Backpropagation and Neural Nets

EECS 442 – David Fouhey

Winter 2023, University of Michigan

http://web.eecs.umich.edu/~fouhey/teaching/EECS442_W23/

So Far: Linear Models

$$L(\boldsymbol{w}) = \lambda \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} \left(y_i - \boldsymbol{w}^T \boldsymbol{x}_i)\right)^2$$

- Example: find **w** minimizing squared error over data
  - Each datapoint represented by some vector **x**
    - Can find optimal **w** with ~10 line derivation

Last Class

$$L(\boldsymbol{w}) = \lambda \|\boldsymbol{w}\|_2^2 + \sum_{i=1}^{n} L(y_i, f(\boldsymbol{x}; \boldsymbol{x}))$$

- What about an arbitrary loss function L?
- What about an arbitrary parametric function f?
- Solution: take the gradient, do gradient descent

$$\boldsymbol{w}_{i+1} = \boldsymbol{w}_{\boldsymbol{i}} - \alpha \nabla_w \underbrace{L(f(\boldsymbol{w}_i))}_{}$$

What if L(f(w)) is complicated?

Today!

Taking the Gradient – Review

$$f(x) = (-x+3)^2$$

$$f = q^2 \qquad q = r + 3 \qquad r = -x$$

$$\frac{\partial f}{\partial q} = 2q \qquad \frac{\partial q}{\partial r} = 1 \qquad \frac{\partial r}{\partial x} = -1$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial r}\frac{\partial r}{\partial x}} = 2q * 1 * -1$$

Chain rule

$$= -2(-x+3)$$

$$= 2x - 6$$

Supplemental Reading

- Lectures can only introduce you to a topic
- You will solidify your knowledge by **doing**
- I highly recommend working through everything in the Stanford CS213N resources
  - http://cs231n.github.io/optimization-2/
- These slides follow the general examples with a few modifications. The primary difference is that I define local variables n, m per-block.

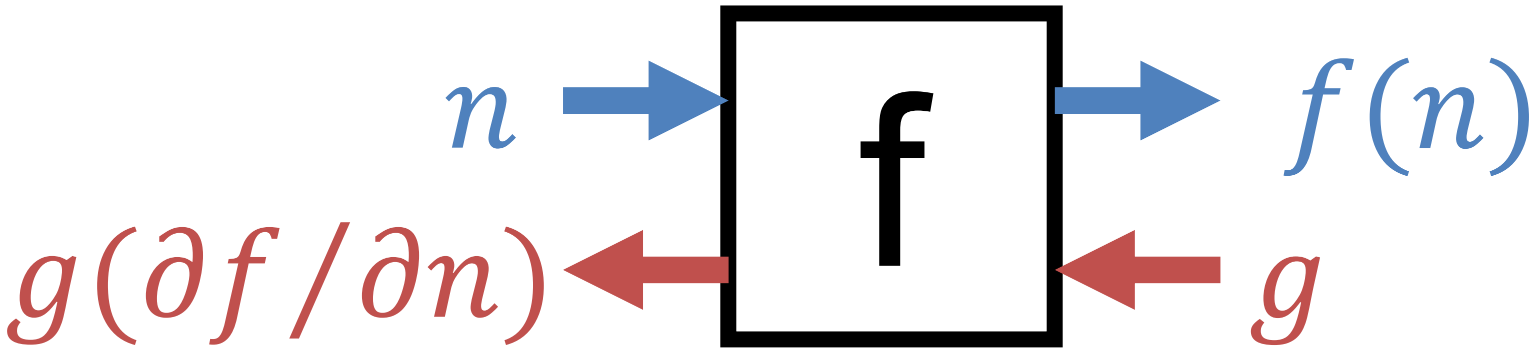
Let's Do This Another Way

Suppose we have a box representing a function f.

This box does two things:

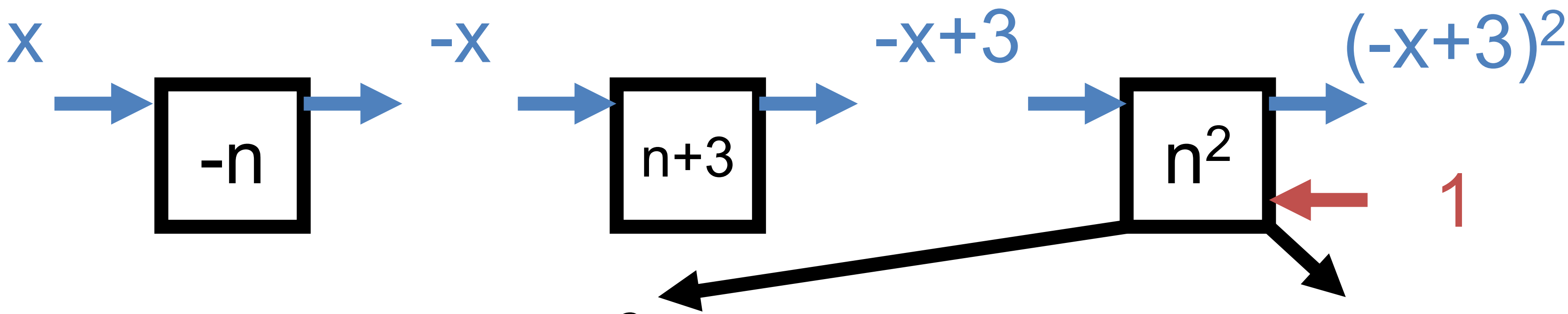
Forward: Given forward input **n**, compute **f(n)**

Backwards: Given backwards input **g**, return **g*df/dn**



Let’s Do This Another Way

$$f(x) = (-x+3)^2$$



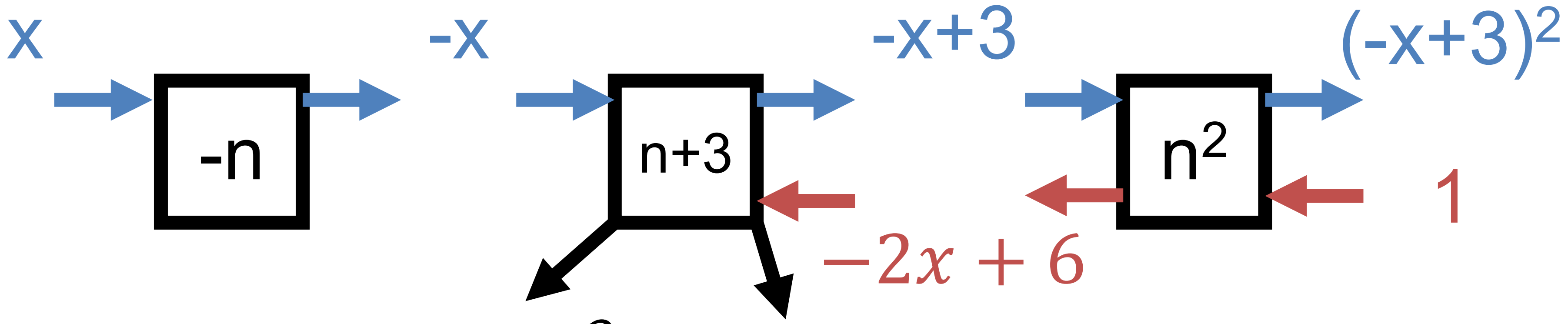
$$\frac{\partial}{\partial n} n^2 = 2n \;= 2(-x+3)$$

$$= -2x + 6$$

$$\frac{\partial}{\partial n} * 1 = (-2x+6) \; * 1$$

Let's Do This Another Way

$$f(x) = (-x + 3)^2$$

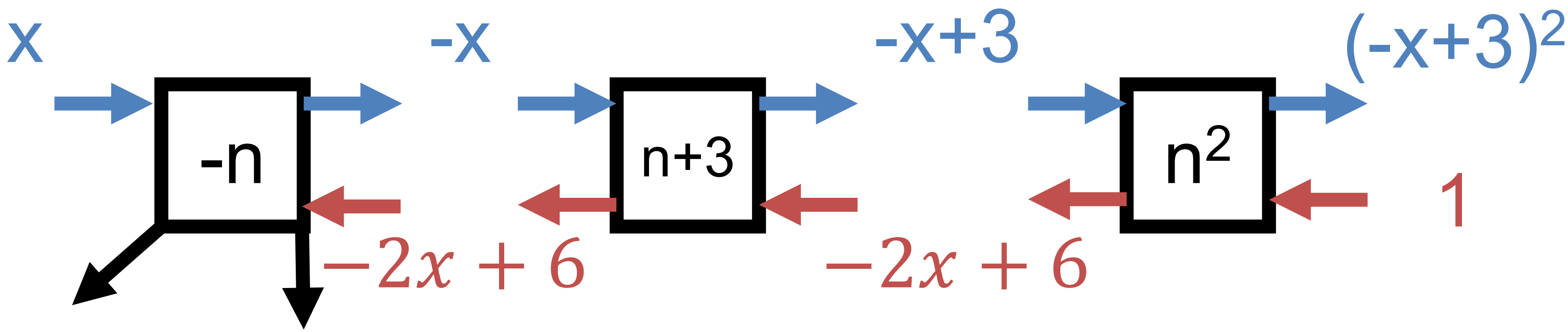


$$\frac{\partial}{\partial n} = 1$$

$$1 * (-2x + 6)$$

Let’s Do This Another Way

$$f(x) = (-x+3)^2$$



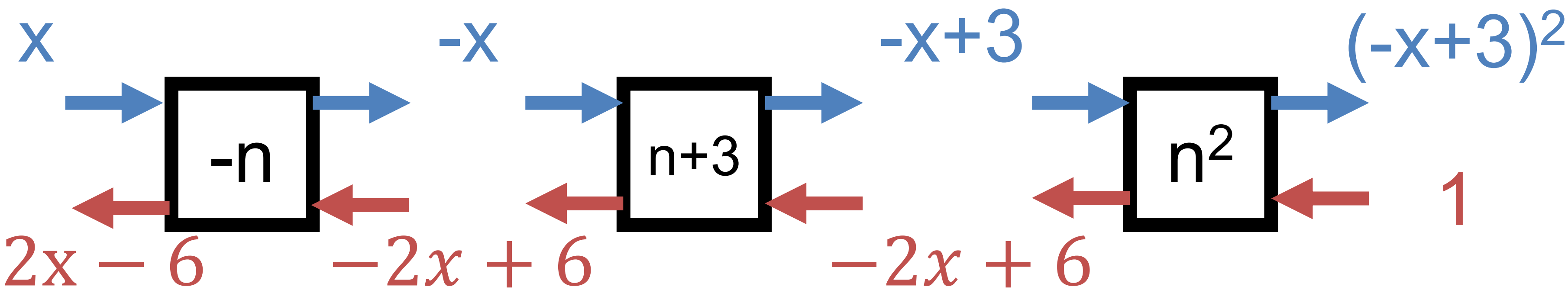
$$\frac{\partial}{\partial n} = -1$$

$$-1 \; * \; (-2\mathrm{x} + 6)$$

$$2\mathrm{x} - 6$$

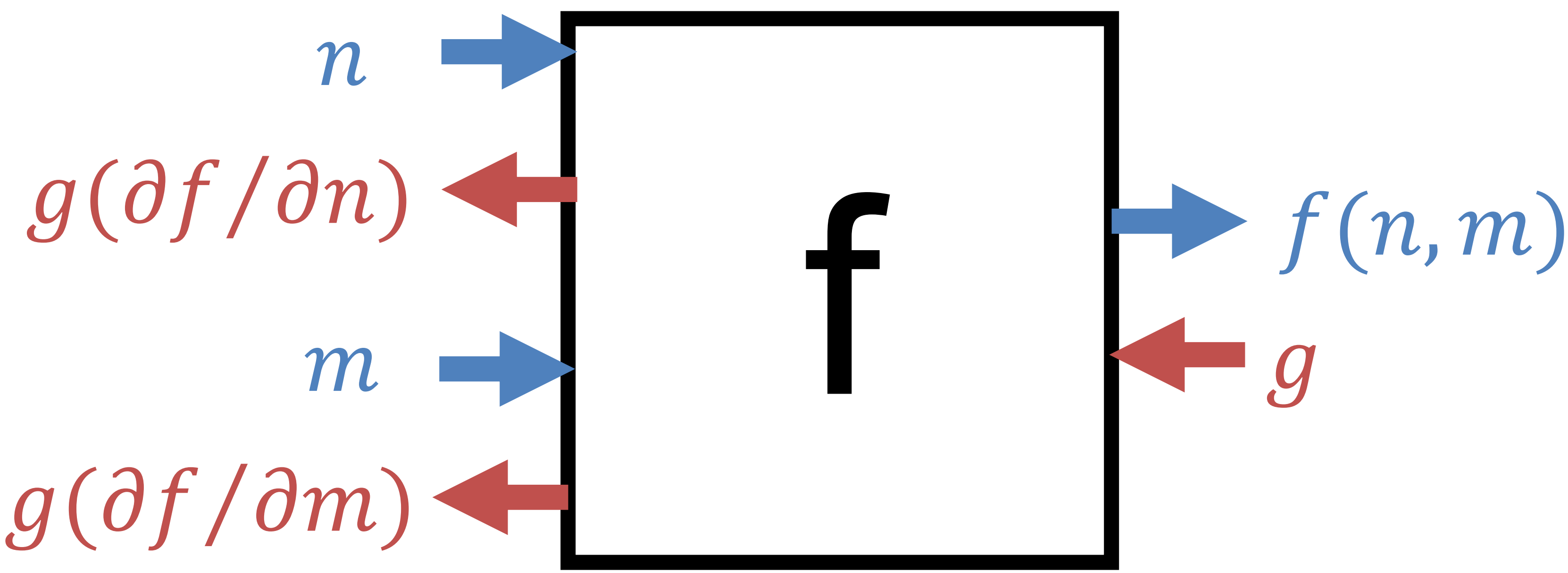
Let's Do This Another Way

$$f(x) = (-x+3)^2$$

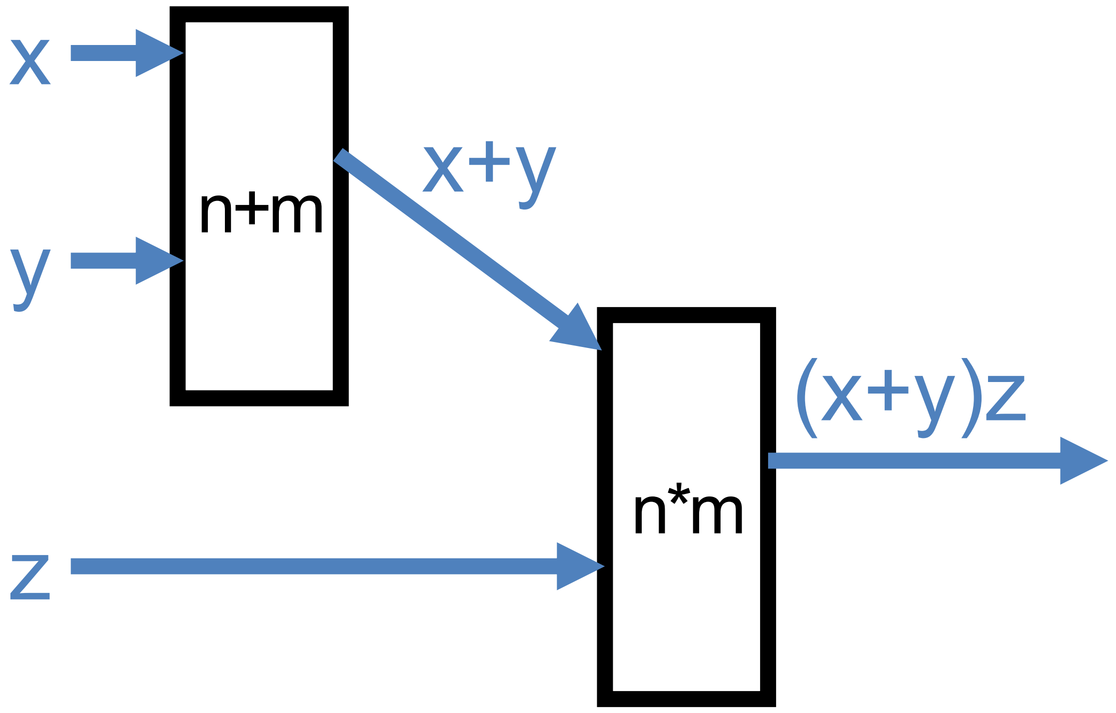


Two Inputs

Given two inputs, just have two input/output wires
Forward: the same
Backward: the same – send gradients with respect to each variable



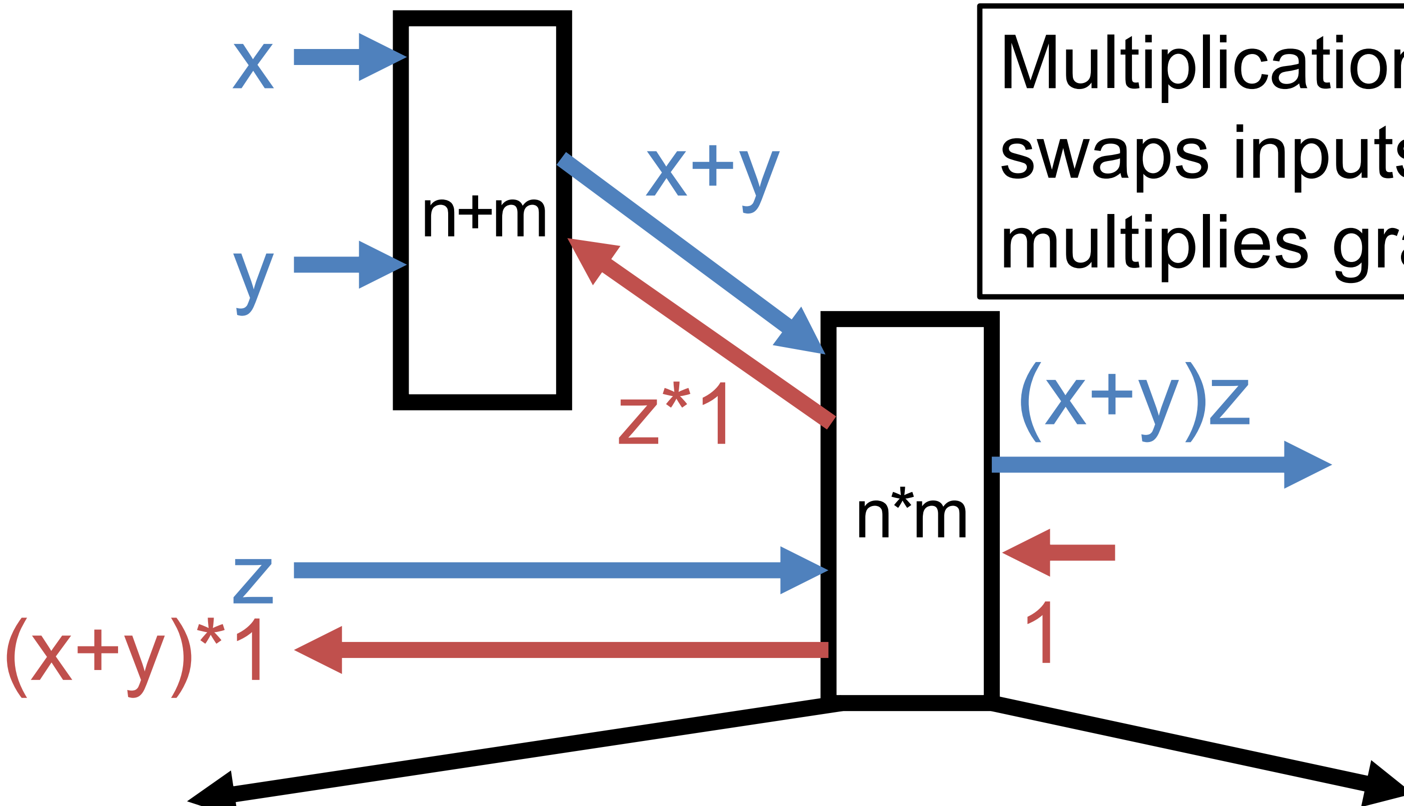
$$f(x,y,z) = (x+y)z$$



Example Credit: Karpathy and Fei-Fei

$$f(x,y,z) = (x+y)z$$

Multiplication swaps inputs, multiplies gradient

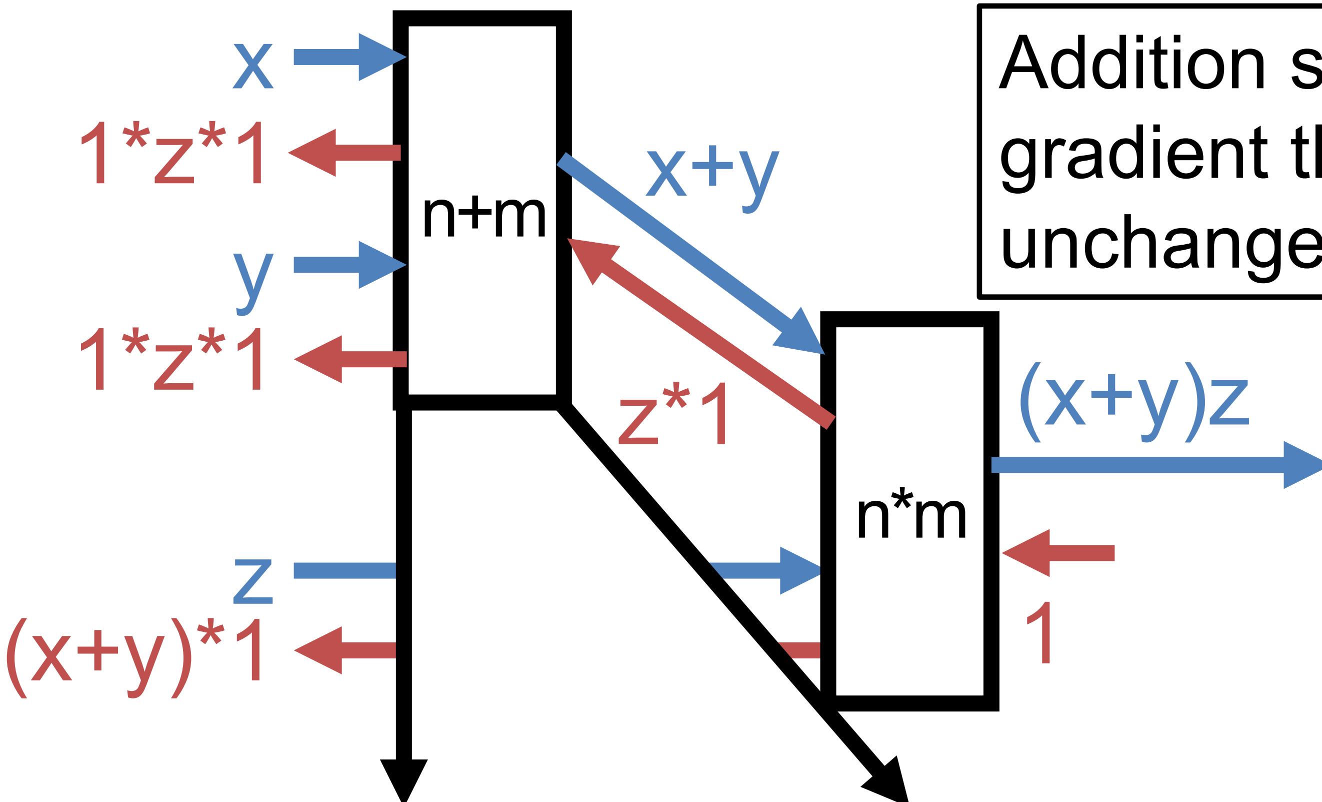


$$\frac{\partial}{\partial n} nm = m$$
$$\rightarrow z * 1$$

$$\frac{\partial}{\partial m} nm = n$$
$$\rightarrow (x+y) * 1$$

Example Credit: Karpathy and Fei-Fei

$$f(x,y,z) = (x+y)z$$



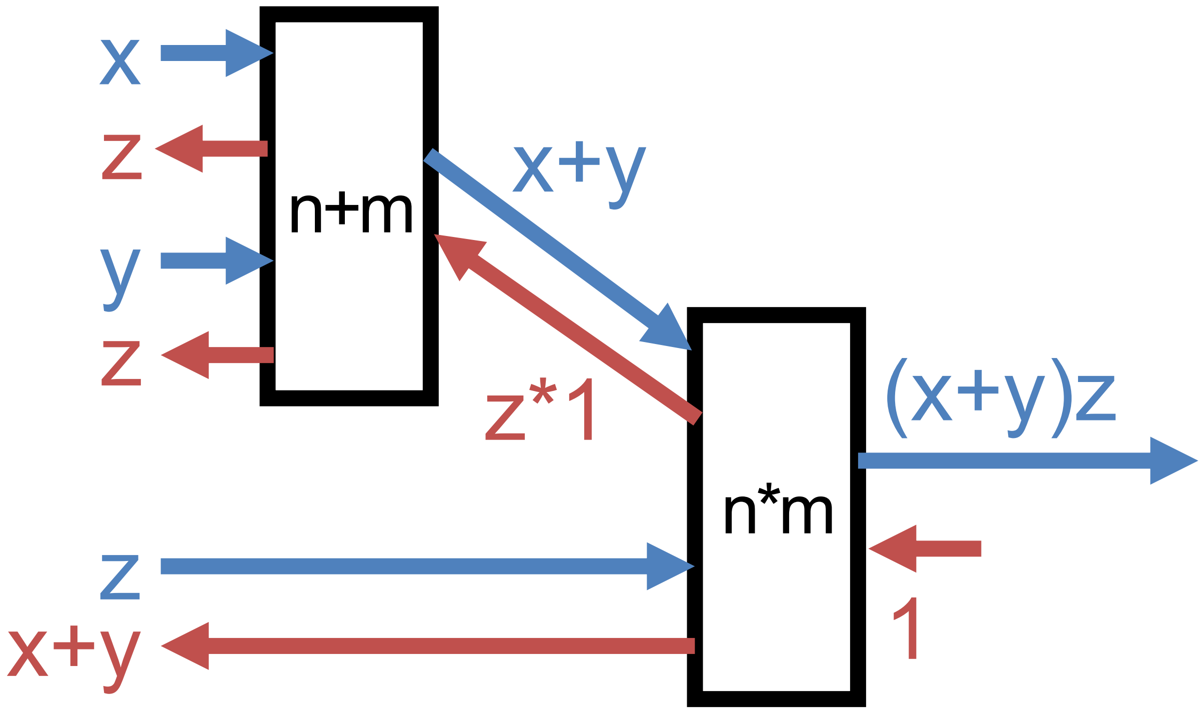
Addition sends gradient through unchanged

$$\frac{\partial}{\partial n} n + m = 1 \qquad \frac{\partial}{\partial m} n + m = 1$$

$$\rightarrow 1 * z * 1 \qquad \rightarrow 1 * z * 1$$

Example Credit: Karpathy and Fei-Fei

f(x,y,z) = (x+y)z

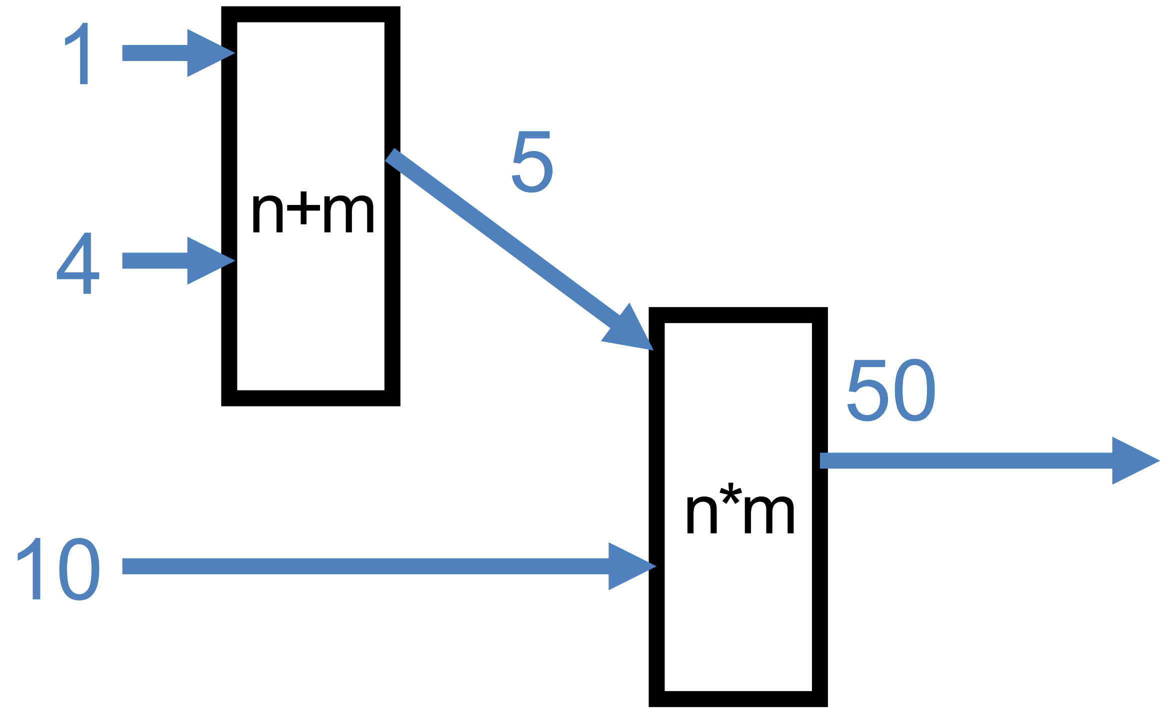


$$\frac{\partial (x+y)z}{\partial x} = z \quad \frac{\partial (x+y)z}{\partial y} = z \quad \frac{\partial (x+y)z}{\partial z} = (x+y)$$

Example Credit: Karpathy and Fei-Fei

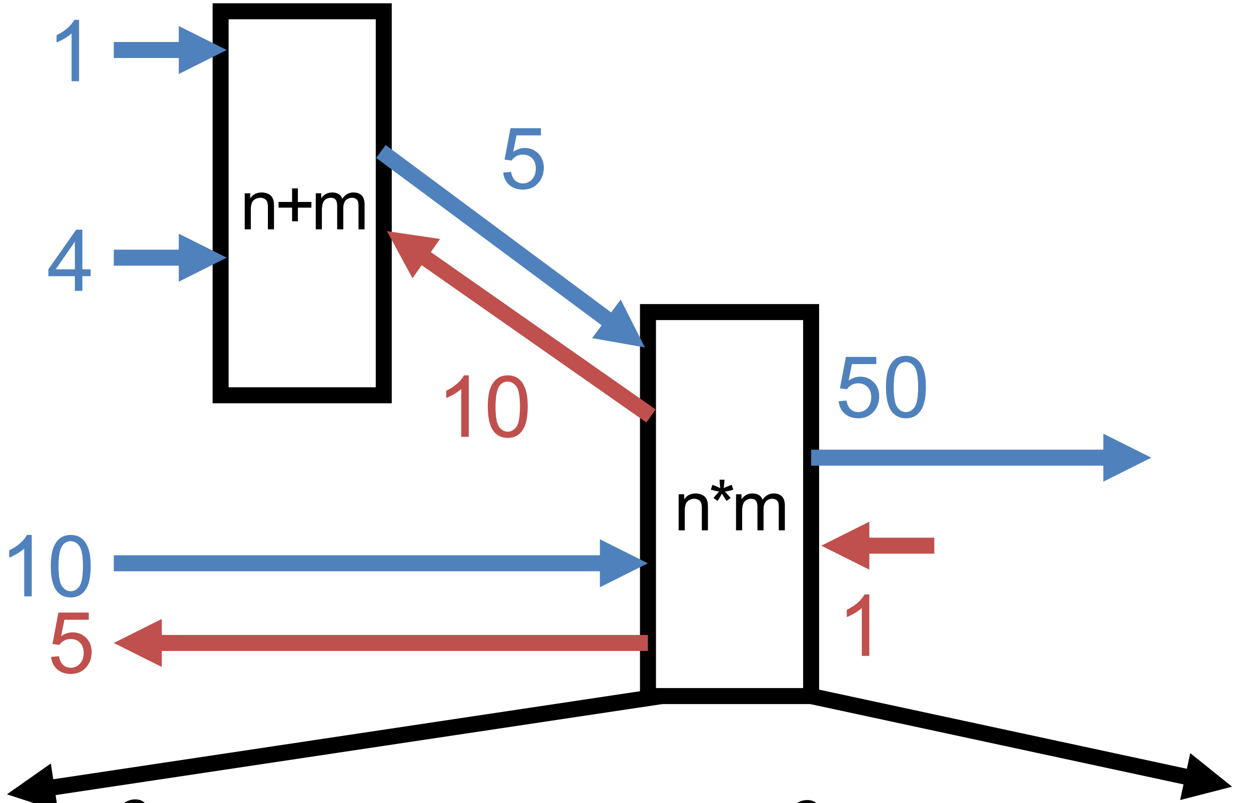
Once More, With Numbers!

$f(x,y,z) = (x+y)z$



Example Credit: Karpathy and Fei-Fei

$$f(x,y,z) = (x+y)z$$

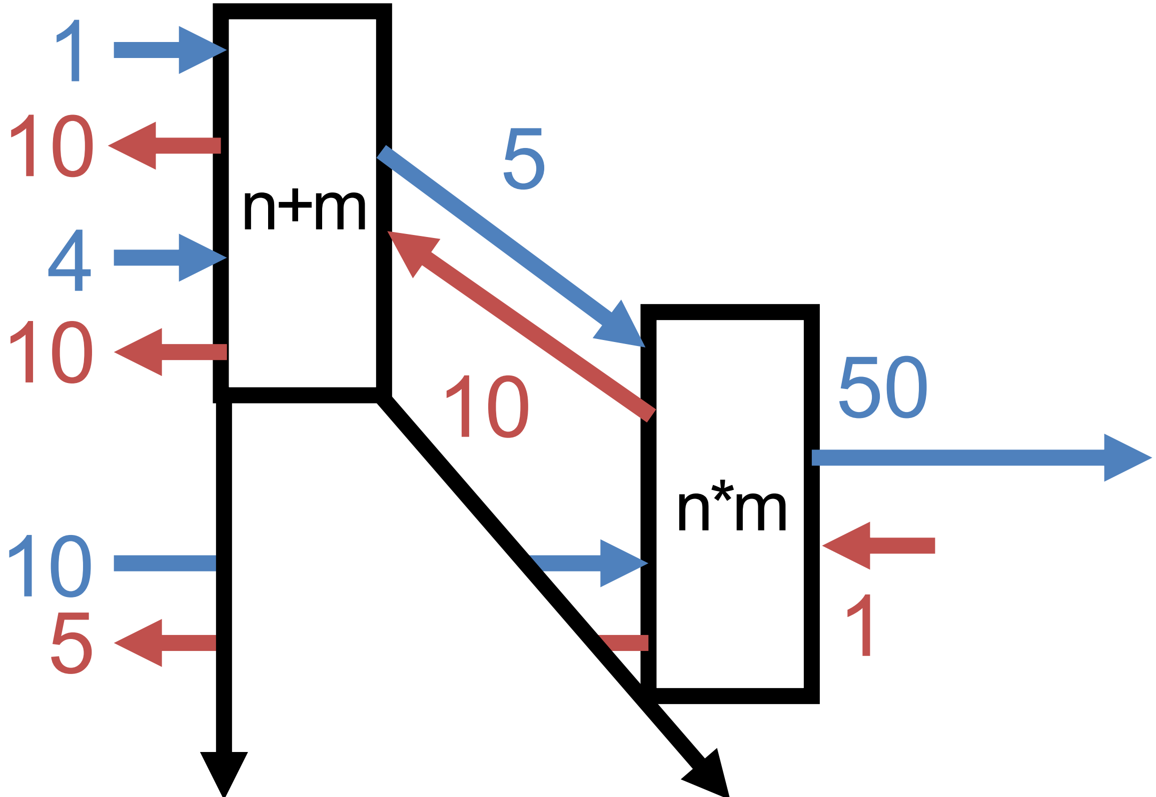


$$\frac{\partial}{\partial n} nm = m \quad \rightarrow 10 * 1$$

$$\frac{\partial}{\partial m} nm = n \quad \rightarrow 5 * 1$$

Example Credit: Karpathy and Fei-Fei

$$f(x,y,z) = (x+y)z$$



$$\frac{\partial}{\partial n} n + m = 1 \qquad \frac{\partial}{\partial m} n + m = 1$$

$$\rightarrow 1 * 10 * 1 \qquad \rightarrow 1 * 10 * 1$$

Example Credit: Karpathy and Fei-Fei

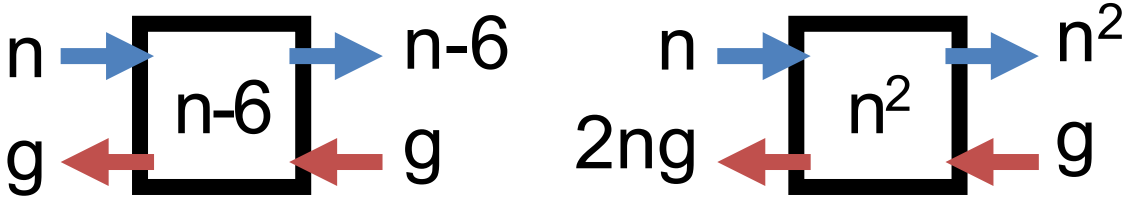
Think You've Got It?

$$L(x) = (w - 6)^2$$

- We want to fit a model w that just will equal 6.
- World's most basic linear model / neural net: no inputs, just constant output.

I'll Need a Few Volunteers

$$L(x) = (w - 6)^2$$



Job #1 (n-6):

Forward:
Compute n-6

Backwards:
Multiply by 1

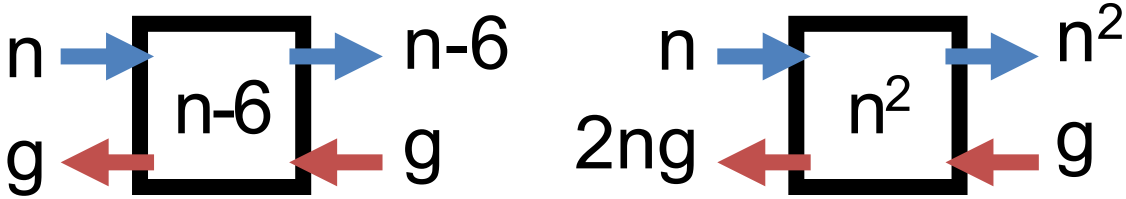
Job #2 (n^2):

Forward:
Compute n^2

Backwards:
Multiply by 2n

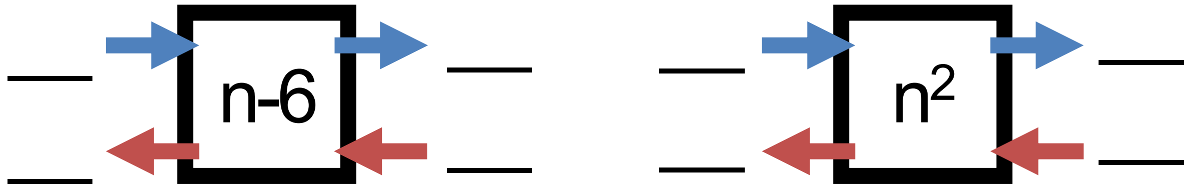
Job #3:

Backwards:
Give me a 1



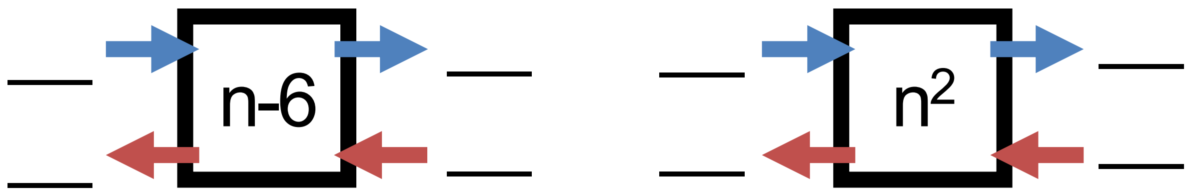
$$L(x) = (w-6)^2$$

w0: ____



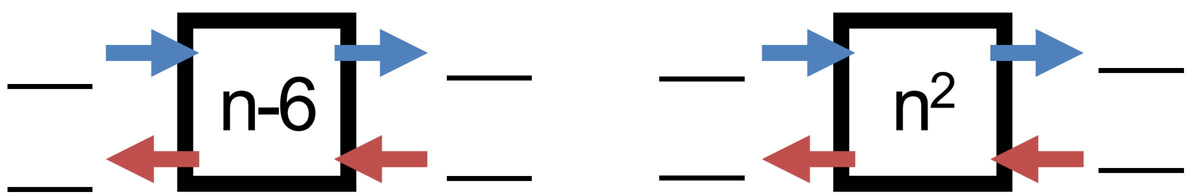
w1 = w0 - (1/4) ____ =

w1: ____



w2 = w1 - (1/4) ____ =

w2: ____



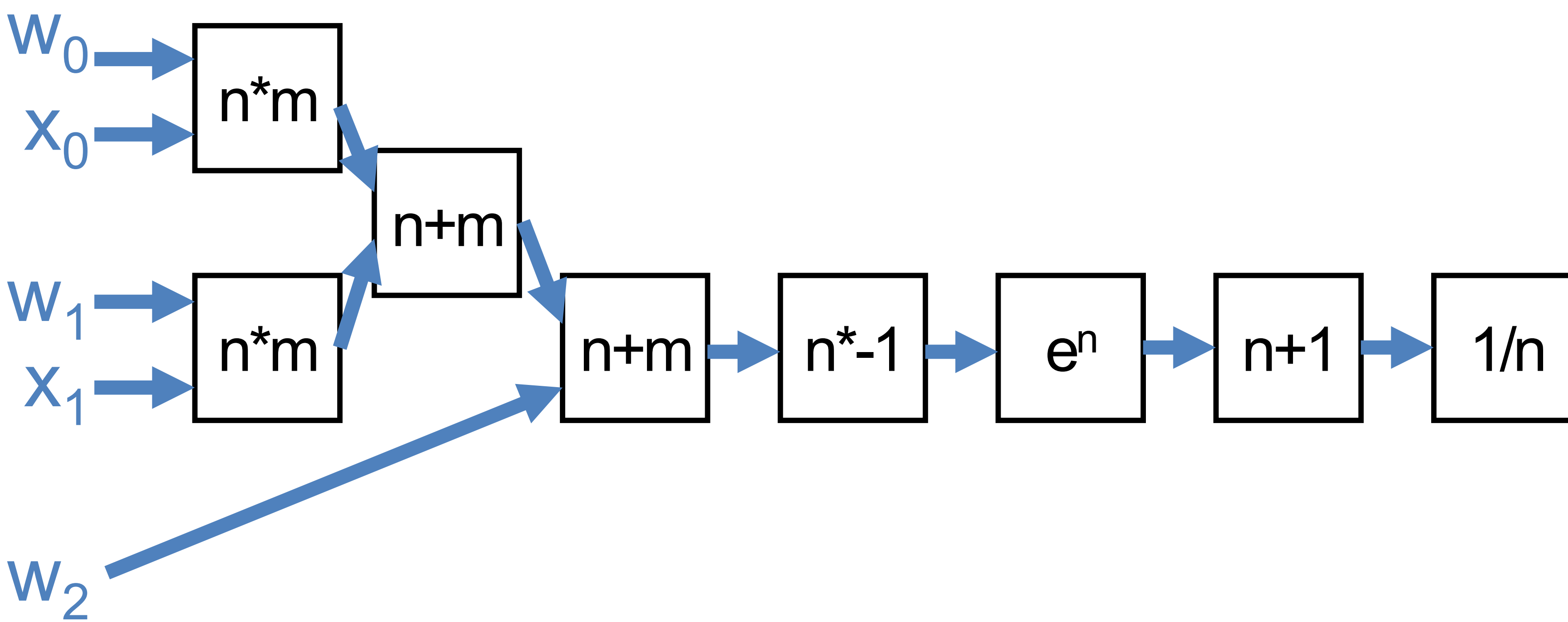
w3 = w2 - (1/4) ____ =

Preemptively

- The diagrams look complex but that’s since we’re covering the details together

Something More Complex

$$f(\boldsymbol{w}, \boldsymbol{x}) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w_2)}}$$



Example Credit: Karpathy and Fei-Fei

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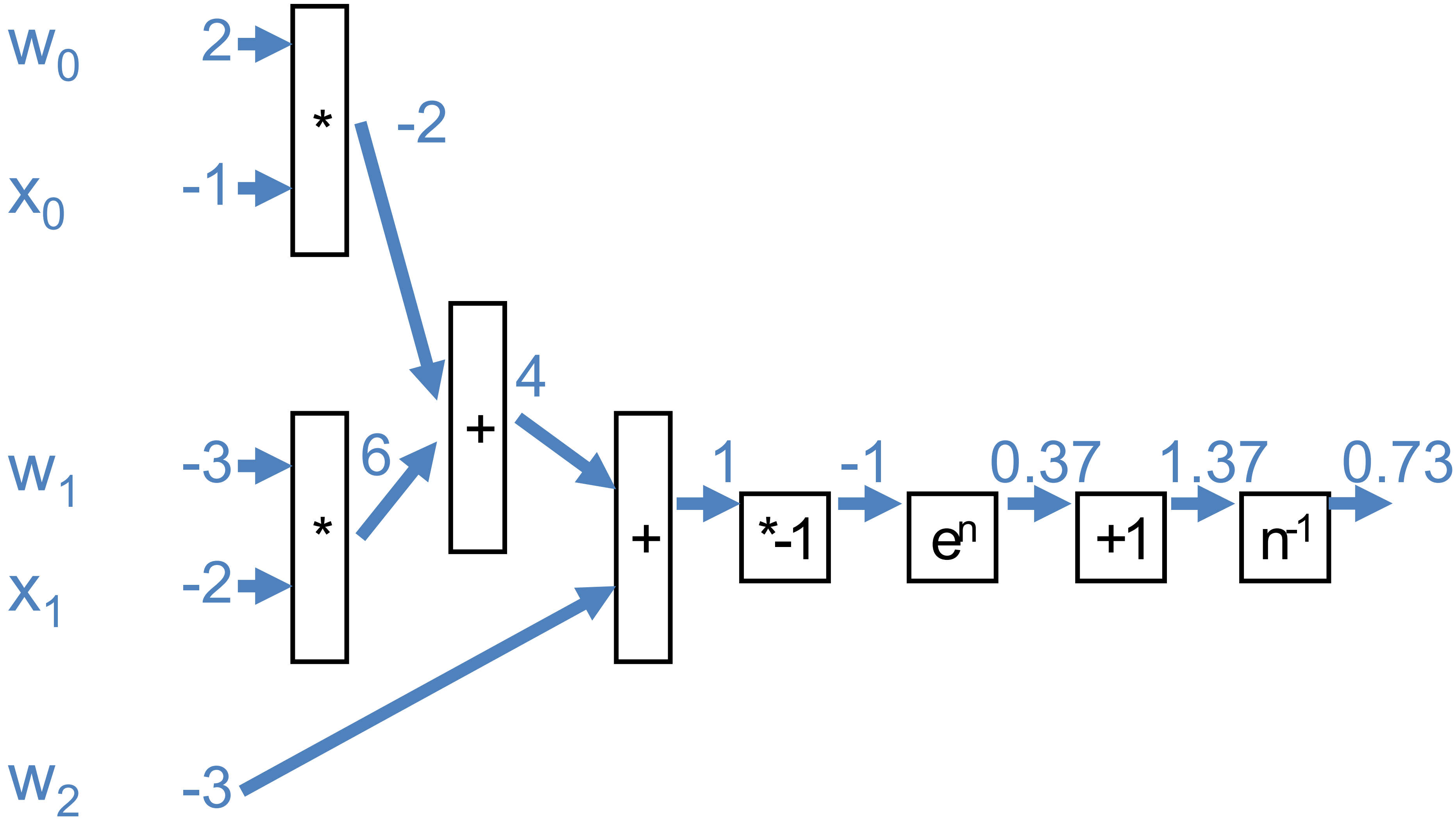


$$f(\boldsymbol{w}, \boldsymbol{x}) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w_2)}}$$

0 $\frac{\partial}{\partial n} m + n = 1$

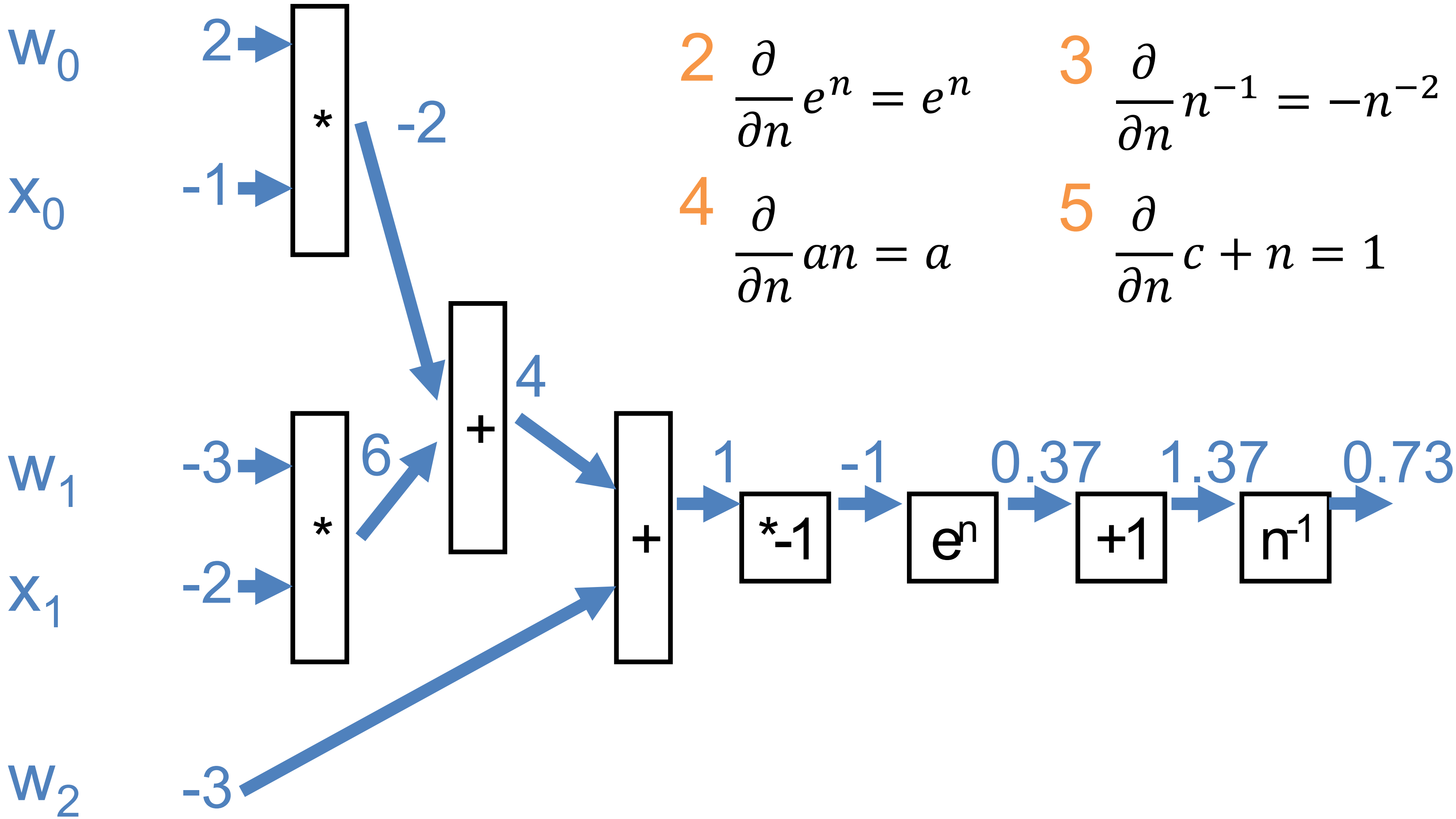
1 $\frac{\partial}{\partial n} mn = m$

2 $\frac{\partial}{\partial n} e^n = e^n$

3 $\frac{\partial}{\partial n} n^{-1} = -n^{-2}$

4 $\frac{\partial}{\partial n} an = a$

5 $\frac{\partial}{\partial n} c + n = 1$



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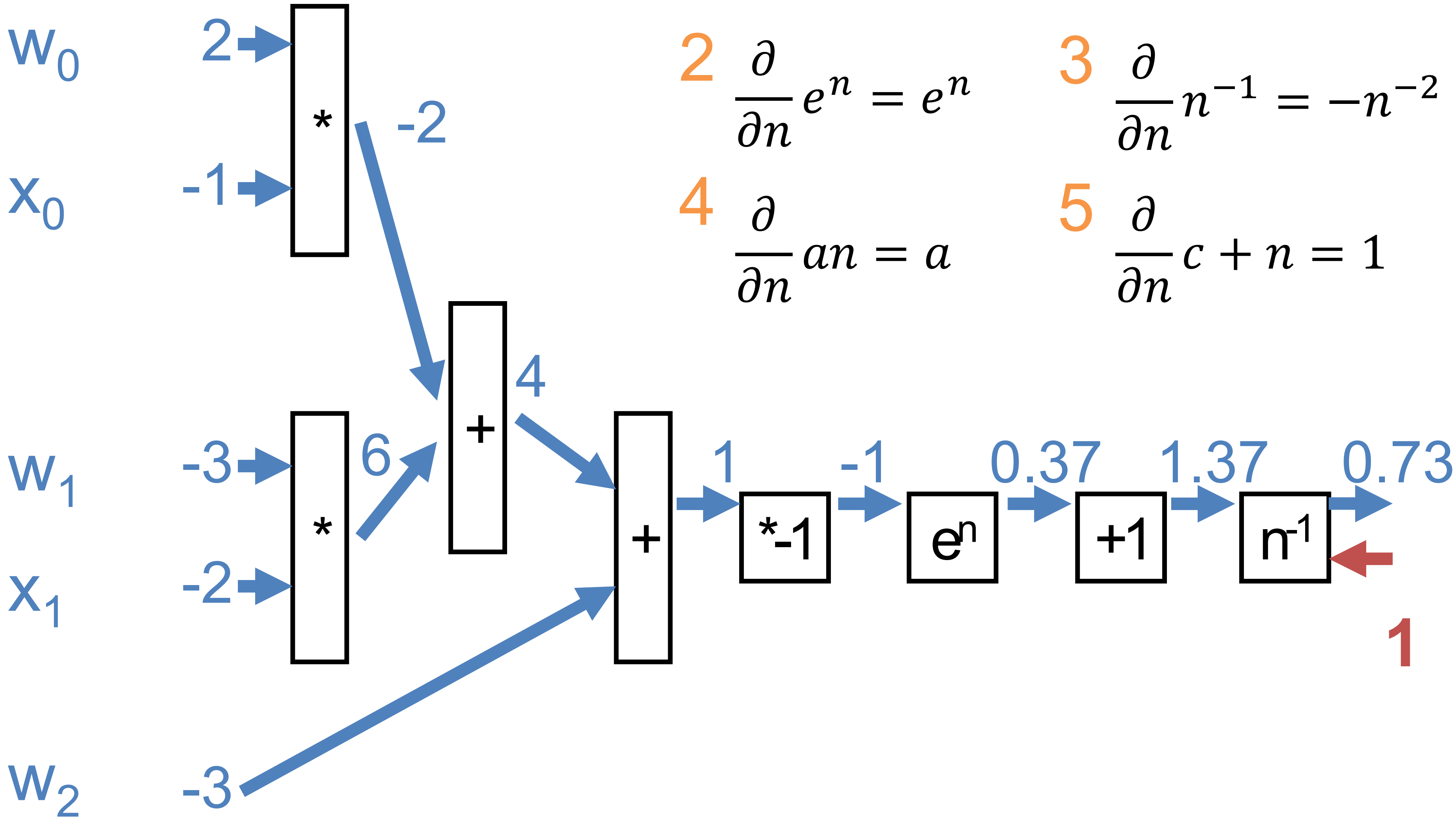

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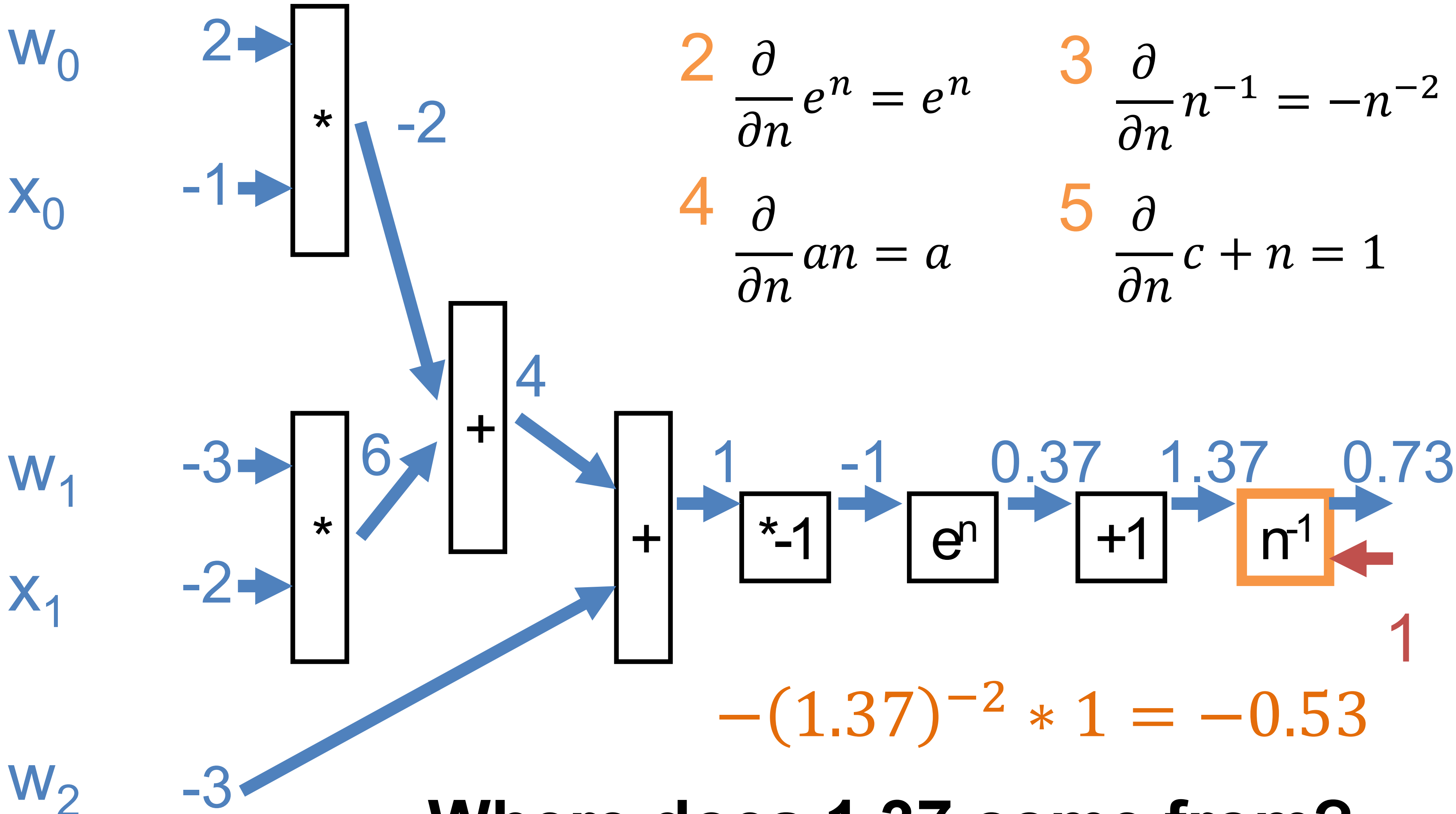

$$-(1.37)^{-2} * 1 = -0.53$$

Where does 1.37 come from?

Example Credit: Karpathy and Fei-Fei

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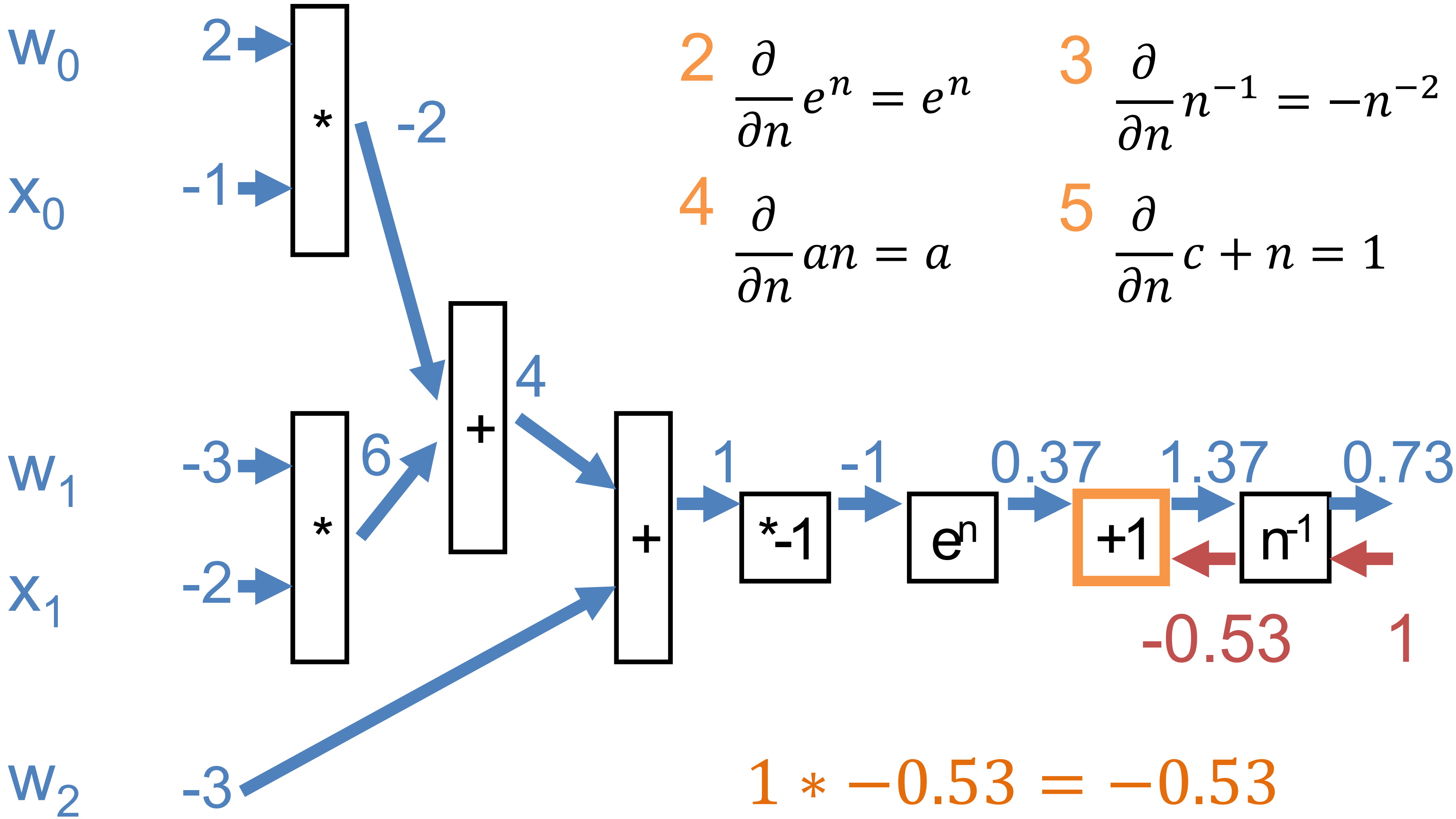

$$1 * -0.53 = -0.53$$

Example Credit: Karpathy and Fei-Fei

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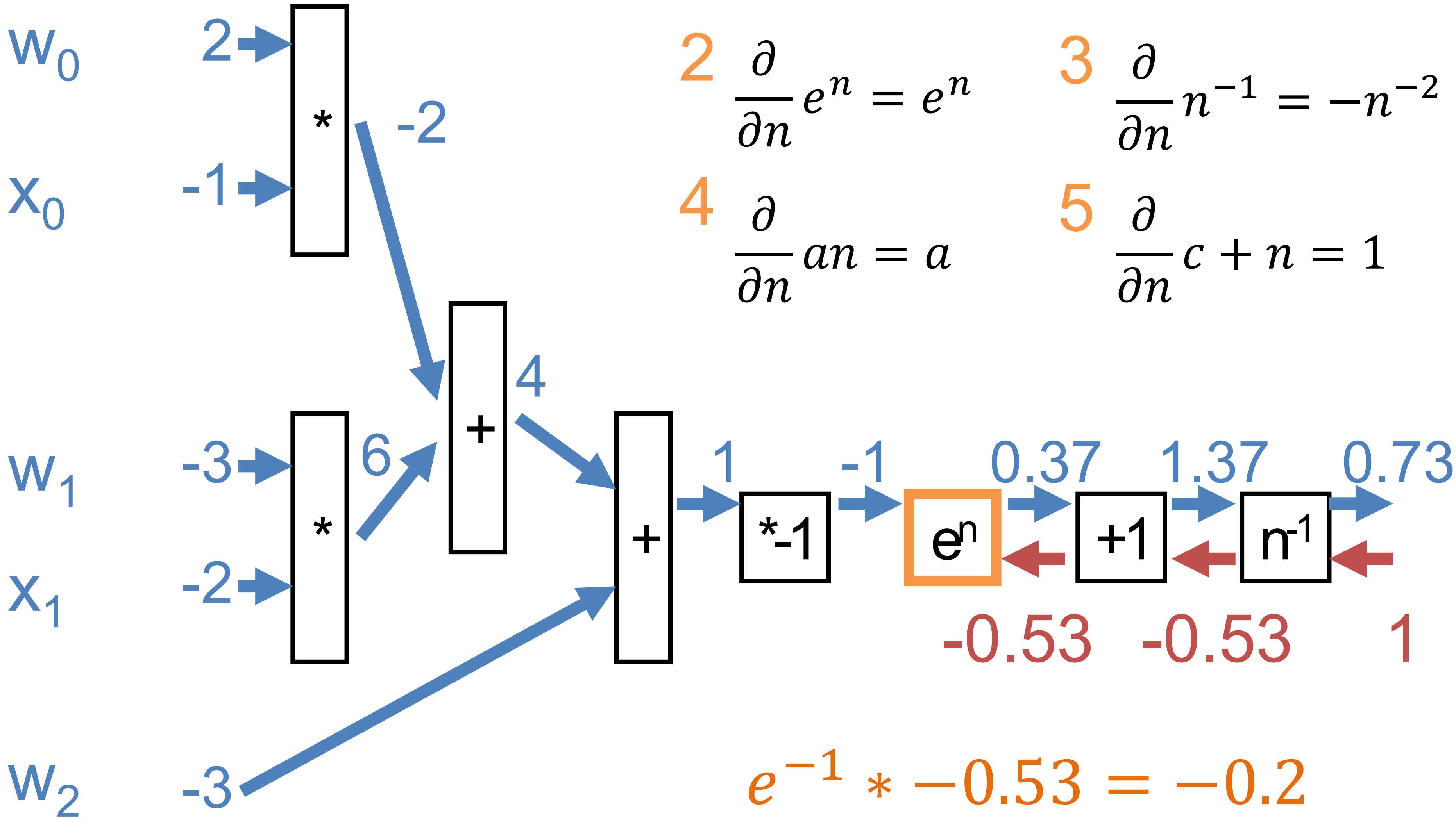

$$e^{-1} * -0.53 = -0.2$$

Example Credit: Karpathy and Fei-Fei

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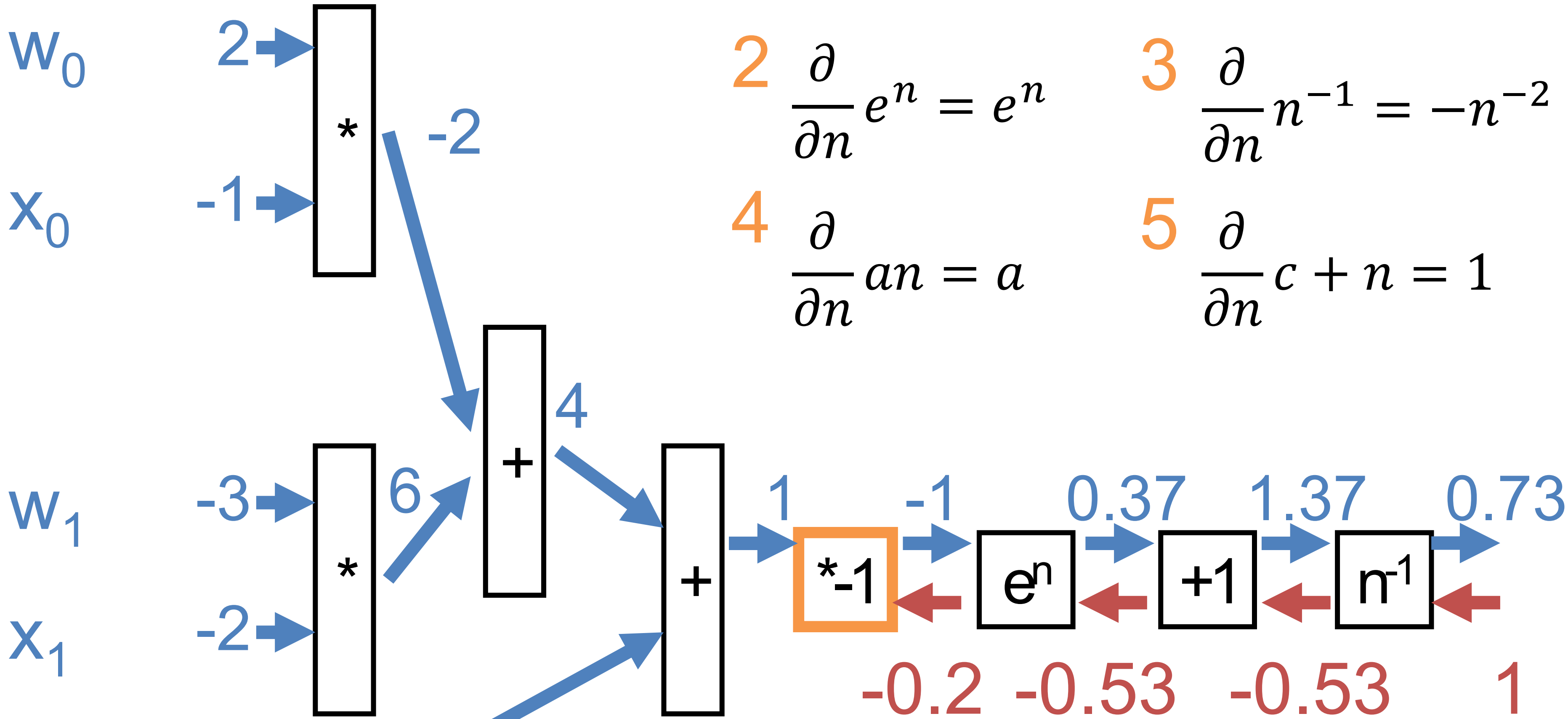

$$-1 * -0.2 = 0.2$$

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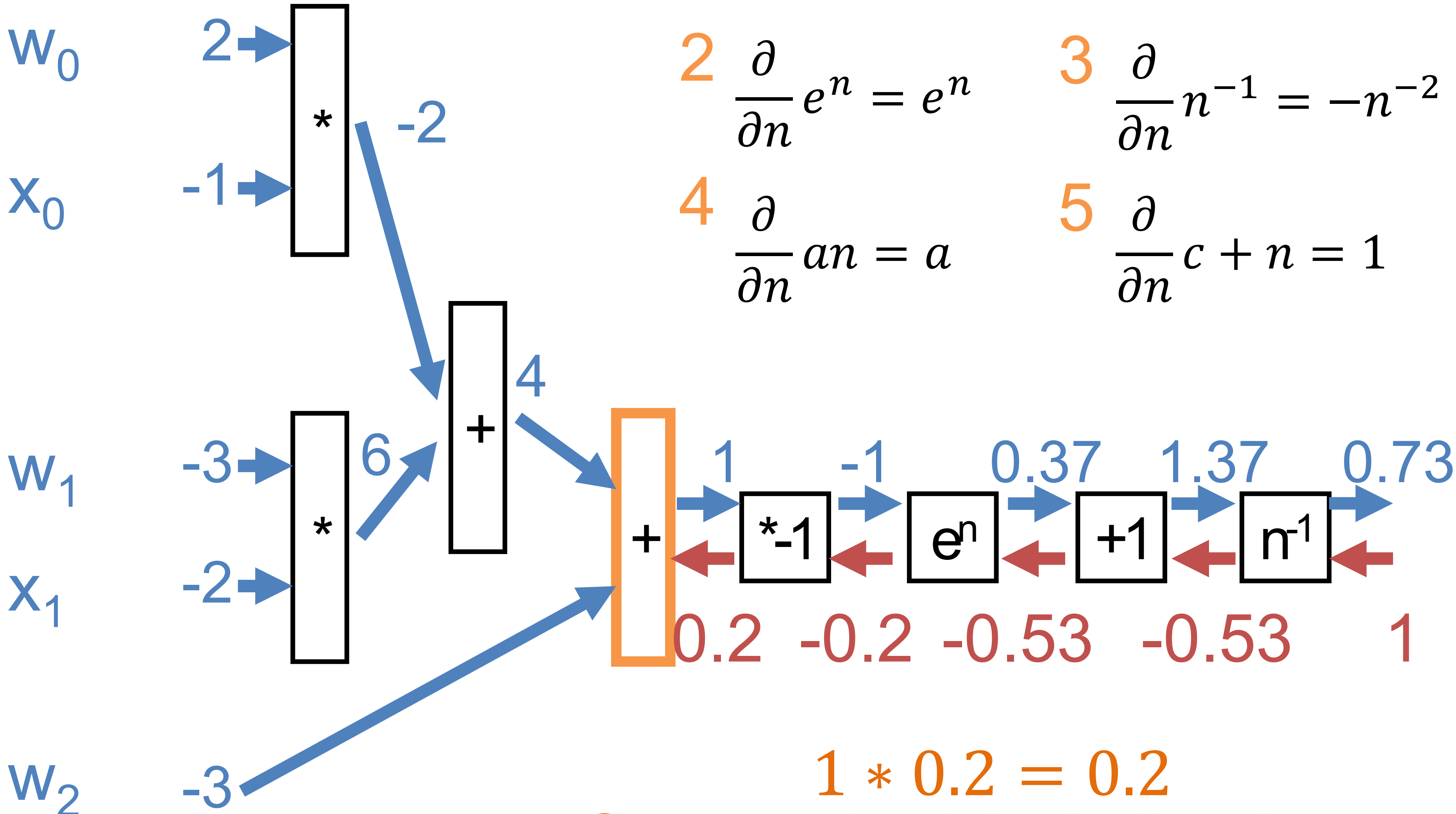

$1 * 0.2 = 0.2$

Gets sent back both directions

Example Credit: Karpathy and Fei-Fei

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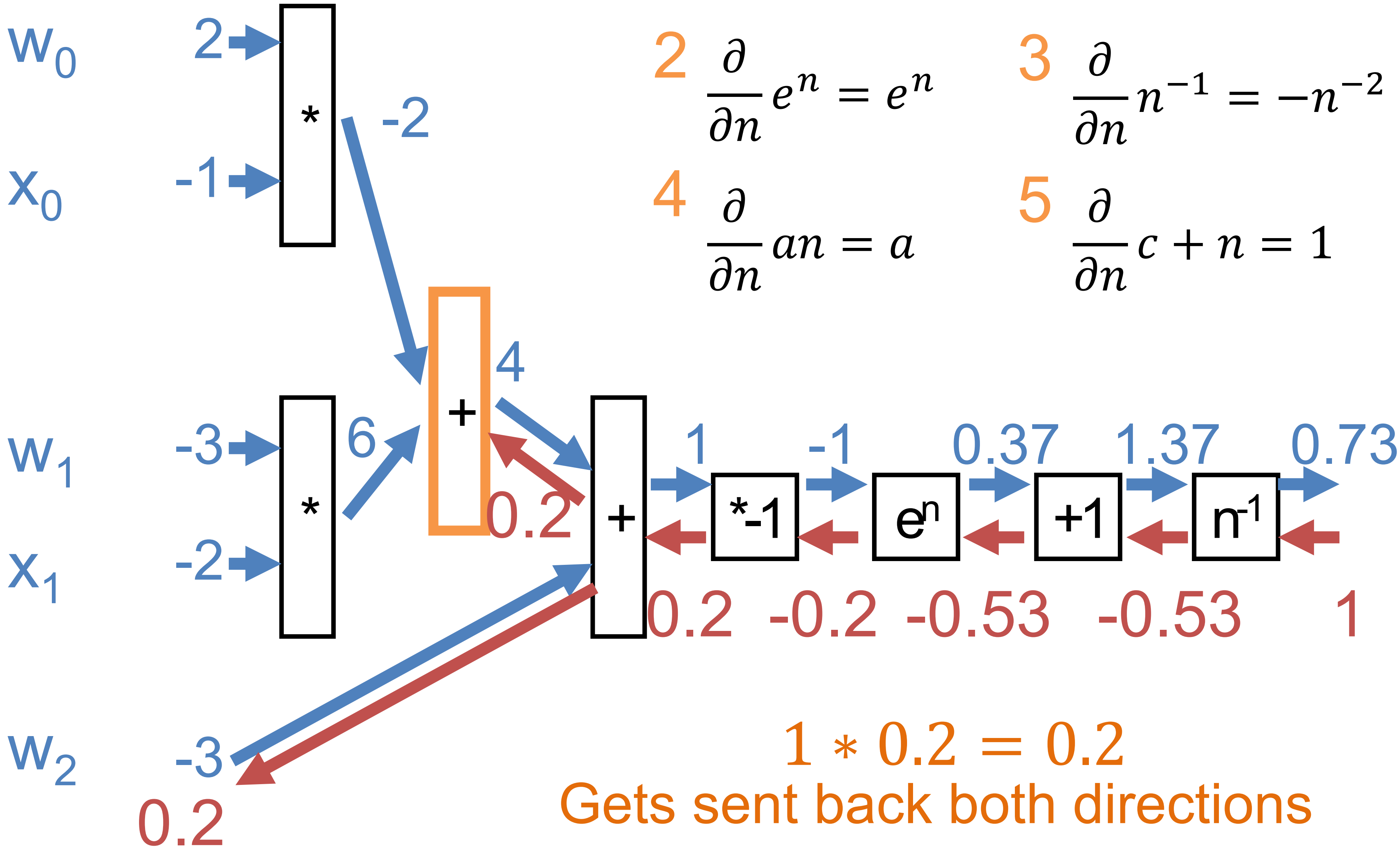

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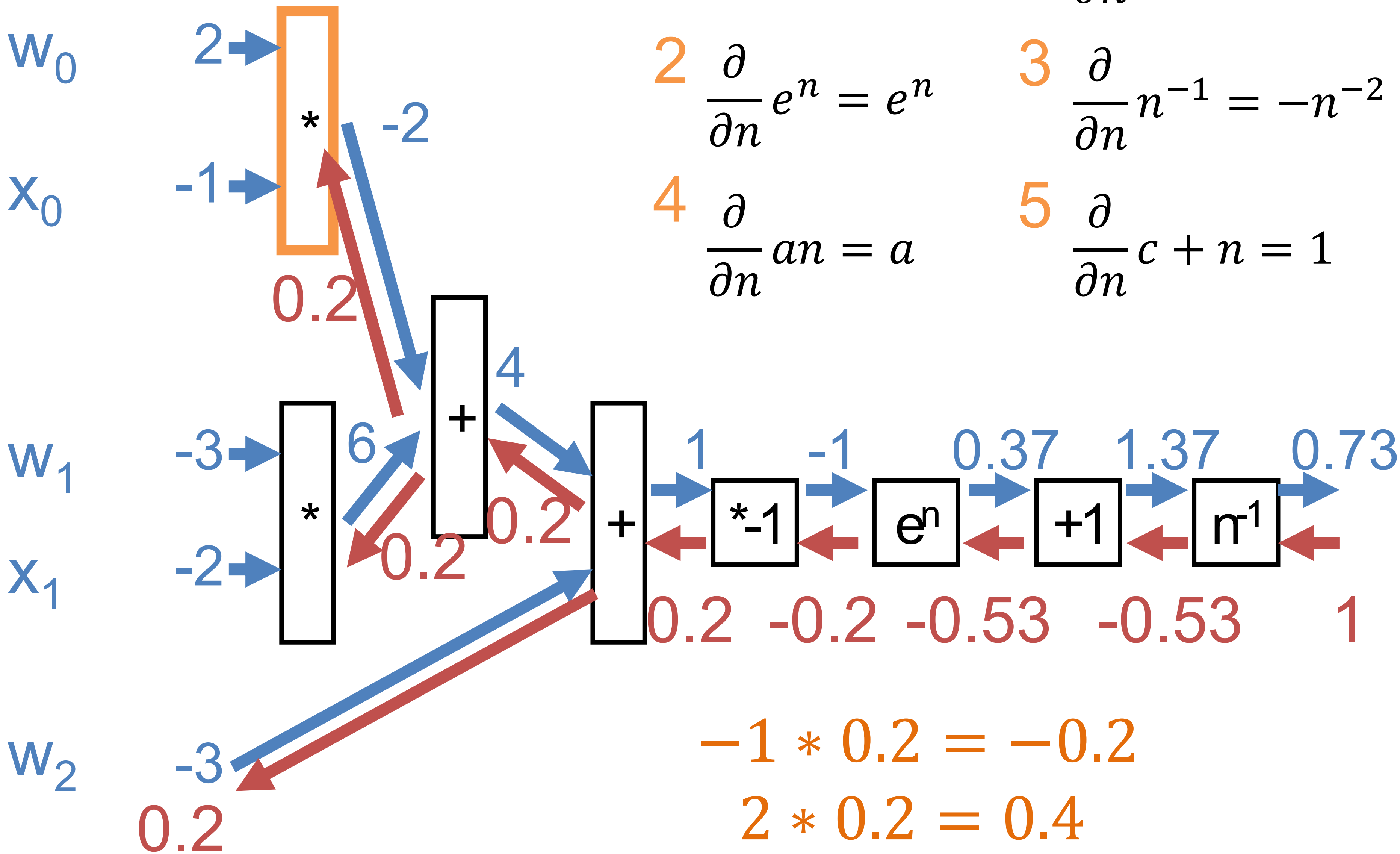

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$$-1 * 0.2 = -0.2$$

$$2 * 0.2 = 0.4$$

Example Credit: Karpathy and Fei-Fei

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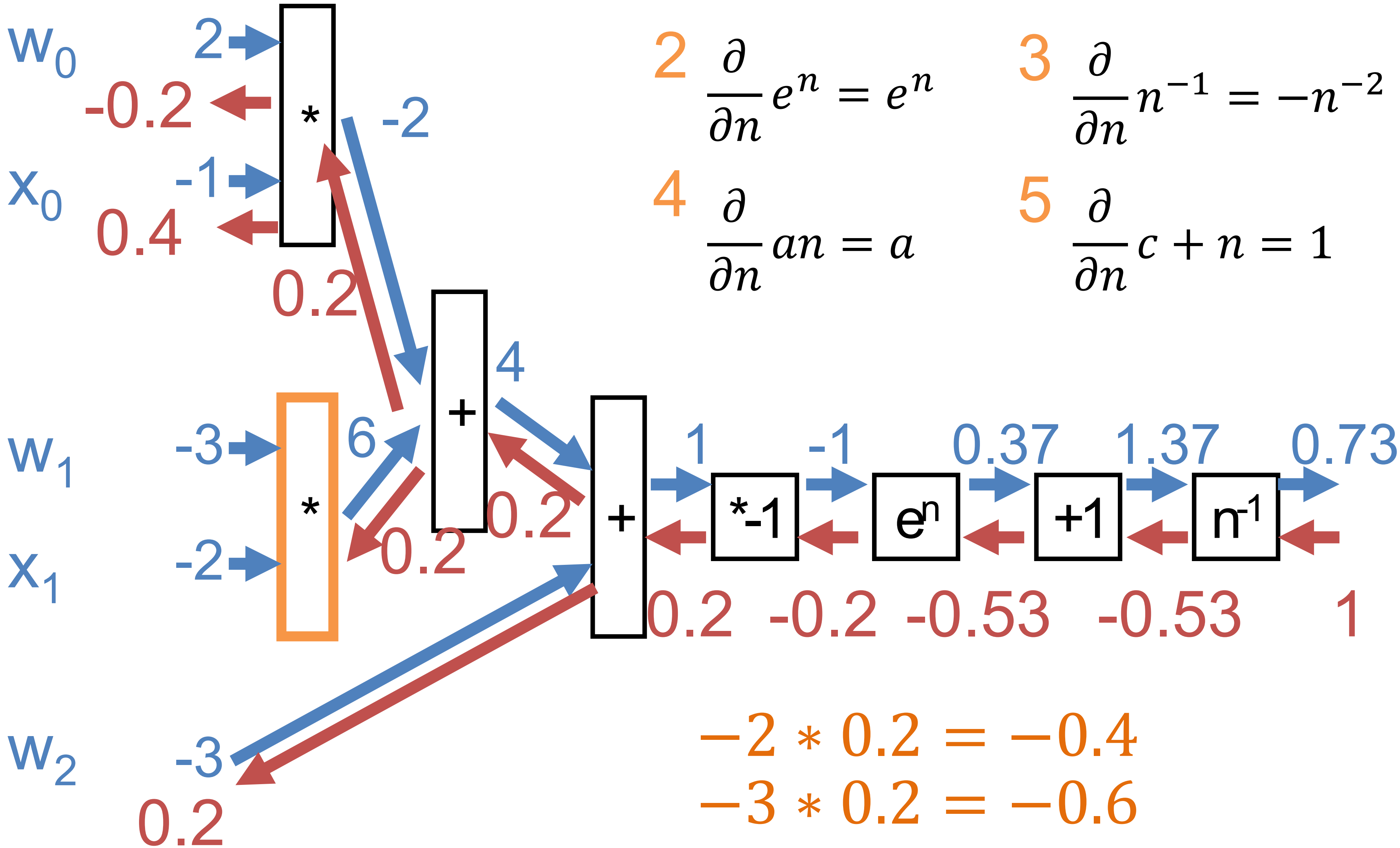

$$-2 * 0.2 = -0.4$$
$$-3 * 0.2 = -0.6$$

Example Credit: Karpathy and Fei-Fei

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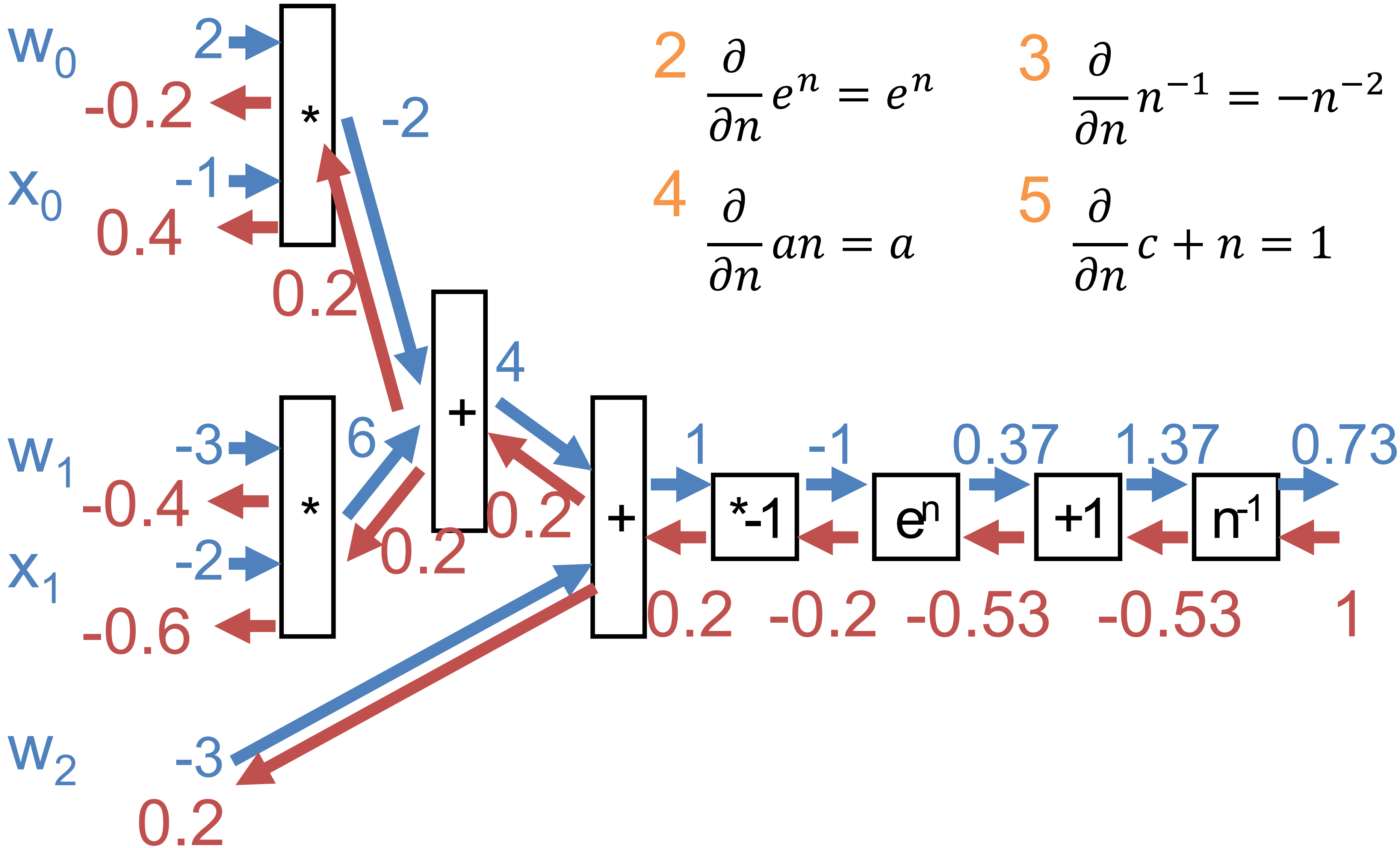

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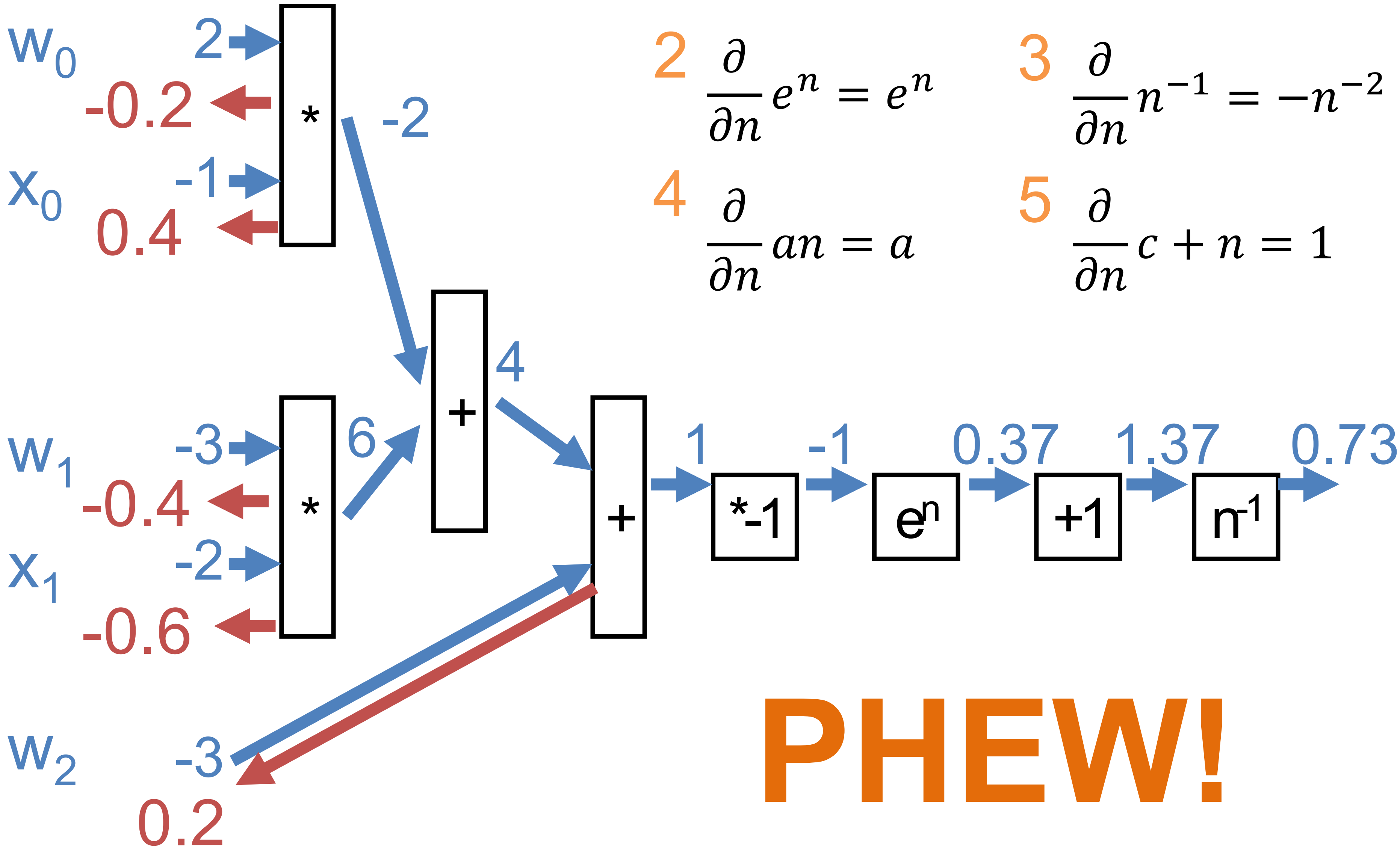
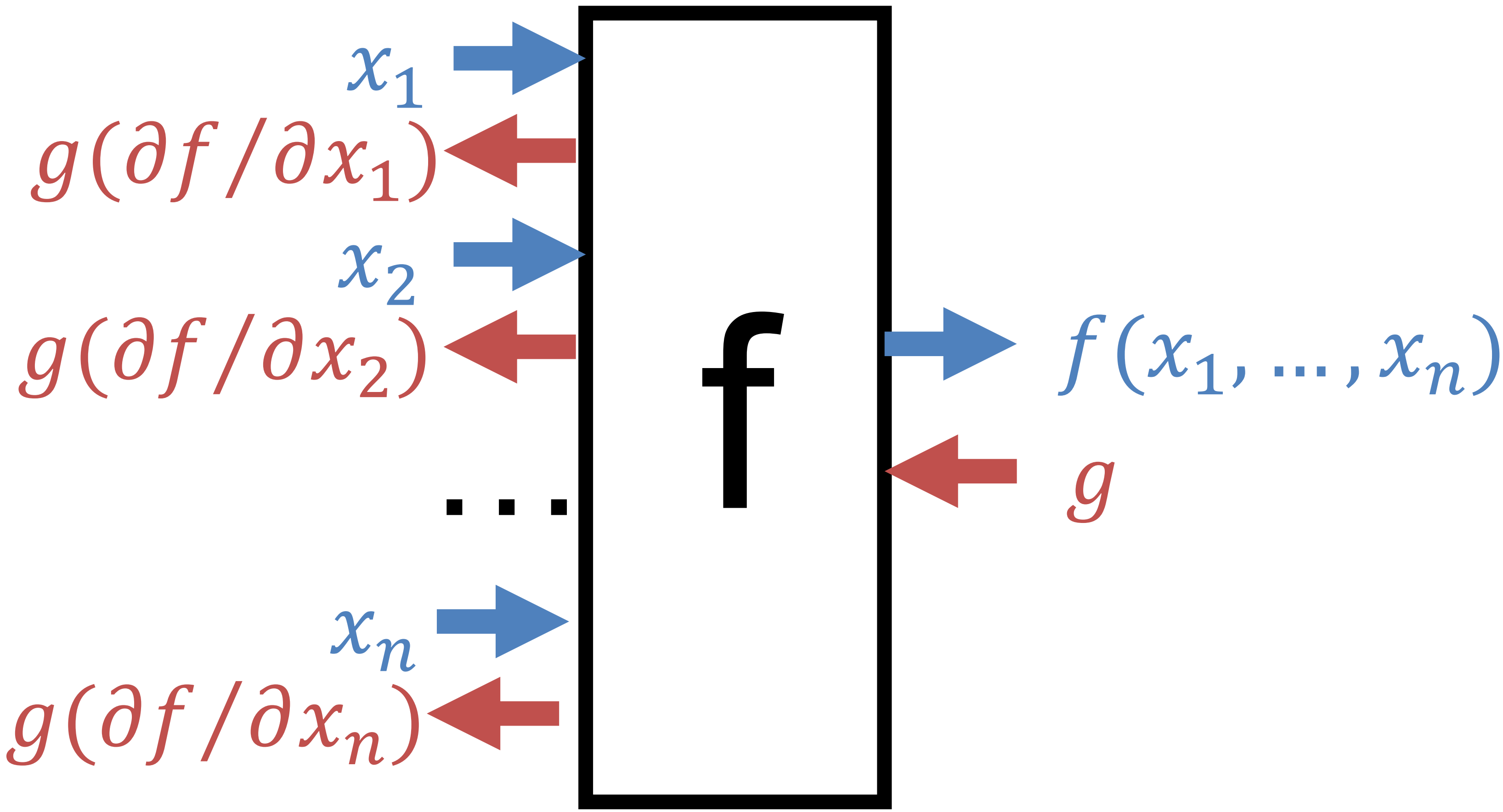


PHEW!

Example Credit: Karpathy and Fei-Fei

Summary

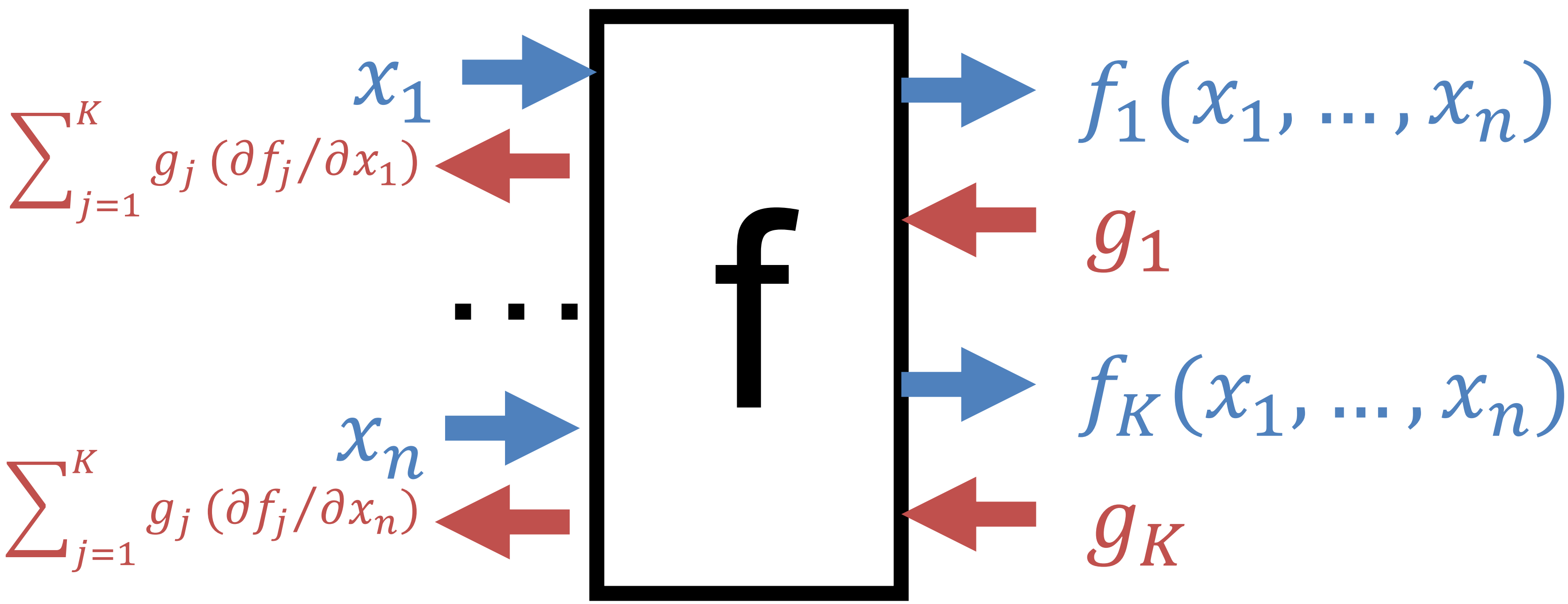
Each block computes backwards (g) * local gradient (df/dx_i) at the evaluation point



Multiple Outputs Flowing Back

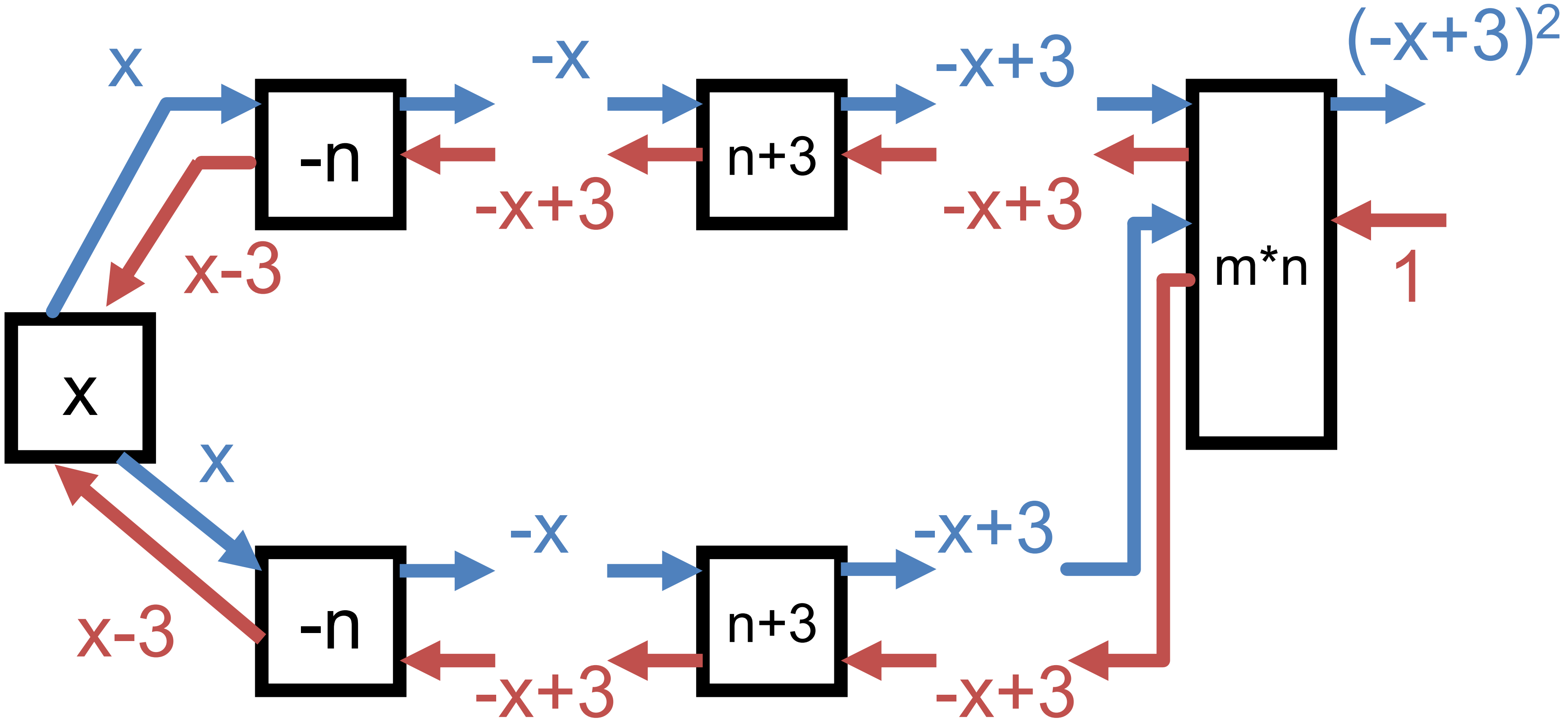
Gradients from different backwards sum up

$$\sum_{j=1}^{K} g_j \left(\partial f_j / \partial x_i\right)$$



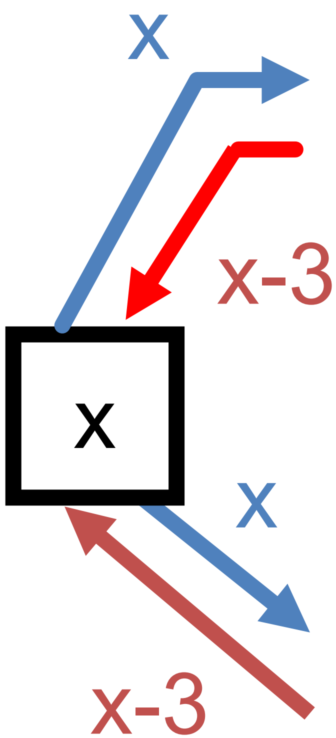
Multiple Outputs Flowing Back

$$f(x) = (-x+3)^2$$



Multiple Outputs Flowing Back

$$f(x) = (-x+3)^2$$

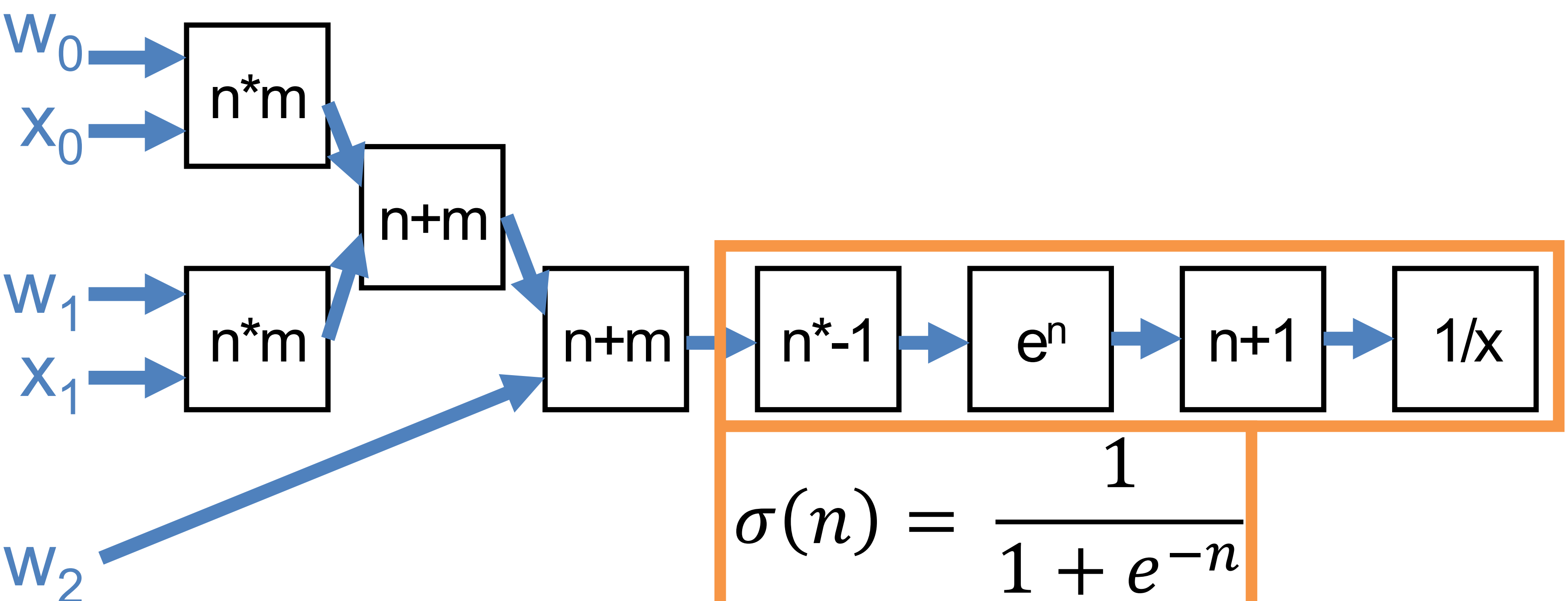


$$\frac{\partial f}{\partial x} = (x-3) + (x-3)$$

$$= 2x - 6$$

Does It Have To Be So Painful?

$$f(\boldsymbol{w}, \boldsymbol{x}) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w_2)}}$$



$$\sigma(n) = \frac{1}{1 + e^{-n}}$$

Example Credit: Karpathy and Fei-Fei

Does It Have To Be So Painful?

$$\sigma(n) = \frac{1}{1+e^{-n}}$$

$$\frac{\partial}{\partial n}\sigma(n) = \frac{e^{-n}}{(1+e^{-n})^2} = \left(\frac{1+e^{-n}-1}{1+e^{-n}}\right)\left(\frac{1}{1+e^{-n}}\right)$$

$$\frac{1+e^{-n}}{1+e^{-n}} - \frac{1}{1+e^{-n}} = 1-\sigma(n) \qquad \sigma(n)$$

$$= \big(1-\sigma(n)\big)\sigma(n)$$

For the curious

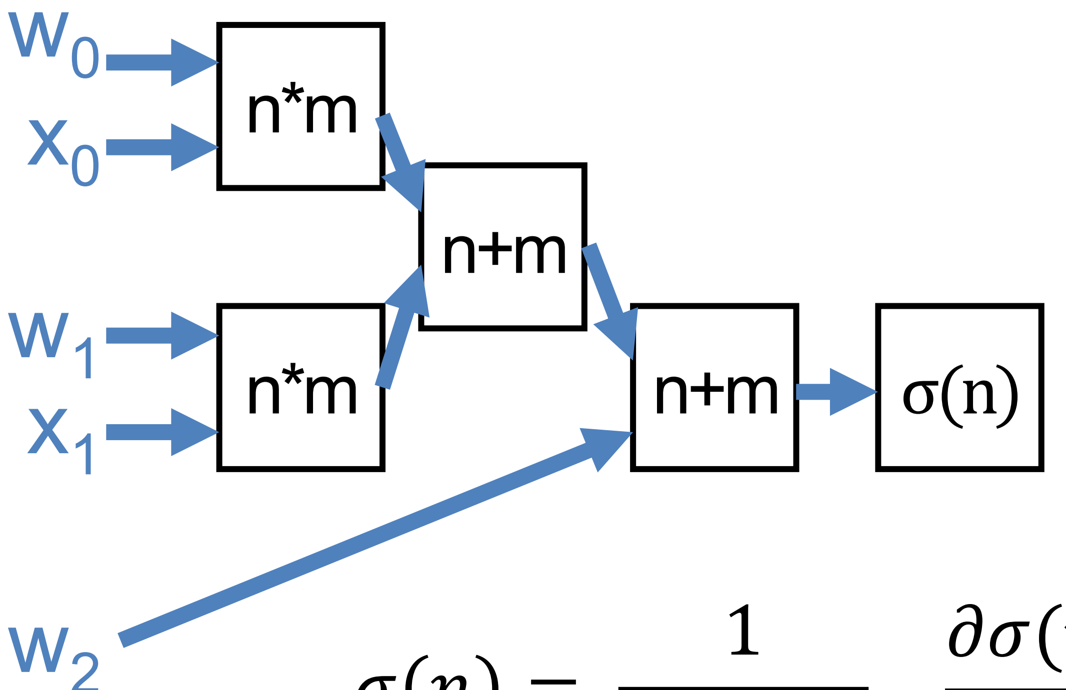
Line 1 to 2: $\frac{\partial}{\partial n}\sigma(n) = \left(\frac{-1}{(1+e^{-n})^2}\right) * 1 * e^{-n} * -1$

Chain rule: d/dx (1/x)*d/dx (1+x)* d/dx (e*x)*d/dx (-x)

Example Credit: Karpathy and Fei-Fei

Does It Have To Be So Painful?

$$f(\boldsymbol{w}, \boldsymbol{x}) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w_2)}}$$



$$\sigma(n) = \frac{1}{1 + e^{-n}} \qquad \frac{\partial \sigma(n)}{\partial n} = (1 - \sigma(n))\sigma(n)$$

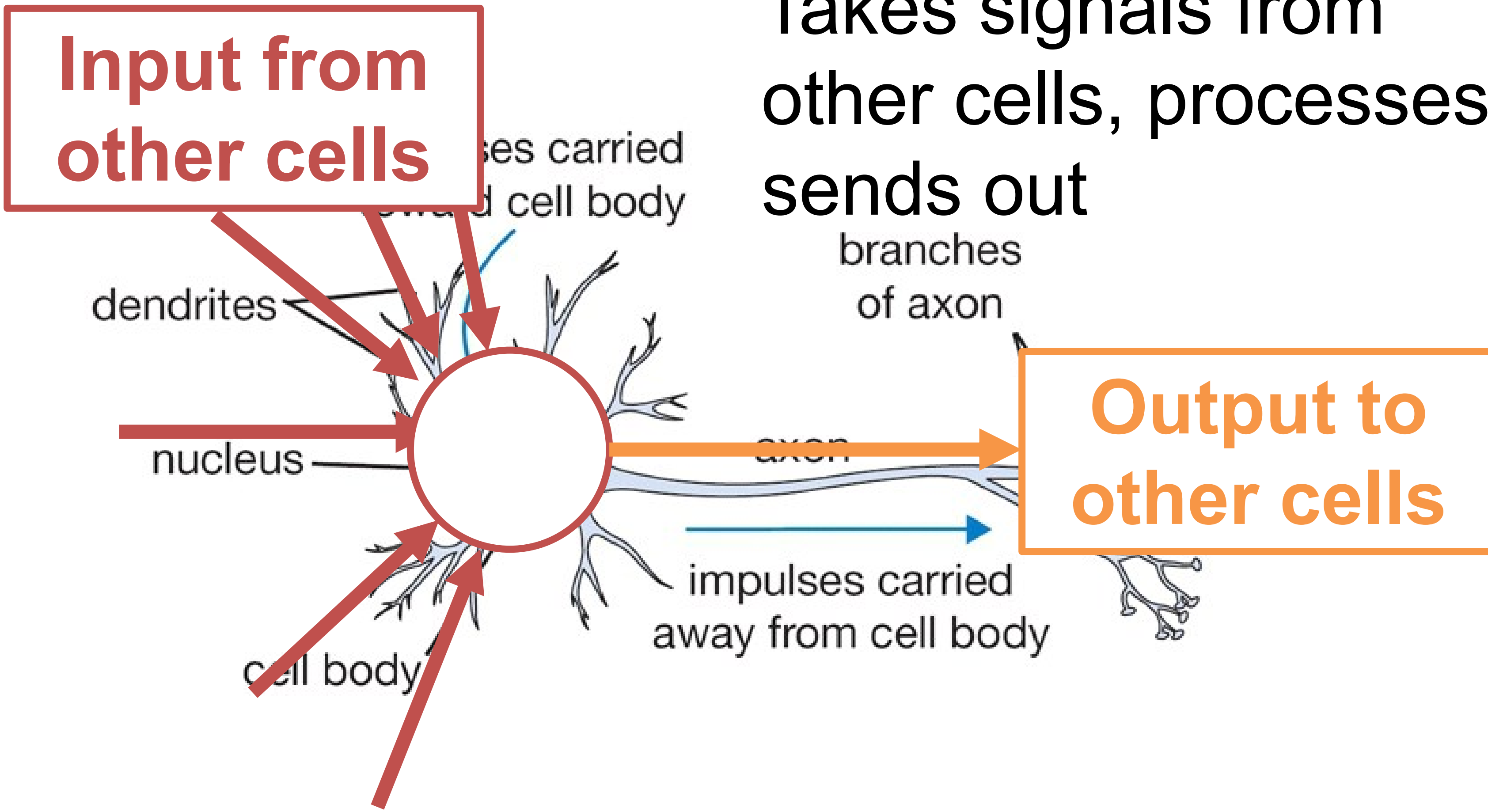
Example Credit: Karpathy and Fei-Fei

Does It Have To Be So Painful?

- Can compute for any function
- Pick your functions carefully: existing code is usually structured into sensible blocks

Building Blocks

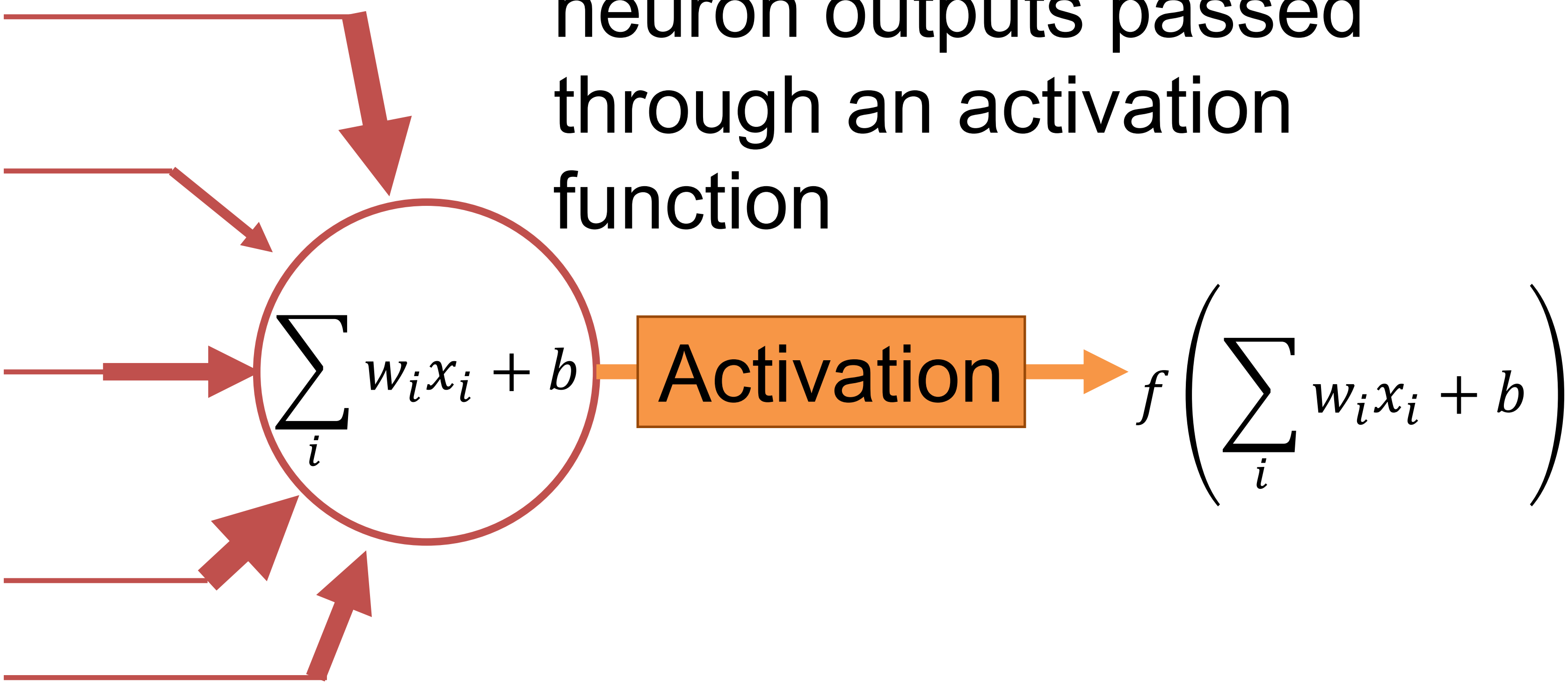
Takes signals from other cells, processes, sends out



Neuron diagram credit: Karpathy and Fei-Fei

Artificial Neuron

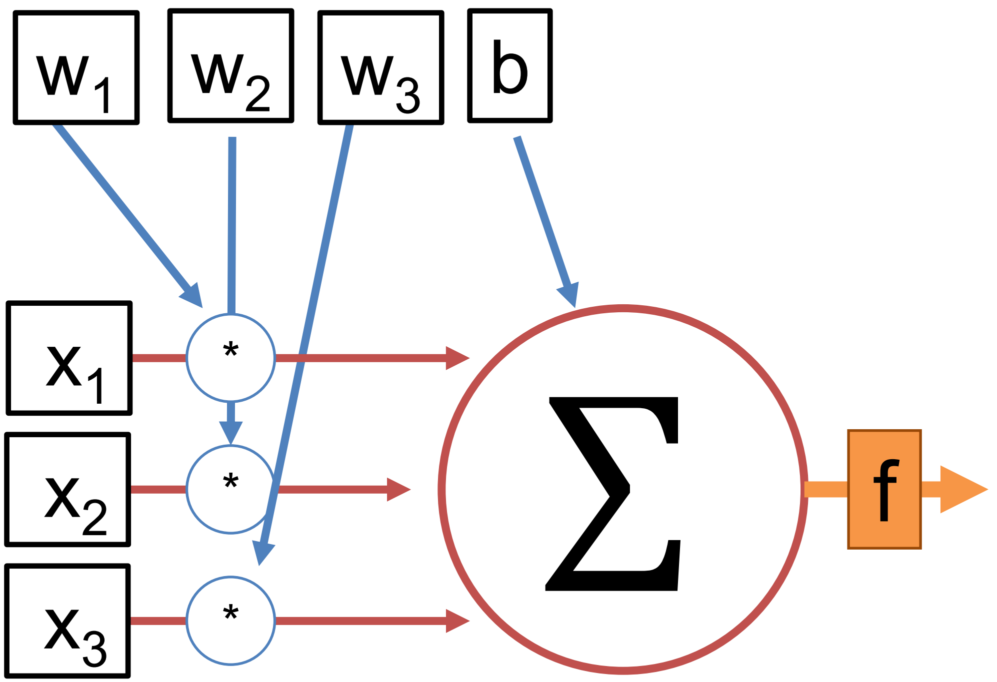
Weighted average of other neuron outputs passed through an activation function



Artificial Neuron

Can differentiate whole thing e.g., dNeuron/dx_1.

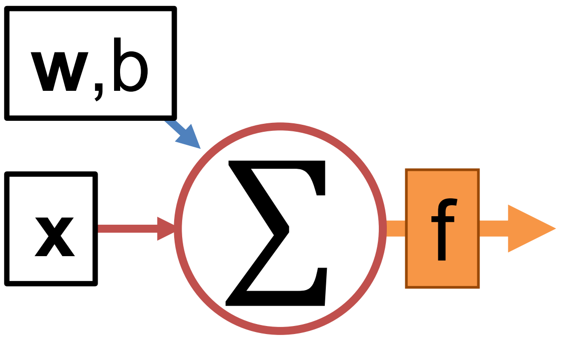
What can we now do?



Artificial Neuron

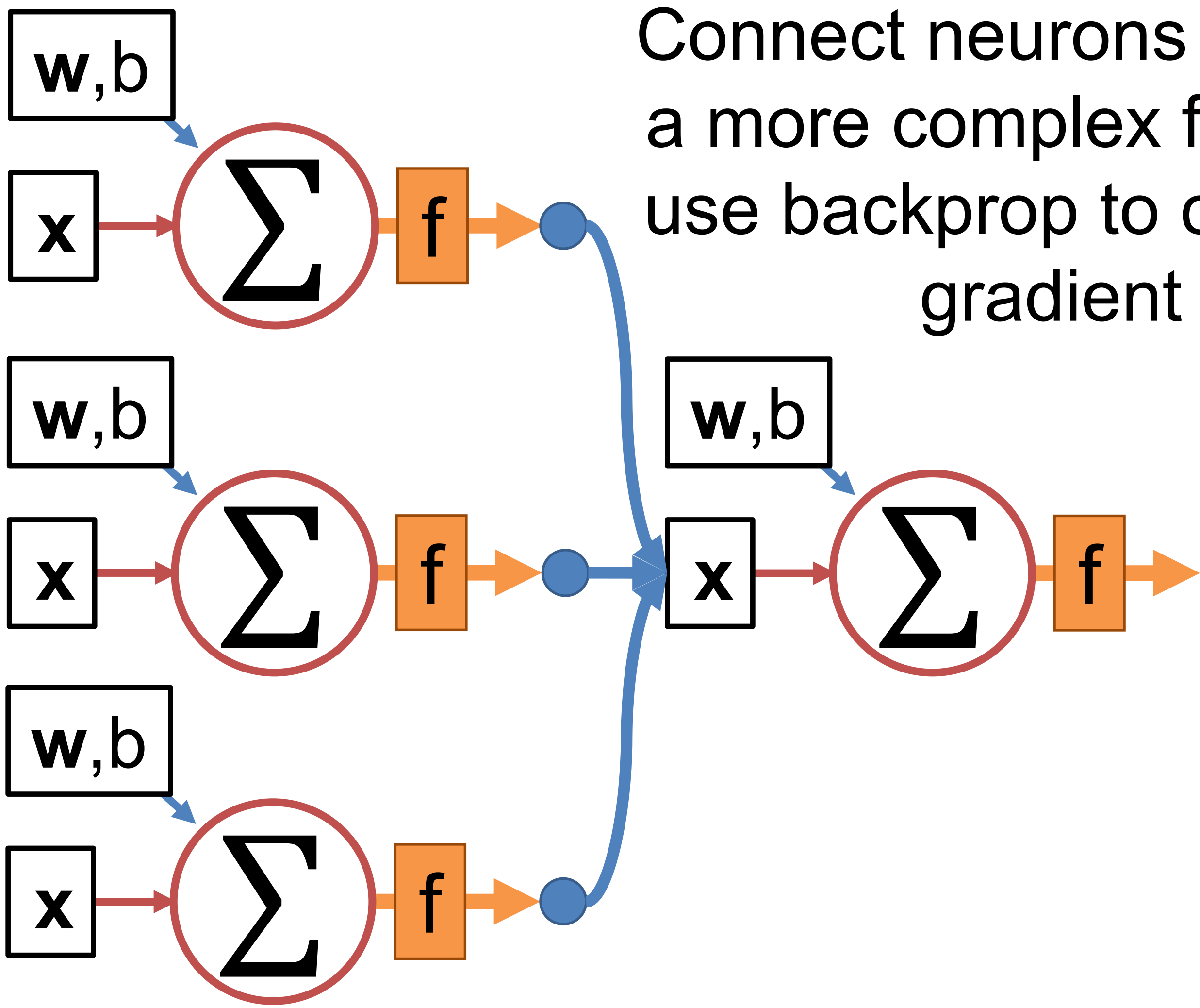
Each artificial neuron is a linear model + an **activation function** f

Can find **w**, b that minimizes a loss function with gradient descent

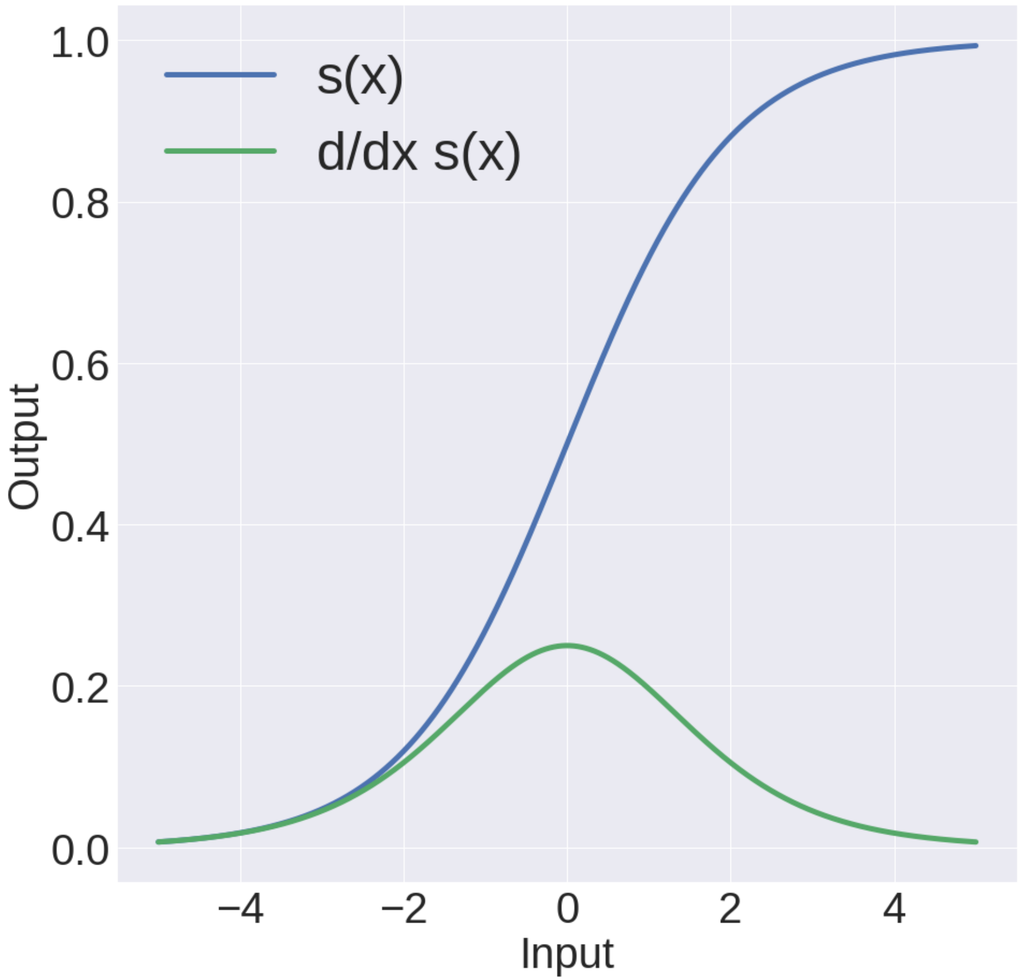


Artificial Neurons

Connect neurons to make a more complex function; use backprop to compute gradient



What's The Activation Function

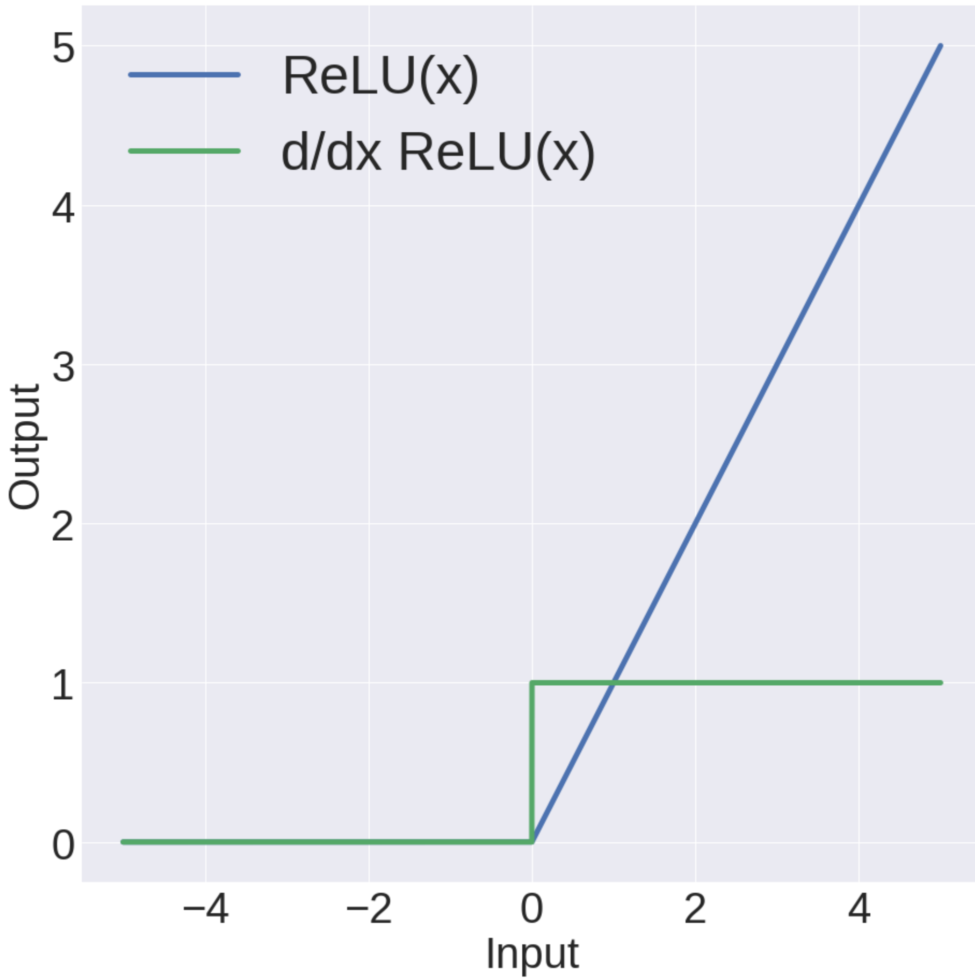


Sigmoid

$$s(x) = \frac{1}{1 + e^{-x}}$$

- Nice interpretation
- Squashes things to (0,1)
- Gradients are near zero if neuron is high/low

What’s The Activation Function



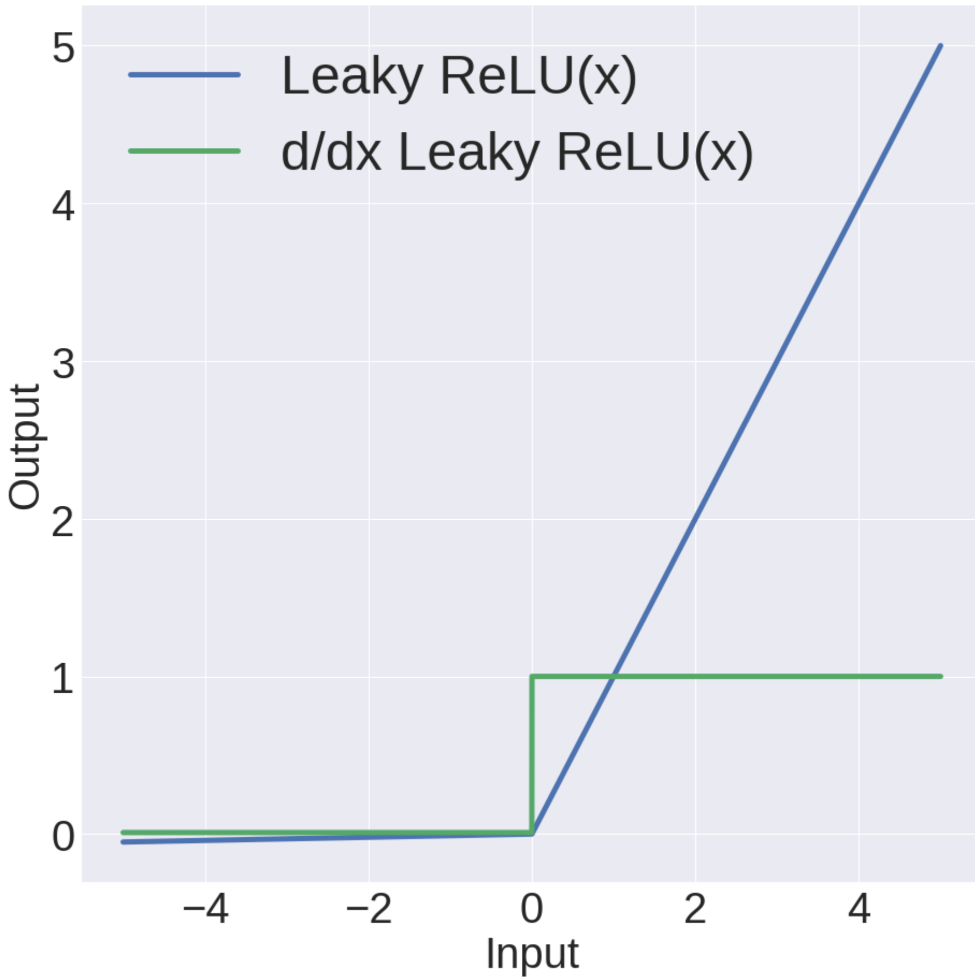
ReLU

(**<u>Re</u>**ctifying **<u>L</u>**inear **<u>U</u>**nit)

$$\max(0, x)$$

- Constant gradient
- Converges ~6x faster
- If neuron negative, zero gradient. Be careful!

What’s The Activation Function



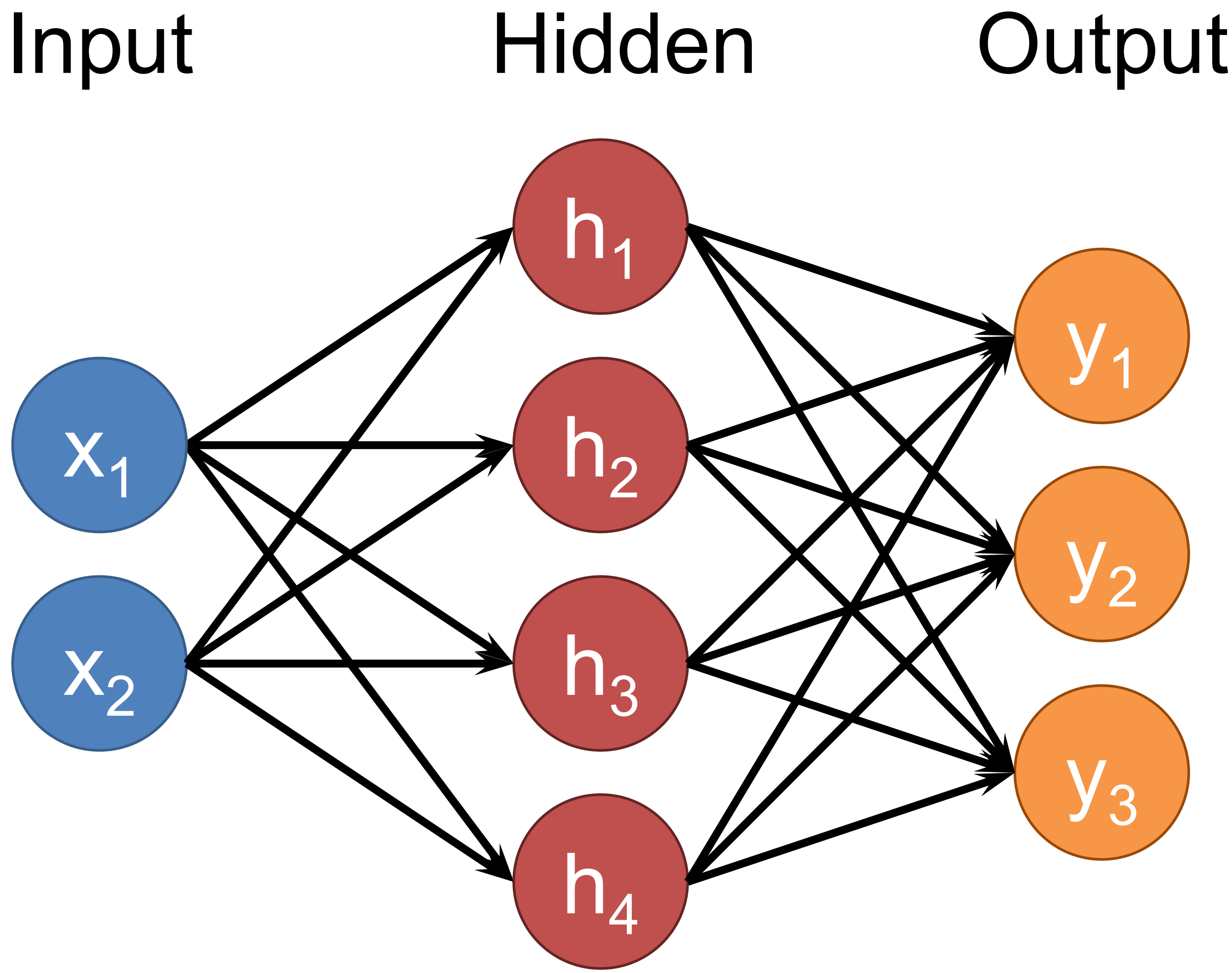
Leaky ReLU

(**Re**ctifying **L**inear **U**nit)

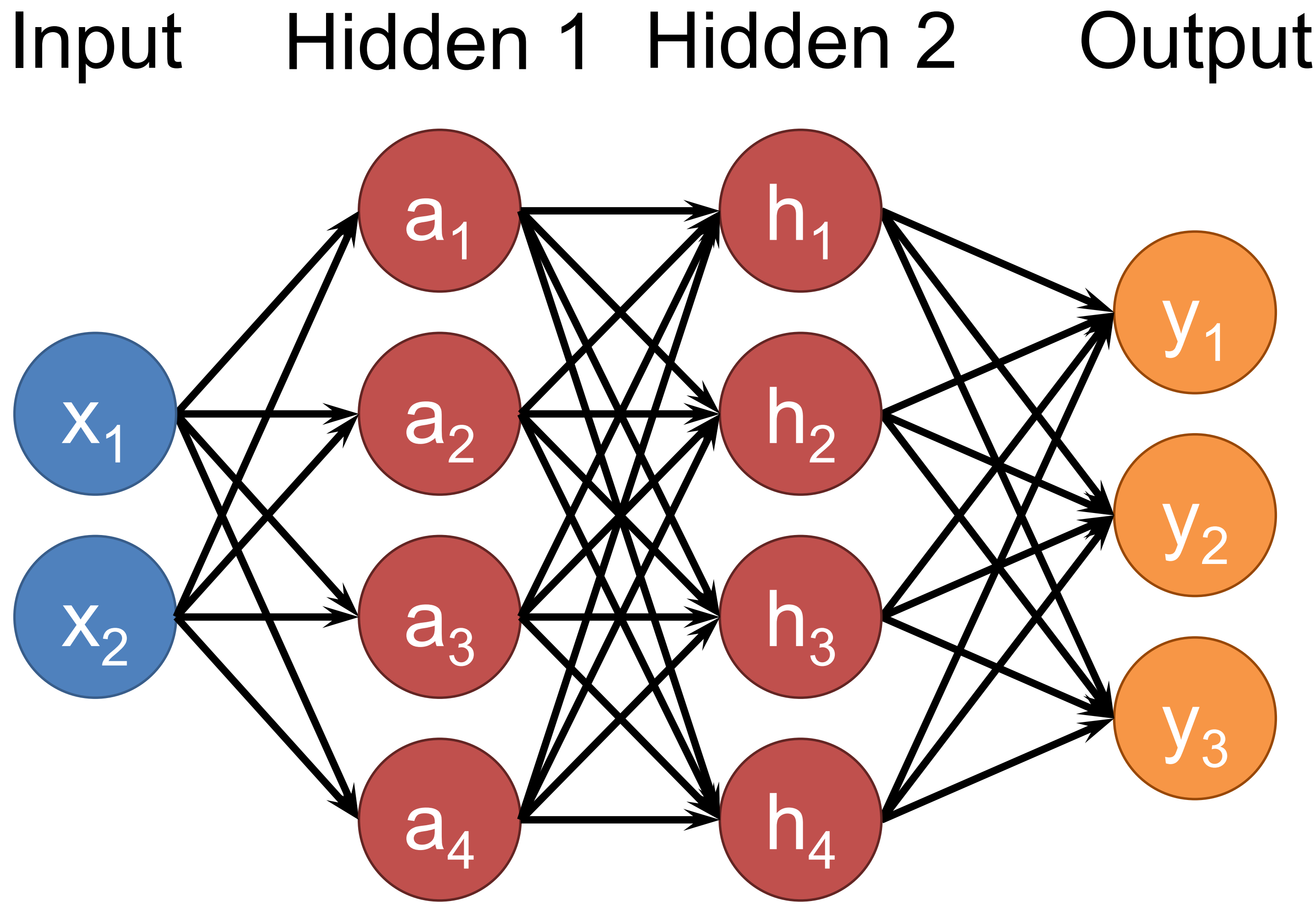
$$\begin{cases} x : x \geq 0 \\ 0.01x : x < 0 \end{cases}$$

- ReLU, but allows some small gradient for negative vales

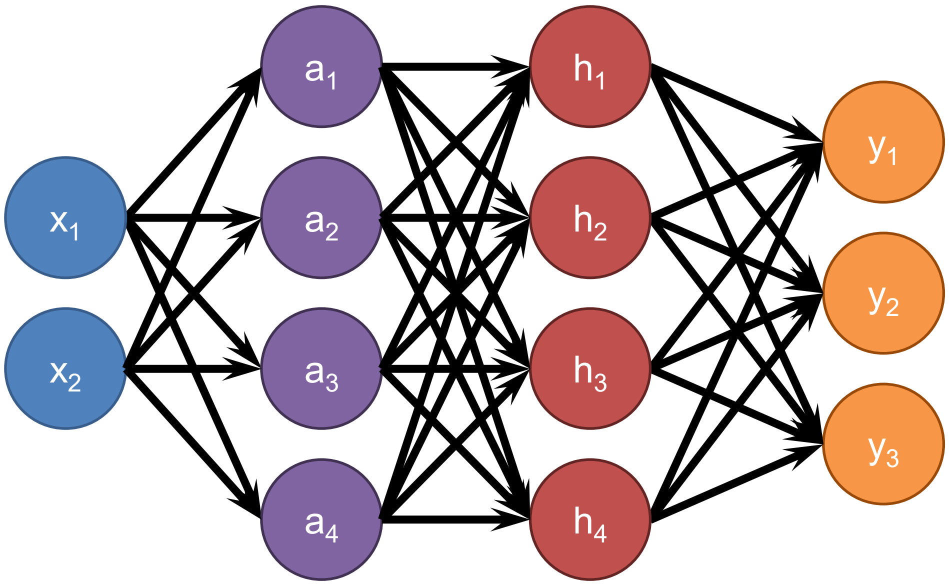
Setting Up A Neural Net



Setting Up A Neural Net

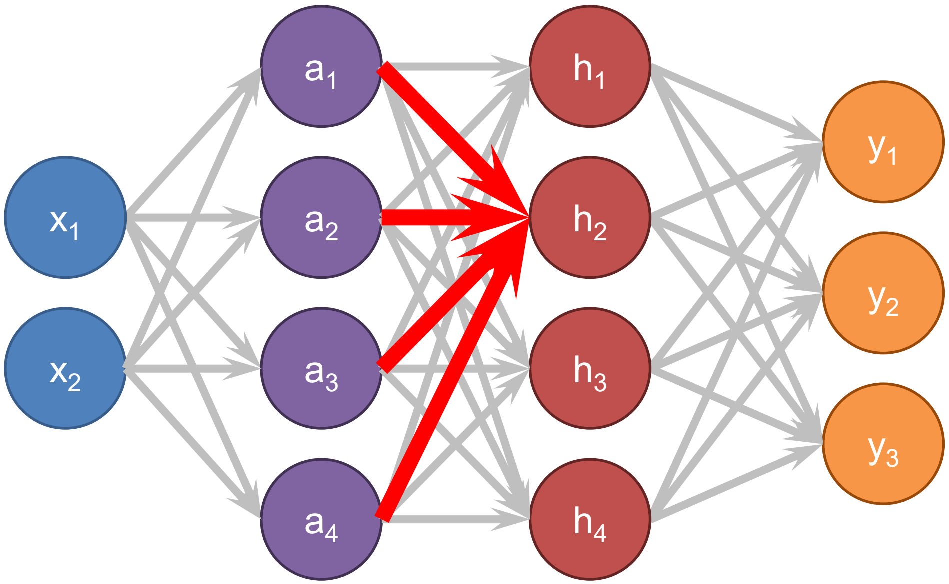


Fully Connected Network



Each neuron connects to each neuron in the previous layer

Fully Connected Network



$\boldsymbol{a}$ All layer a values

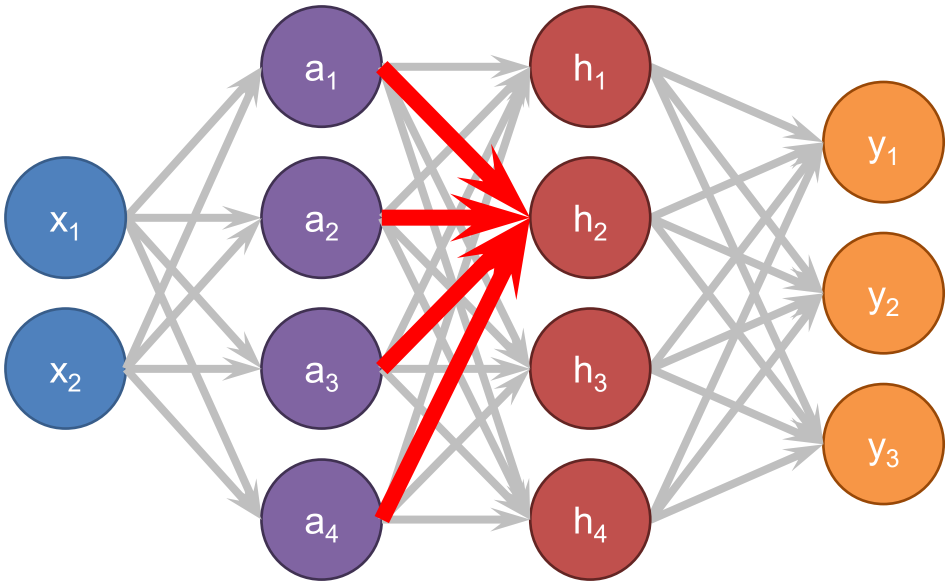
$\boldsymbol{w_i}, b_i$ Neuron i weights, bias

f Activation function

$$h_i = f(\boldsymbol{w_i^T a} + b_i)$$

How do we do all the neurons all at once?

Fully Connected Network



$\boldsymbol{a}$ All layer *a* values

$\boldsymbol{w_i}, b_i$ Neuron i weights, bias

f Activation function

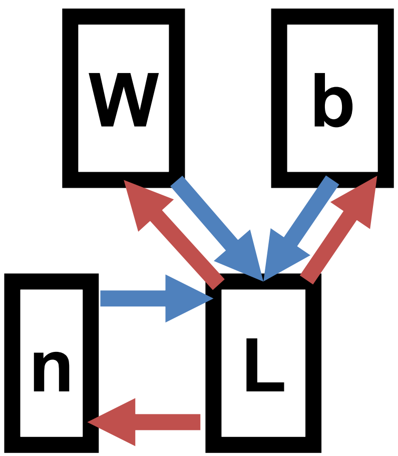
$$\boldsymbol{h} = f(\boldsymbol{W}\boldsymbol{a} + \boldsymbol{b})$$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = f\left(\begin{bmatrix} \leftarrow w_1^T \rightarrow \\ \leftarrow w_2^T \rightarrow \\ \leftarrow w_3^T \rightarrow \\ \leftarrow w_4^T \rightarrow \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \right)$$

Fully Connected Network

Define New Block: “Linear Layer”

(Ok technically it’s Affine)

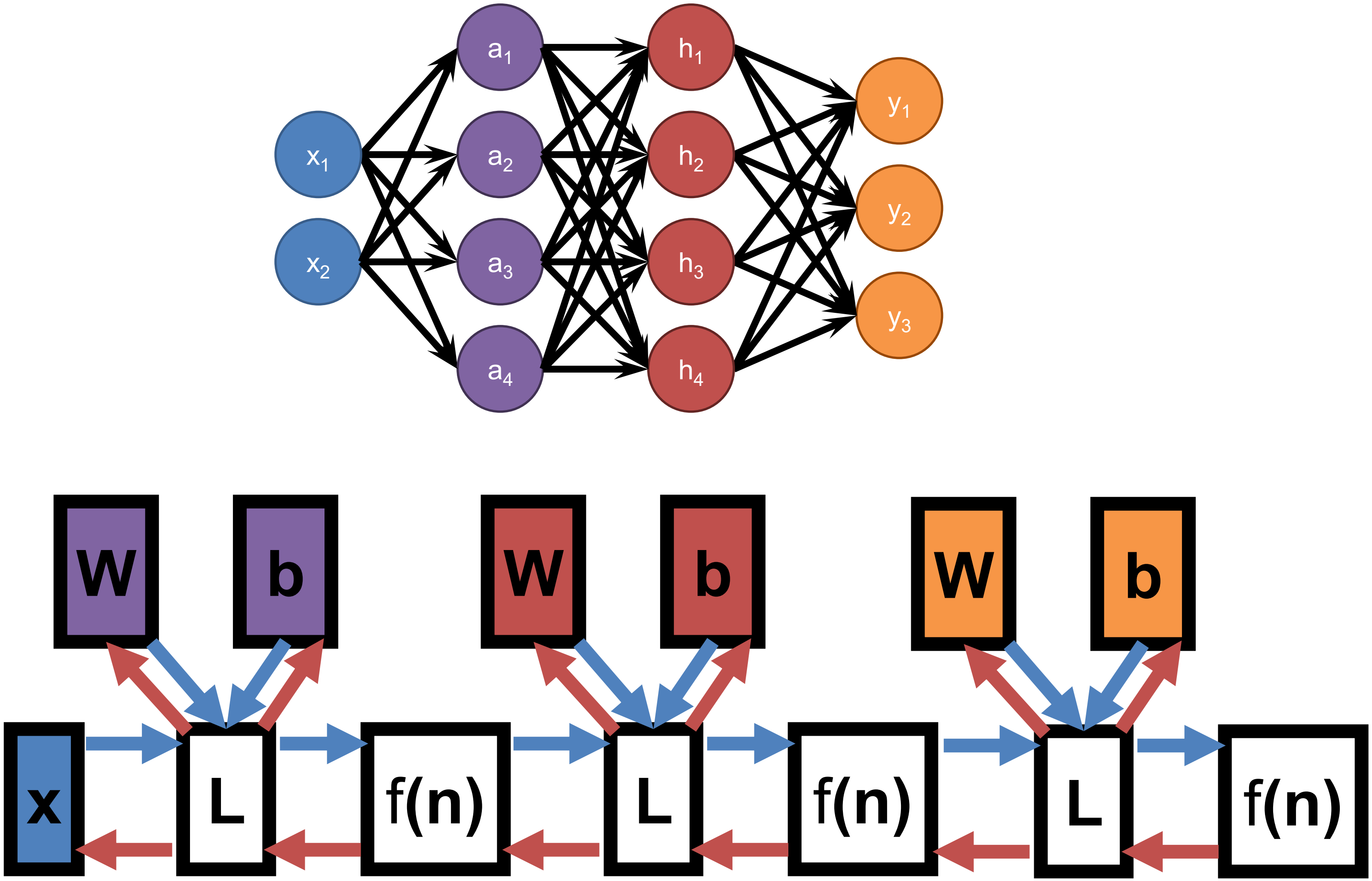


$$L(\boldsymbol{n}) = \boldsymbol{W}\boldsymbol{n} + \boldsymbol{b}$$

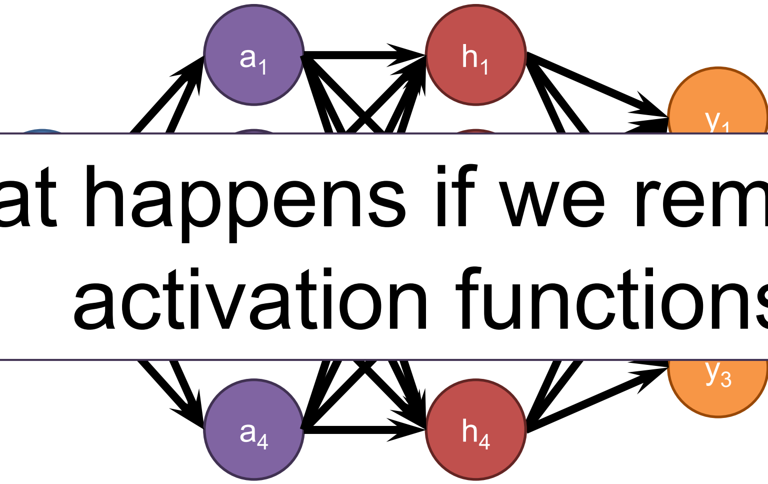
Can get gradient with respect to all the inputs

(do on your own; useful trick: have to be able to do matrix multiply)

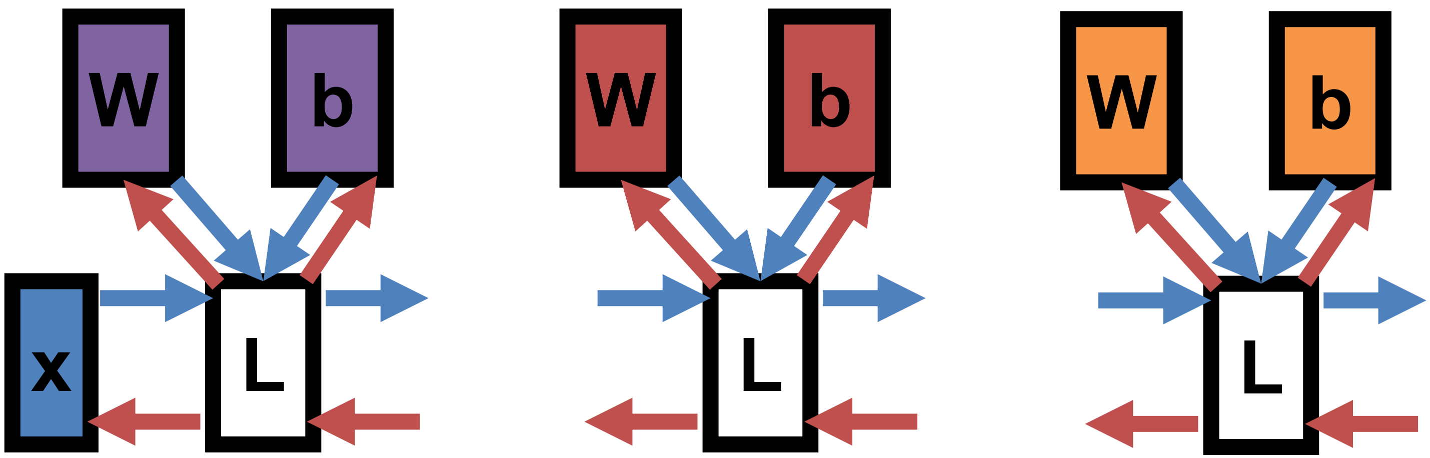
Fully Connected Network



Fully Connected Network



What happens if we remove the activation functions?



Demo Time

https://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html