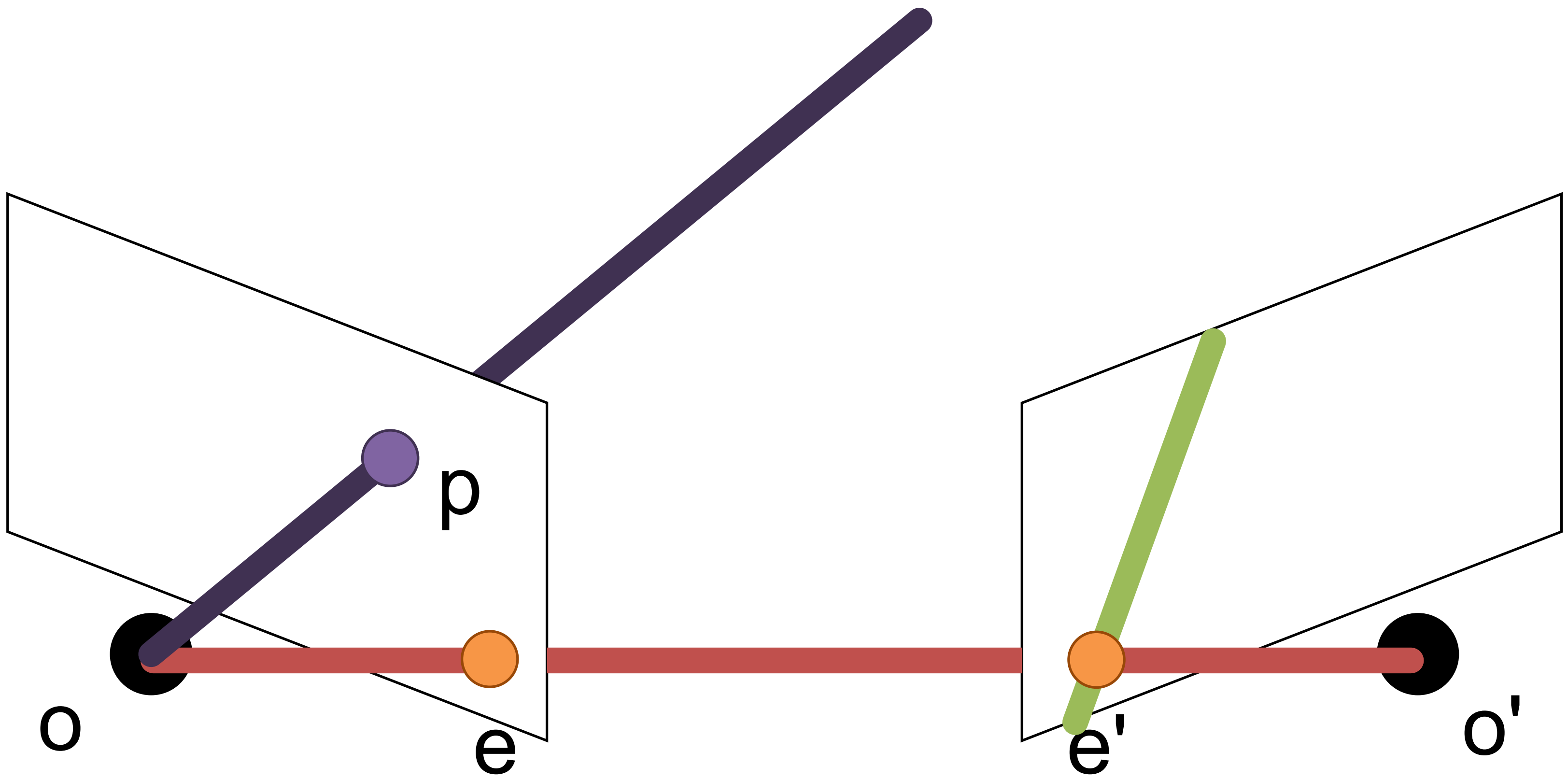


Image 2 points and epipolar lines from image 1
0
200
400
600
800
1000
1200

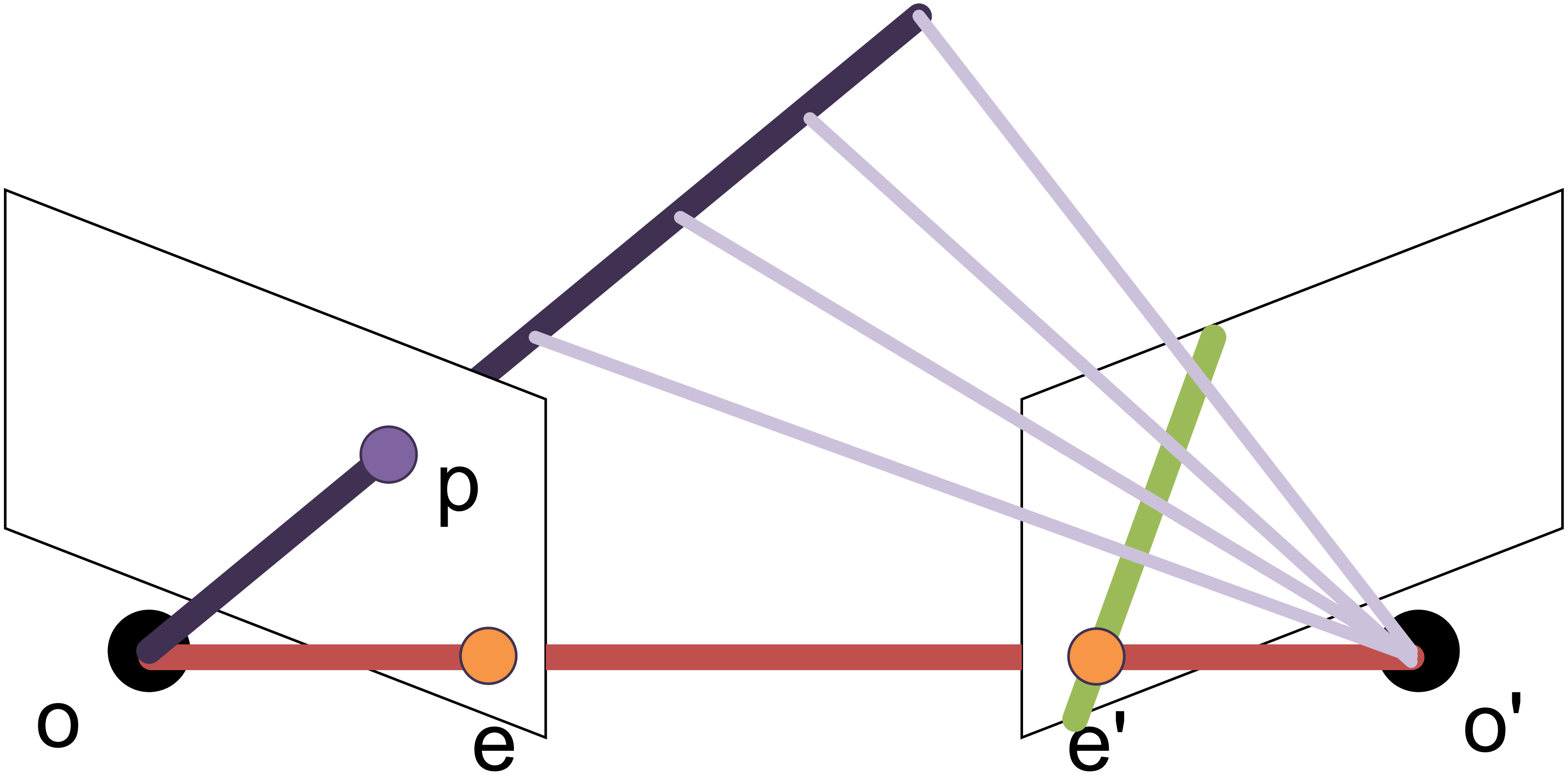
So?

Epipolar Geometry



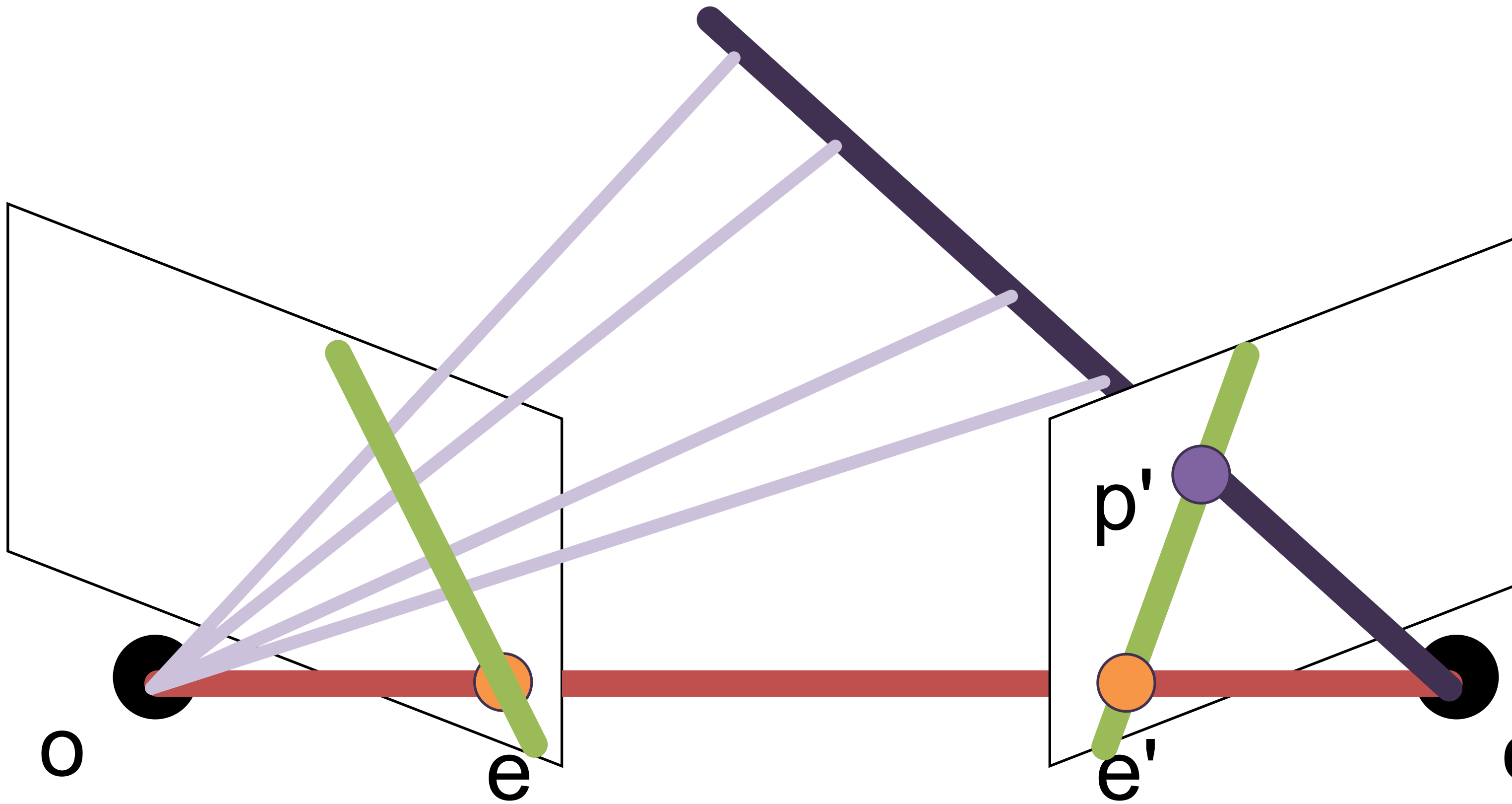
- Suppose we don’t know X and just have p
- Can construct the epipolar line in the other image

Epipolar Geometry



- Suppose we don't know X and just have p
- Corresponding p' is on corresponding epipolar line

Epipolar Geometry

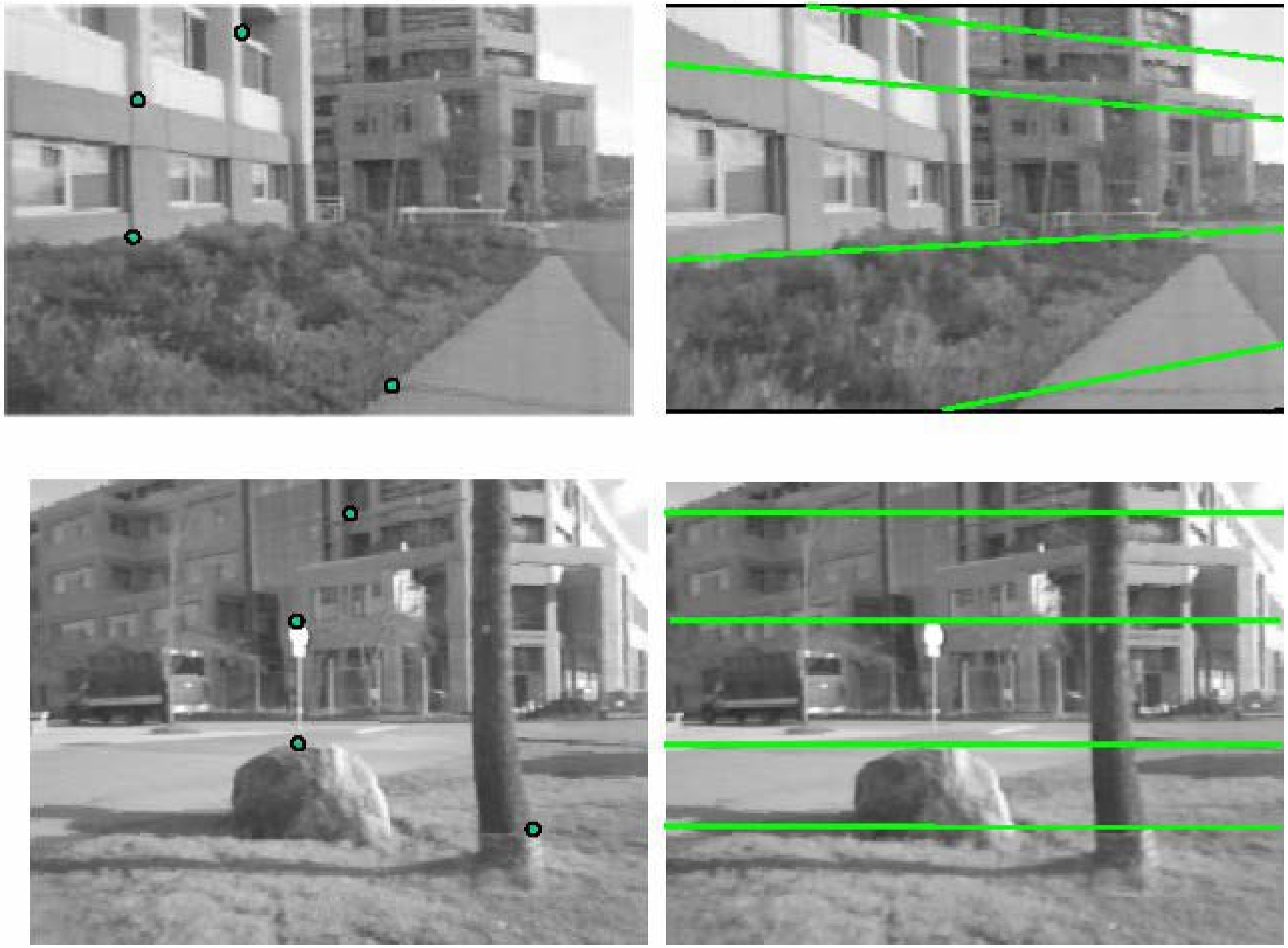


- Suppose we don’t know X and just have p’
- Corresponding p is on corresponding epipolar line

Epipolar Geometry

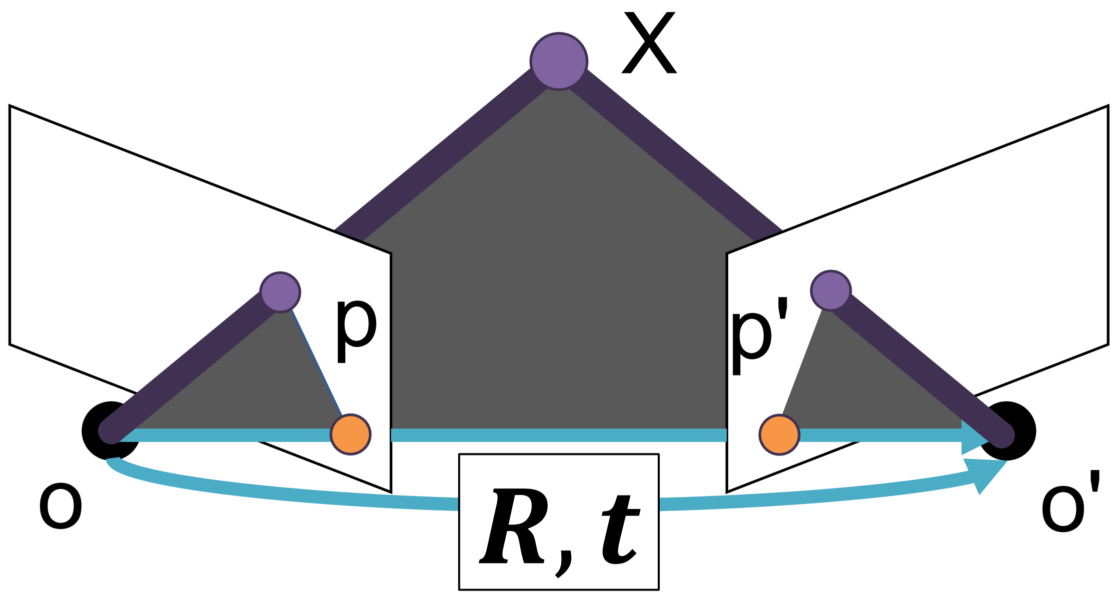
- If I want to do stereo, I want to find a corresponding pixel for each pixel in the image:
- Naïve search:
  - For each pixel, search every other pixel
- With epipolar geometry:
  - For each pixel, search along each line (1D search)

Epipolar constraint example



Slide Credit: S. Lazebnik

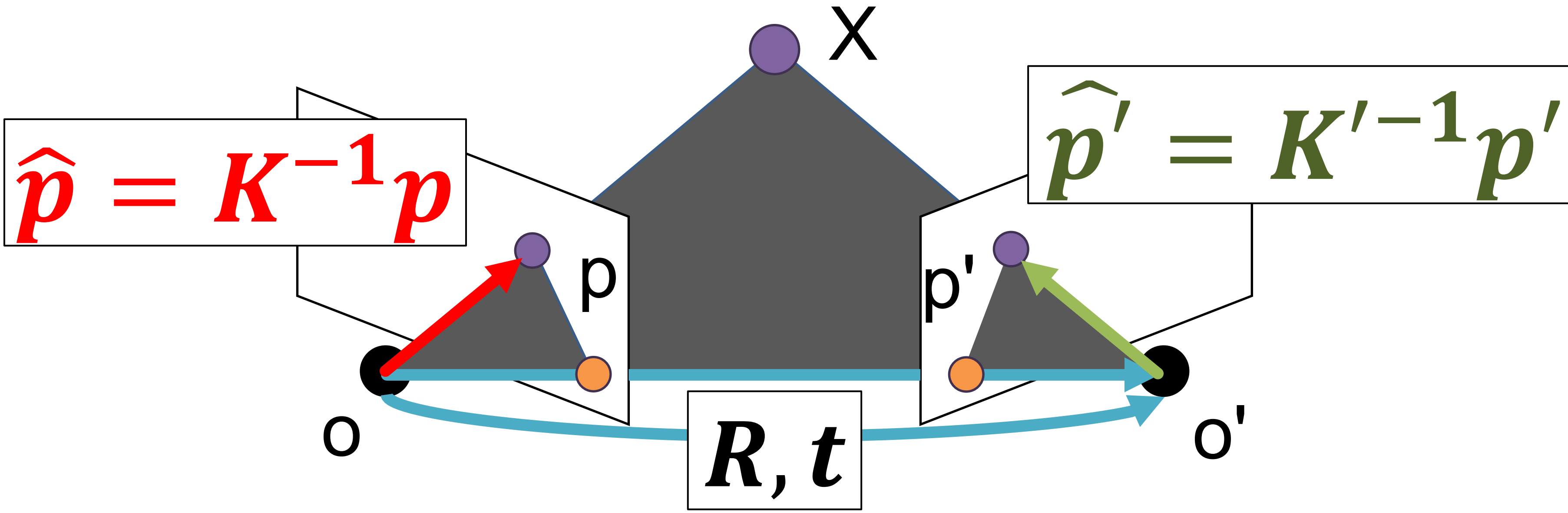
Epipolar Constraint: Calibrated Case



- If we know intrinsic and extrinsic parameters, set coordinate system to first camera
- Projection matrices: $\boldsymbol{M}_1 = \boldsymbol{K}[\boldsymbol{I}, \boldsymbol{0}]$ and $\boldsymbol{M}_2 = \boldsymbol{K}'[\boldsymbol{R}, \boldsymbol{t}]$
- **What are:**

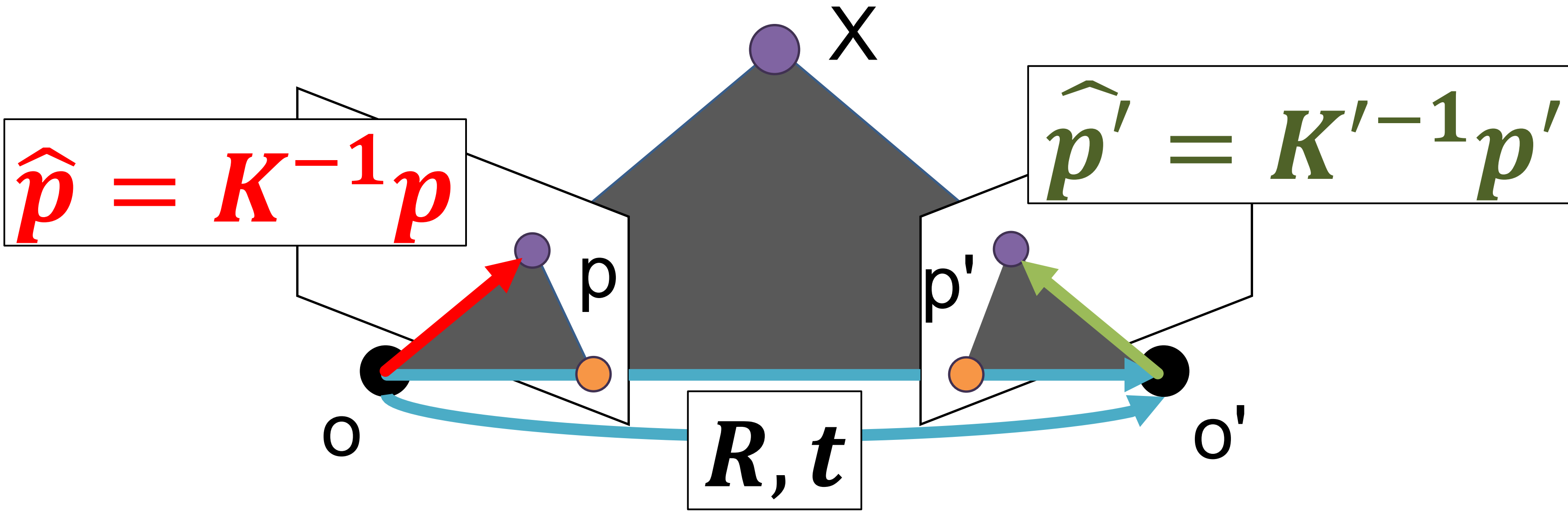
$$\boldsymbol{M}_1\boldsymbol{X} \qquad \boldsymbol{M}_2\boldsymbol{X} \qquad \boldsymbol{K}^{-1}\boldsymbol{p} \qquad \boldsymbol{K}'^{-1}\boldsymbol{p}'$$

Epipolar Constraint: Calibrated Case



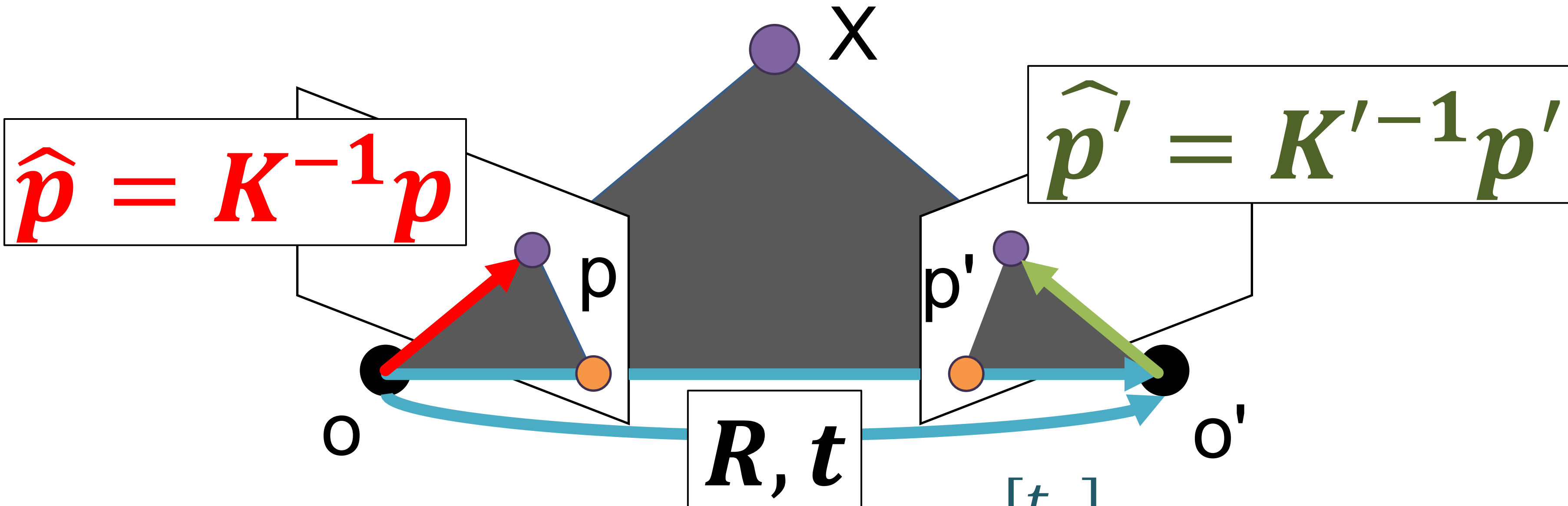
- Given calibration, $\hat{p} = K^{-1}p$ and $\widehat{p'} = K'^{-1}p'$ are "normalized coordinates"
- Note that $\widehat{p'}$ is actually translated and rotated to o'

Epipolar Constraint: Calibrated Case



- The following are all co-planar: $\mathbf{R}\hat{p}$, t, $\hat{p'}$ (can ignore translation for co-planarity here)
- One way to check co-planarity (triple product): $\hat{p'}^T(t \times R\hat{p}) = 0$

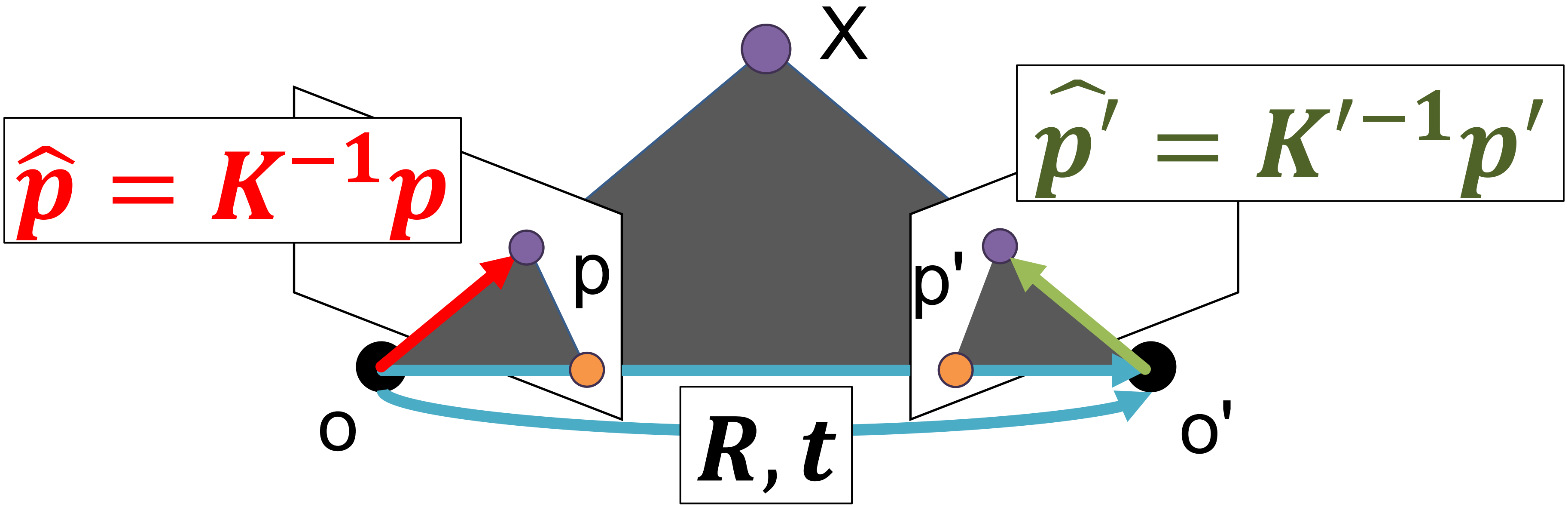
Epipolar Constraint: Calibrated Case



$$\hat{p}'^T(t \times R\hat{p}) = 0 \rightarrow \hat{p}'^T \underbrace{\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}}_{[t_x]} R\hat{p} = 0$$

Want something like $\mathbf{x}^T\mathbf{A}\mathbf{y}=0$. **What's A?**

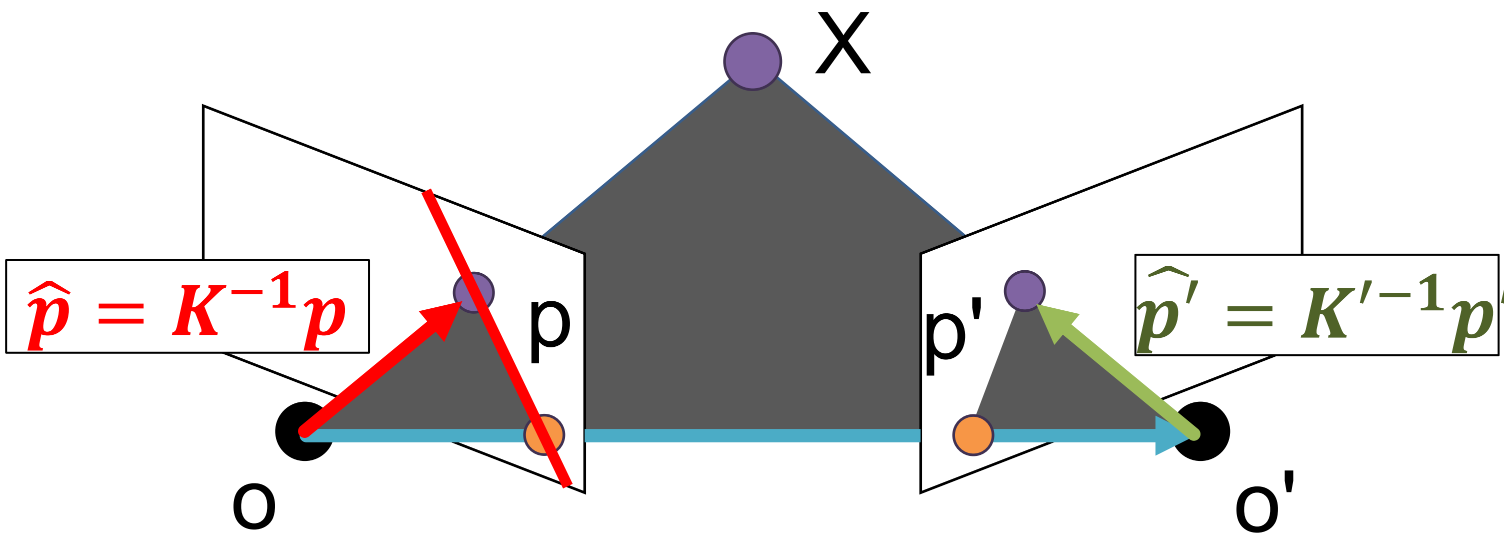
Epipolar Constraint: Calibrated Case



Essential matrix (Longuet-Higgins, 1981): $E = [t_x]R$

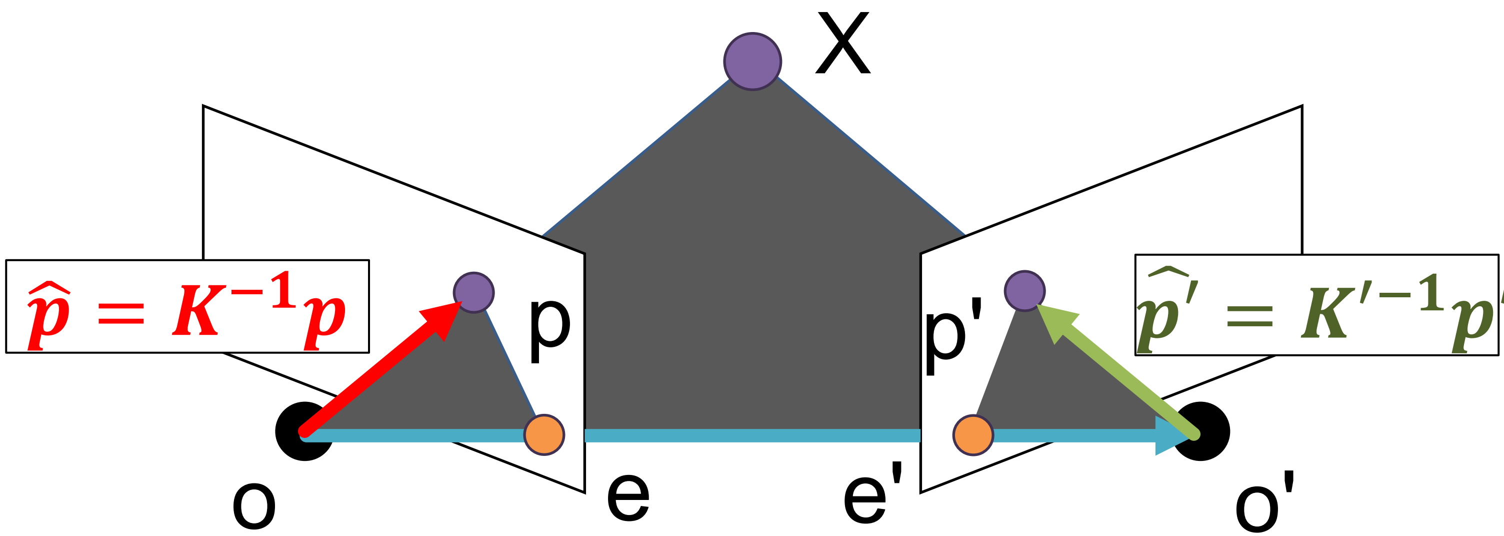
If you have a normalized point $\hat{p}$, its correspondence $\hat{p}'$ **must** satisfy $\hat{p}'^T E \hat{p} = 0$

Essential Essential Matrix Facts



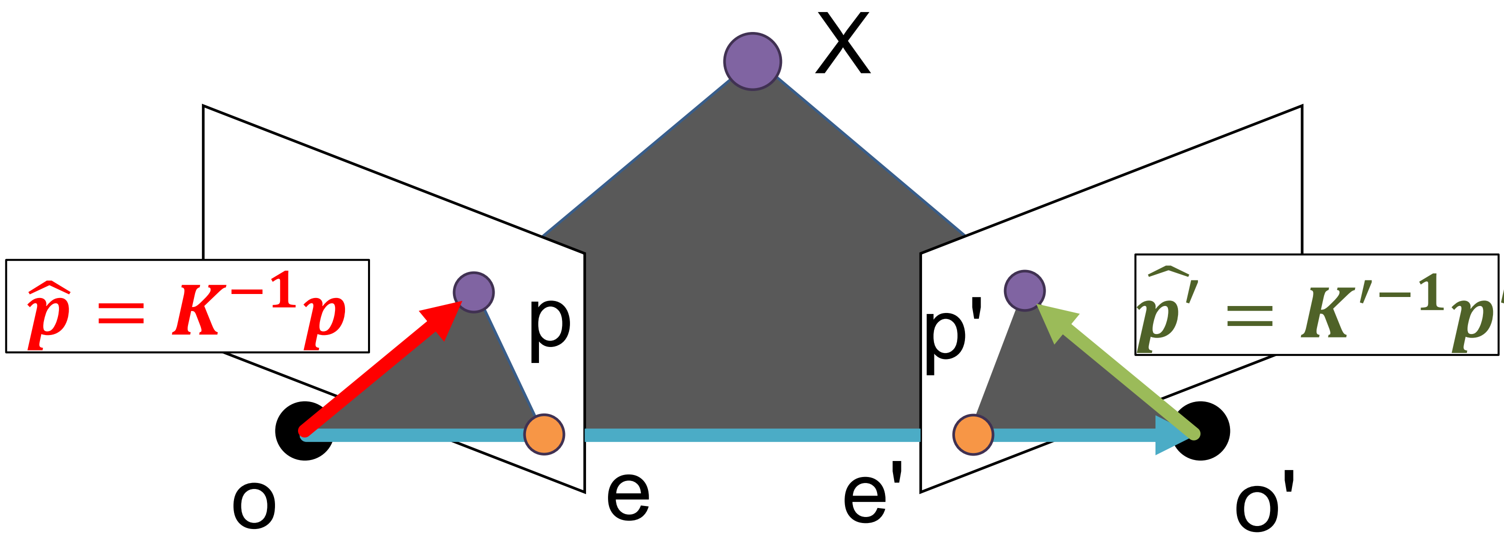
- Suppose we know **E** and $\widehat{\boldsymbol{p}}'^{T}\boldsymbol{E}\widehat{\boldsymbol{p}} = 0$. What is the set $\{\boldsymbol{x}: \widehat{\boldsymbol{p}'}^{T}\boldsymbol{E}\boldsymbol{x} = \boldsymbol{0}\}$?
- $\widehat{\boldsymbol{p}'}^{T}\boldsymbol{E}$ gives equation of the epipolar line (in ax+by+c=0 form) in image for o.
- What's $\boldsymbol{E}^{T}\widehat{\boldsymbol{p}}'$?

Essential Essential Matrix Facts



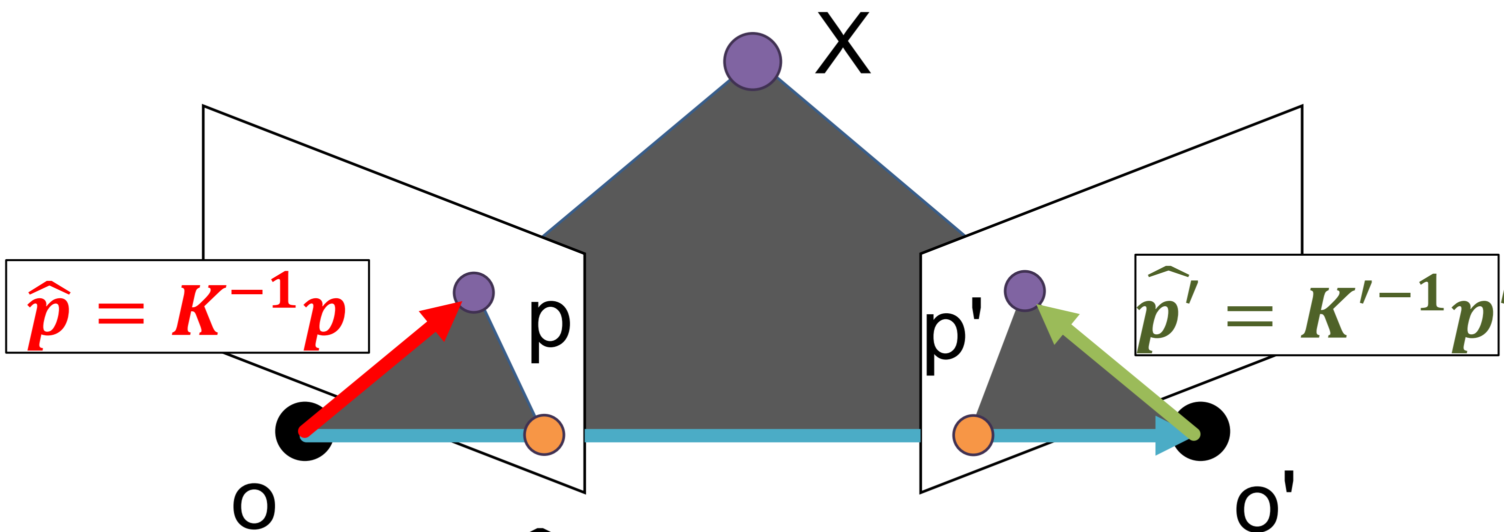
- $\boldsymbol{E\hat{e}} = 0$ and $\boldsymbol{E^T\hat{e}'} = 0$ (epipoles are the nullspace of E – note all epipolar lines pass through epipoles)
- **Degrees of freedom (Recall $\boldsymbol{E = [t_x]R}$)?**
- 5 – 3 (R)+ 3 (t) – 1 due to scale ambiguity
- E is singular (rank 2); it has two non-zero and *identical* singular values

Essential Essential Matrix Facts



- One nice thing: if I estimate E from two images (more on this later), it's uniquely decomposable into R and t up to easy symmetries

What if we don't know K?



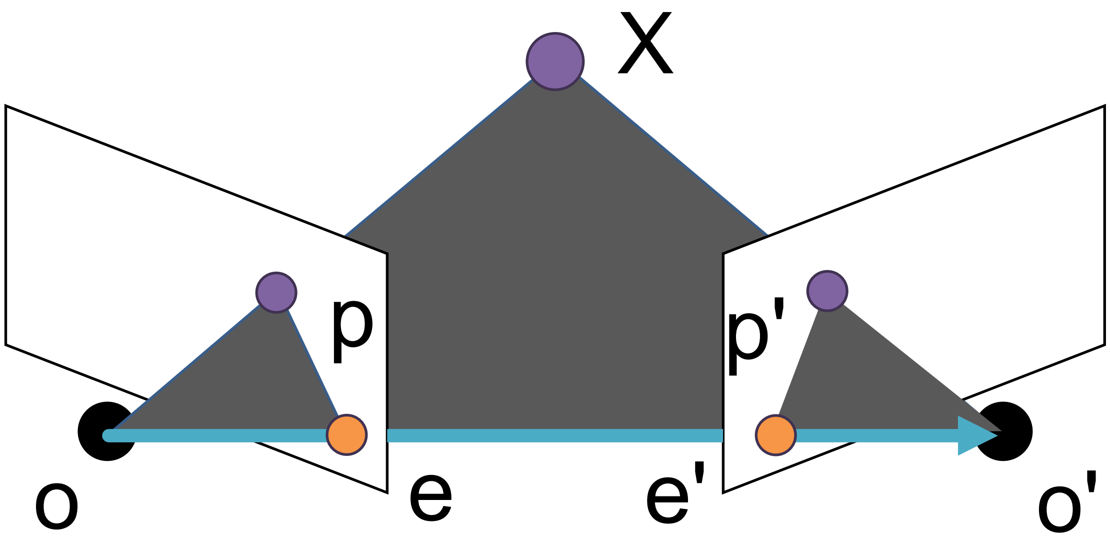
Have: $\hat{p} = K^{-1}p$, $\hat{p}' = K'^{-1}p'$, $\hat{p}'^T E \hat{p} = 0$

$$\left(K'^{-1}p'\right)^T E(K^{-1}p) = \mathbf{0} \rightarrow p'^T K'^{-T} E K^{-1} p = 0$$

Set: $\underbrace{F = K'^{-T} E K^{-1}}$ Then: $p'^T F p = 0$

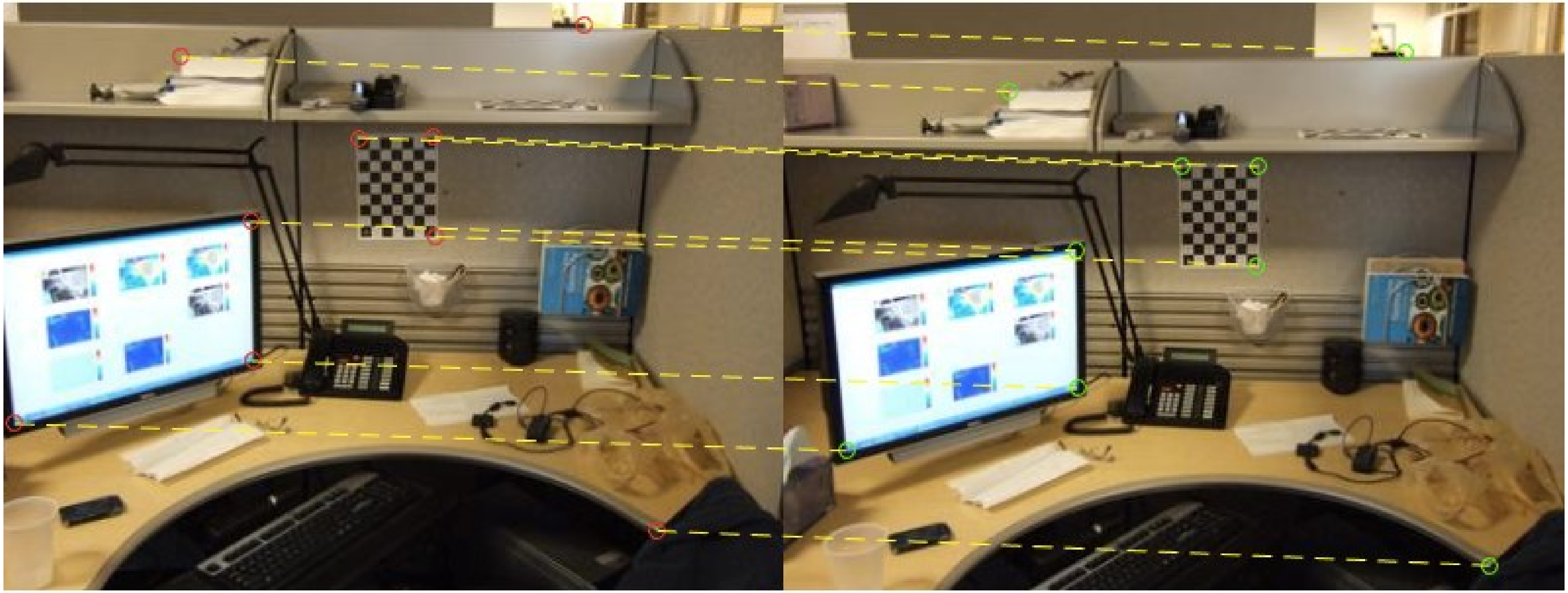
Fundamental Matrix (Faugeras and Luong, 1992)

Fundamental Matrix Fundamentals



- $\boldsymbol{Fp}, \boldsymbol{F^T p'}$ are epipolar lines for p', p
- $\boldsymbol{Fe} = 0, \boldsymbol{F^T e'} = 0$
- F is singular (rank 2)
- F has seven degrees of freedom
- F definitely does not have unique decomposition

Estimating the fundamental matrix



Slide Credit: S. Lazebnik

Estimating the fundamental matrix

- F has 7 degrees of freedom so it’s in principle possible to fit F with seven correspondences, but it’s a slightly more complex and typically not taught in regular vision classes

Estimating the fundamental matrix

Given correspondences $\boldsymbol{p} = [u, v, 1]$ and $\boldsymbol{p}' = [u', v', 1]$ (e.g., via SIFT) we know: $\boldsymbol{p}'^{T}\boldsymbol{F}\boldsymbol{p} = 0$

$$[u', v', 1]\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

$$\begin{matrix}[u'u, u'v, u', v'u, v'v, v', u, v, 1] \cdot \\ [f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}]\end{matrix} = 0$$

How do we solve for f?

How many correspondences do we need?

Leads to the **eight point algorithm**

Eight Point Algorithm

Each point gives an equation:

$$[u'u, u'v, u', v'u, v'v, v', u, v, 1] \cdot [f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}] = 0$$

Stack equations to yield **U**:

$$\boldsymbol{U} = \begin{bmatrix} & & & & \vdots & & & & \\ u_i'u_i & u_i'v_i & u_i' & v_i'u_i & v_i'v_i & v_i' & u_i & v_i & 1 \\ & & & & \vdots & & & & \end{bmatrix}$$

Usual eigenvalue stuff to find **f** (**F** unrolled):

$$\arg \min_{\|\boldsymbol{f}\|=1} \|\boldsymbol{U}\boldsymbol{f}\|_2^2$$

→ Eigenvector of $\boldsymbol{U}^T\boldsymbol{U}$ with smallest eigenvalue

Eight Point Algorithm – Difficulty 1

If we estimate F, we get some 3x3 matrix F.
We know F needs to be singular/rank 2. How do we force F to be singular?

$$U\Sigma V^T = F_{init}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \rightarrow \Sigma' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F = U\Sigma' V^T$$

Open it up with SVD, mess with singular values, put it back together.

See Eckart–Young–Mirsky theorem if you're interested

Eight Point Algorithm – Difficulty 1

Estimated F
(Wrong)

Estimated+SVD'd F
(Correct)

Slide Credit: S. Lazebnik

Eight Point Algorithm – Difficulty 2

$$[u'u, u'v, u', v'u, v'v, v', u, v, 1] \cdot [f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}] = 0$$

Recall: u,u' are in pixels. Suppose image is 1Kx1K

How big might uu' be? How big might u' be?

Each row looks like:

$$\boldsymbol{U} = \begin{bmatrix} & & & & \vdots & & & & \\ 10^6 & 10^6 & 10^3 & 10^6 & 10^6 & 10^3 & 10^3 & 10^3 & 1 \\ & & & & \vdots & & & & \end{bmatrix}$$

Then: $\boldsymbol{U}^T\boldsymbol{U}_{1,1}$ is ~10^{12}, $\boldsymbol{U}^T\boldsymbol{U}_{2,9}$ is ~10^3

Eight Point Algorithm – Difficulty 2

Numbers of varying magnitude → instability

Remember: a floating point number (float/double) isn't a "real" number: for sign, coefficient, exponent integers

$(-1)^{sign} * \text{coefficient} * 2^{exponent}$

Exercise to see how this screws up: add up Gaussian noise (mean=100, std=10), divide by number you added up

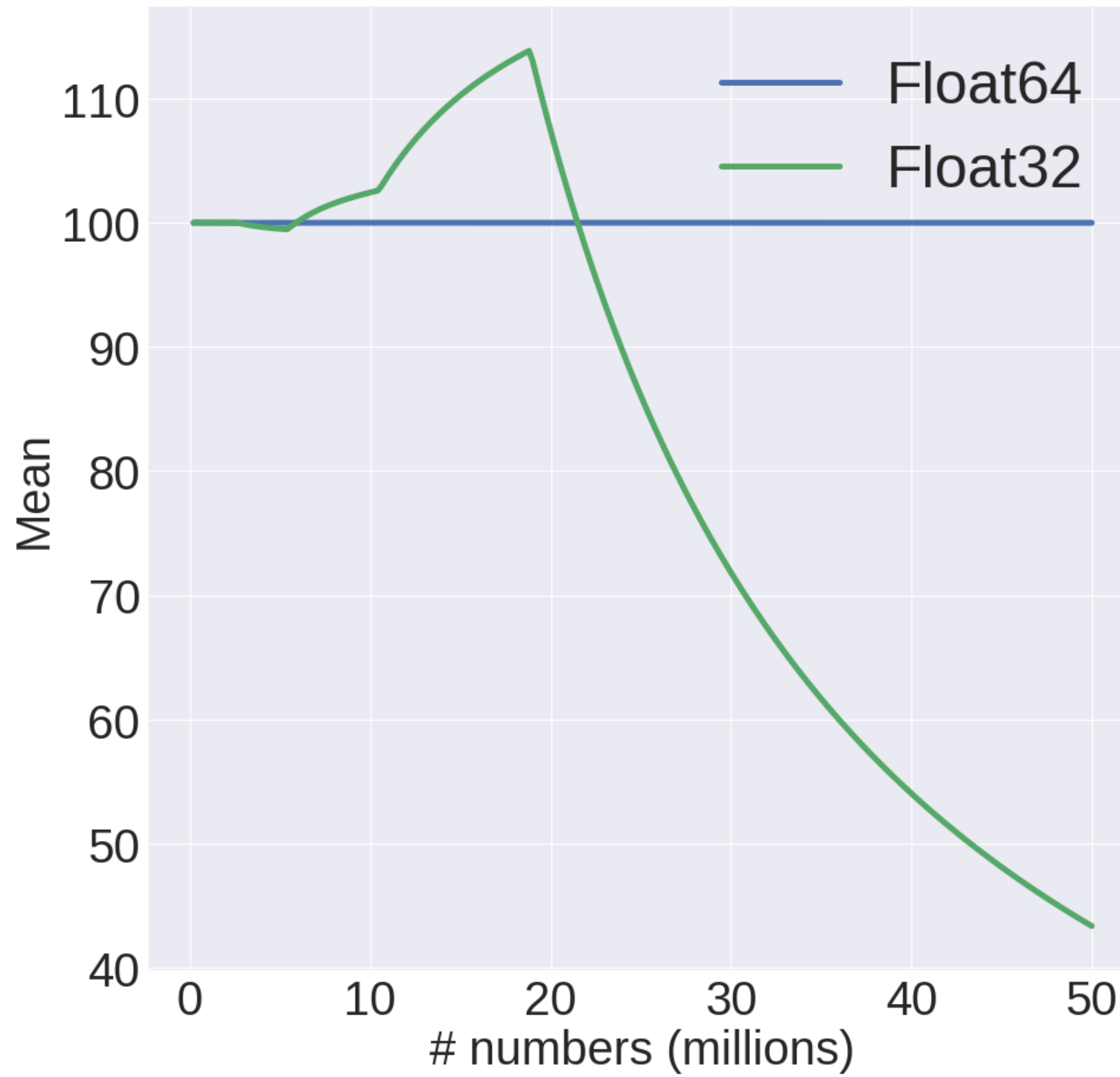
Remember Numerical Instability?

Code:

```
x += N(100,10)
i += 1
mean = x/I
```

Only change is the # of bits in accumulator x

Note: 50M is 50 1Kx1K images



Solution: Normalized 8-point

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute $\boldsymbol{F}$ from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of $\boldsymbol{F}$ and throw out the smallest singular value)
- Transform fundamental matrix back to original units: if $\boldsymbol{T}$ and $\boldsymbol{T}'$ are the normalizing transformations in the two images, than the fundamental matrix in original coordinates is $\boldsymbol{T}'^{T} \boldsymbol{F} \boldsymbol{T}$

R. Hartley
In defense of the eight-point algorithm TPAMI 1997

Slide Credit:
S. Lazebnik

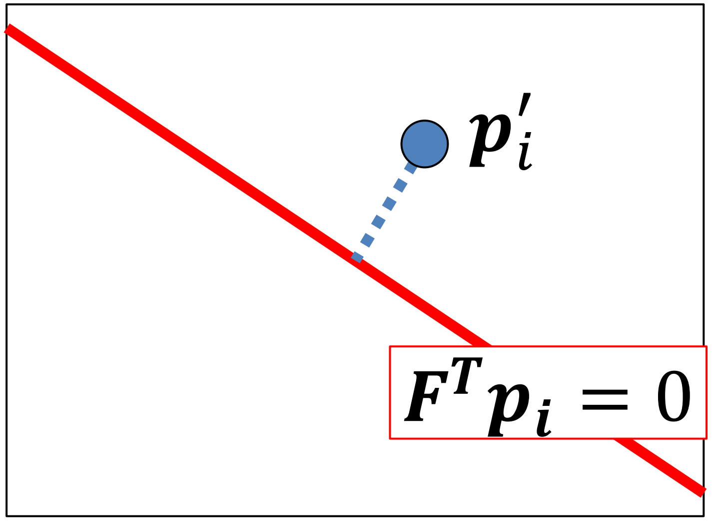
Last Trick

Minimizing via U^TU minimizes sum of squared *algebraic* distances between points $\boldsymbol{p}_i$ and epipolar lines $\boldsymbol{Fp'}_i$ (or points $\boldsymbol{p'}_i$ and epipolar lines $\boldsymbol{F}^T\boldsymbol{p}_i$):

$$\sum_i \left(\boldsymbol{p}'^T_i \boldsymbol{F} \boldsymbol{p}_i\right)^2$$

May want to minimize *geometric* distance:

$$\sum_i d(\boldsymbol{p}'_i, \boldsymbol{F}\boldsymbol{p}_i)^2 + d\left(\boldsymbol{p}_i, \boldsymbol{F}^T\boldsymbol{p}'_i\right)^2$$



Slide Credit: S. Lazebnik

Comparison

	8-point	Normalized 8-point	Nonlinear least squares
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel

Slide Credit: S. Lazebnik

The Fundamental Matrix Song

http://danielwedge.com/fmatrix/

From Epipolar Geometry to Calibration

- Estimating the fundamental matrix is known as “weak calibration”
- If we know the calibration matrices of the two cameras, we can estimate the essential matrix: $\boldsymbol{E} = \boldsymbol{K}'^{T}\boldsymbol{F}\boldsymbol{K}$
- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters
- Alternatively, if the calibration matrices are known, the five-point algorithm can be used to estimate relative camera pose

Slide Credit: S. Lazebnik