Lecture 7
Transcendental Computation

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Overview

We give a brief introduction to transcendental number theory, and issues of transcendental computation. Then we describe a recent result showing a first non-trivial transcendental geometric computation that is computable in the EGC sense.

- 0. Review
- I. Intro to Transcendental Number Theory
- II. A Solved Problem that Isn't: Shortest Path amidst Discs

0. REVIEW

ANSWERS and DISCUSSIONS

- REMEMBER: the prize for best exercises

- SHOW: If a reduced rational p/q is the zero of an integer polynomial $A(X) = \sum_{i=0}^{m} a_i X^i$ then $q|a_m$ and $p|a_0$
  - $*$ Corollary: if p/q is algebraic integer, then $q = 1$
  - $*$ Corollary: $\sqrt{2}$ is irrational

What have we learned so far?

- EGC is an effective method to achieve robust numerical algorithms
- The central problem of EGC are the ZERO PROBLEMS
- EGC can be achieved for all algebraic problems
- This lecture: Which non-algebraic problems can we solve?

I. Transcendental Numbers

Introduction

- What is between $\mathbb{A}$ and $\mathbb{R}$?
  - DEFINE: A transcendental number is a non-algebraic number.
  - Is e and π algebraic?
  - This is the topic of transcendental number theory

- Easier questions
  - Are there any transcendental numbers? Yes (Cantor)
  - Is e rational?
  - Whiteboard Aside: Proof that e is irrational
  - Whiteboard Aside: Proof that e is not quadratic irrational

- Louisville's Theorem (1844)
  - ∗ If α is algebraic of degree $m > 1$ then for all $p/q \in \mathbb{Q}$, $|\alpha - (p/q)| > Cq^{-2}$
  - ∗ Proof: let $A(X)$ be minimal polynomial of α
  - ∗ Then $q^{-m} \le |A(p/q)| = |A(p/q) - A(\alpha)| = |(p/q) - \alpha| \cdot |A'(\beta)|$
  - ∗ But $|A'(\beta)| \le C$ for some constant depending on α

- Corollary: $\sum_{n=1}^{\infty} 2^{-n!}$ is transcendental
  - ∗ Proof: take $q = 2^{n!}$ for sufficiently large n

- Progress is slow:
  - ∗ Hermite 1873, e is transcendental
  - ∗ Lindemann 1882, π is transcendental

* Roth 1955 (culmination of Thue, Siegel)
* Gelfond Schneider: e^{π} is transcendental. But is π^{e}?

PART II. A SOLVED-PROBLEM THAT ISN'T

(Joint with E.Chien, S.Choi, D.Kwon, H.Park)

Shortest Path Amidst Disc Obstacles

- Given: Points $p, q \in \mathbb{R}^2$ and a collection S of discs
- Find: shortest path from p to q which avoids the obstacles in S
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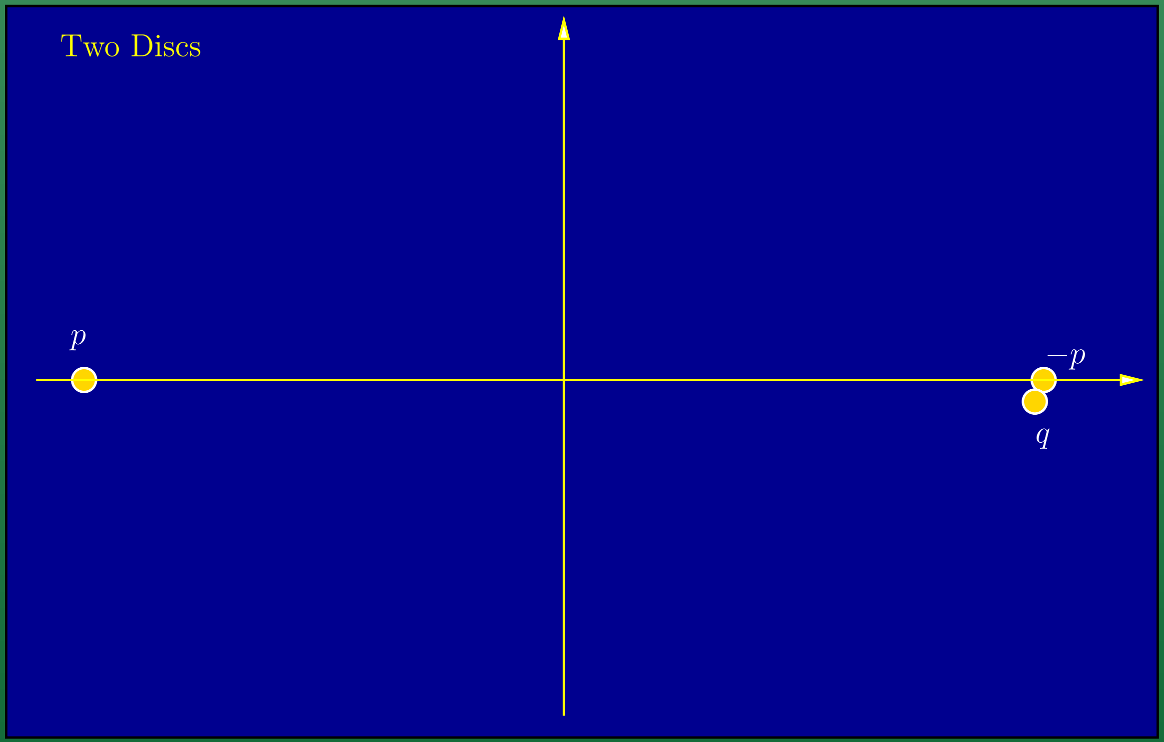
- 

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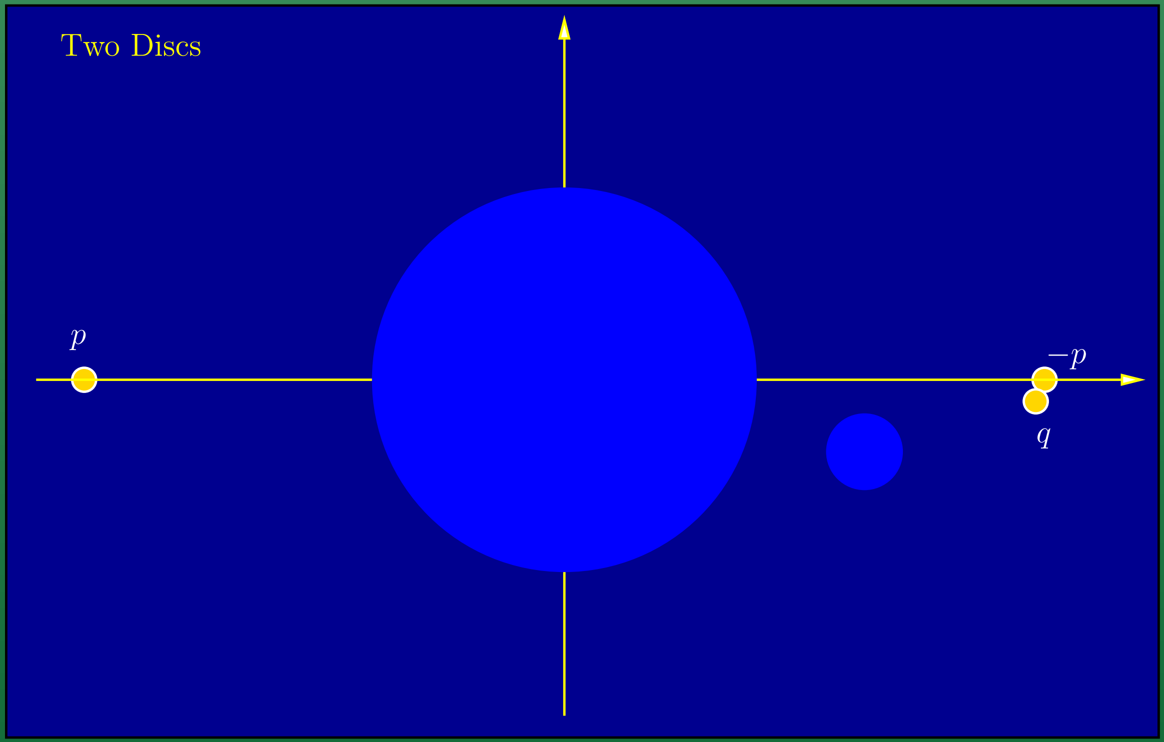
- 

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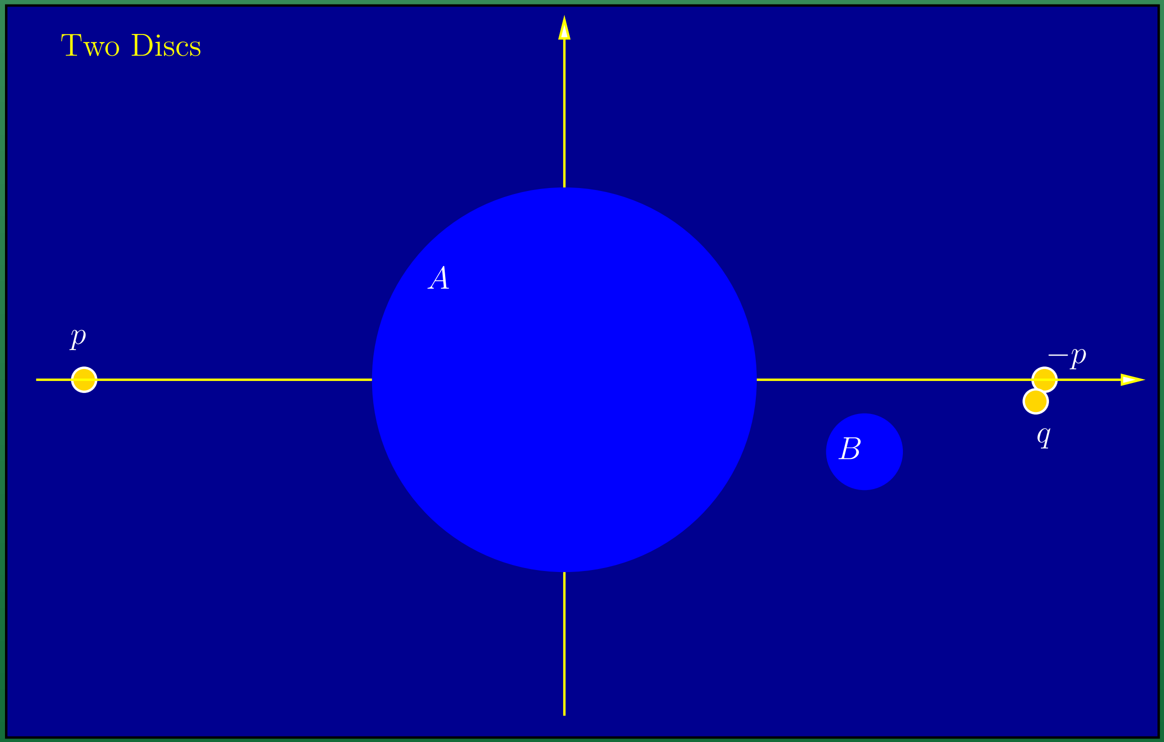
- 

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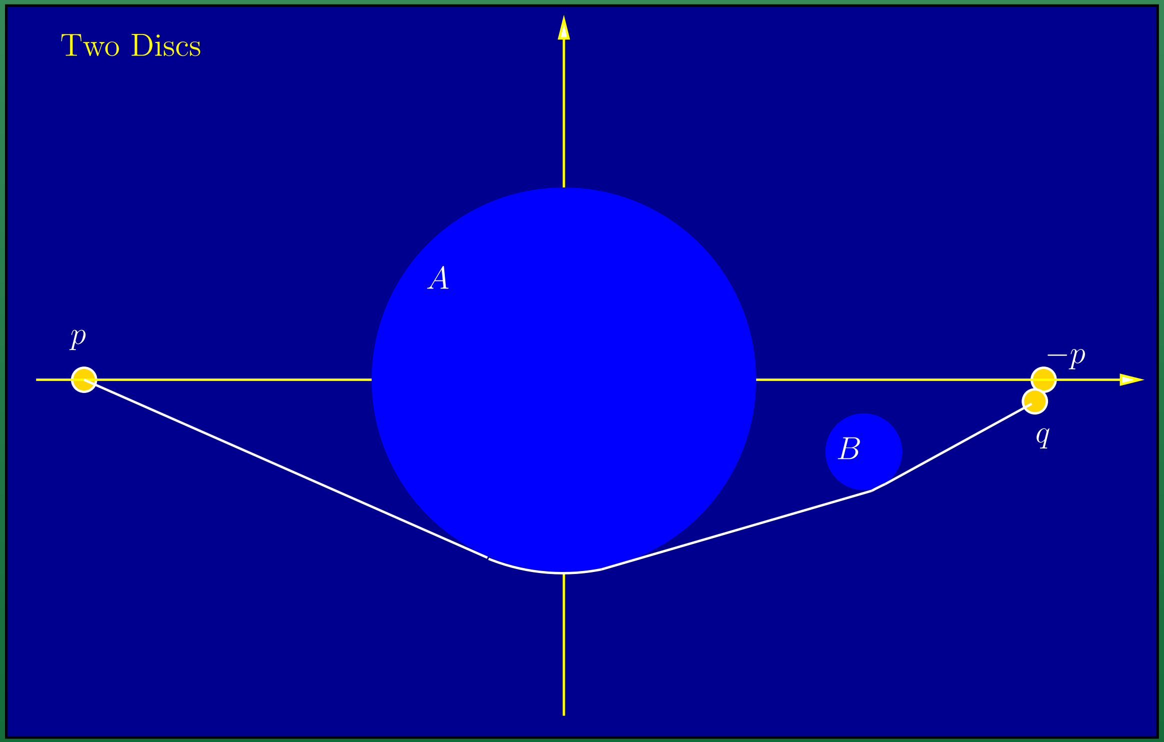

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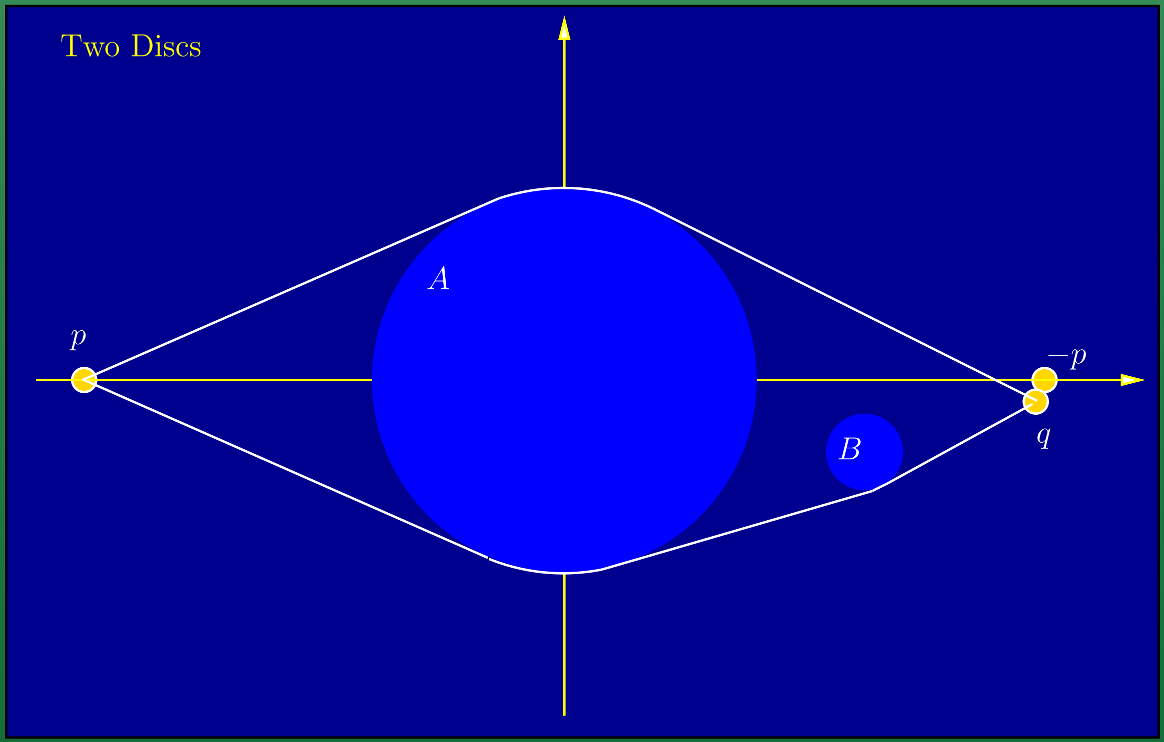
- 

Standard Solution: Reduce to Dijkstra's Algorithm

- Feasible paths: $\mu = \mu_1; \mu_2; \cdots; \mu_k$
  - ∗ μ_i is a straightline segment iff μ_{i+1} is an arc
  - ∗ Straightline segments are common tangents to 2 discs
- Apply Dijkstra's shortest path algorithm to a combinatorial graph $G = (V, E)$
- Size of G is $O(n^2)$ and algorithm is $O(n^2 \log n)$.

What is Wrong?

- Real RAM model assumed!
- Length of a feasible path is

$$d(\mu) = \sum_{i=1}^{k} d(\mu_i) = \alpha + \sum_{i=1}^{m} \theta_i r_i \qquad (1)$$

  * $\alpha \geq 0$ is algebraic
  * $0 < r_1 < \cdots < r_m$ are distinct radii of discs
  * θ_i is total angle (in radians) around discs of radii r_i

Is it really Transcendental?

- E.g., if $\theta = \pi$, then transcendental.
- LEMMA: $\cos\theta_i$ is algebraic
- COROLLARY (Lindemann): A non-zero θ_i is transcendental

Approach for Comparing Lengths

- Let $d(\mu) = \alpha + \theta$, and $d(\mu') = \alpha' + \theta'$
  - E.g., all discs have unit radius
- LEMMA: $d(\mu) = d(\mu')$ iff $\alpha = \alpha'$ and $\theta = \theta'$
- Hence, we need to ability to add arc lengths

Representation of Arc Lengths

- Let A be a directed arc of a circle C
  - Represent A by $[C, p, q, n]$.
- $$Val[C, p, q, r] \;=\; (\phi(p, q) + n\pi)r$$

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Representation of arc length by $[C, p, q, n]$.

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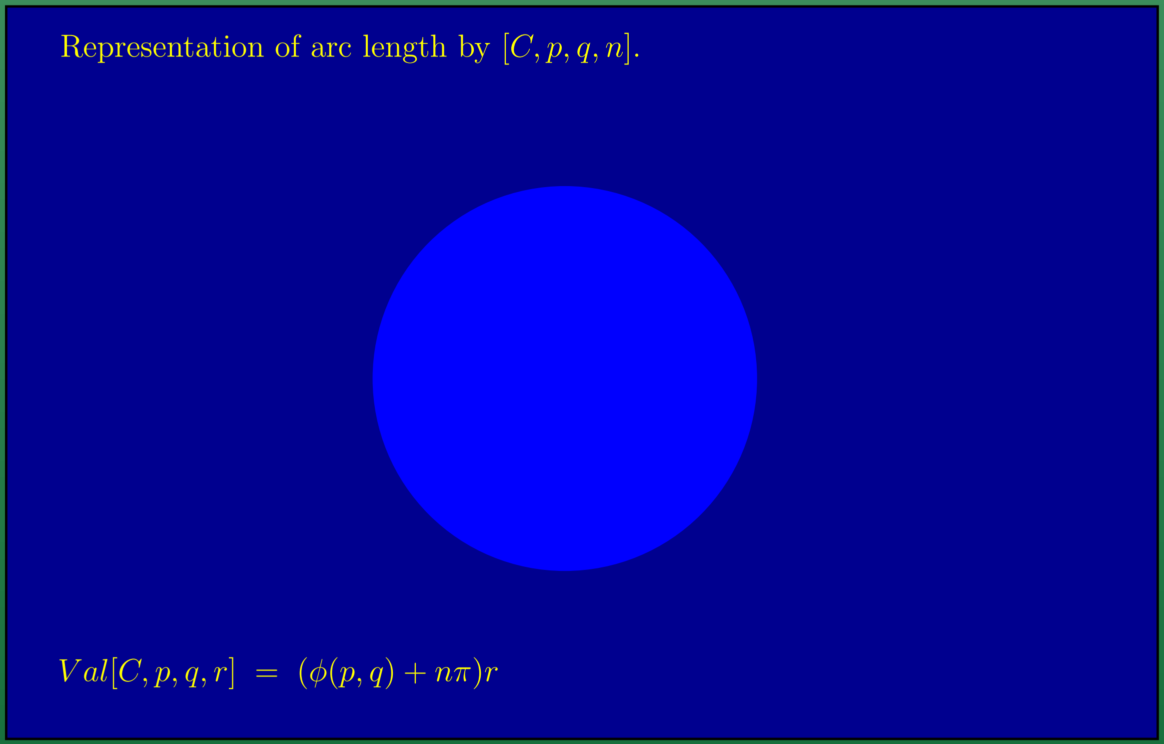

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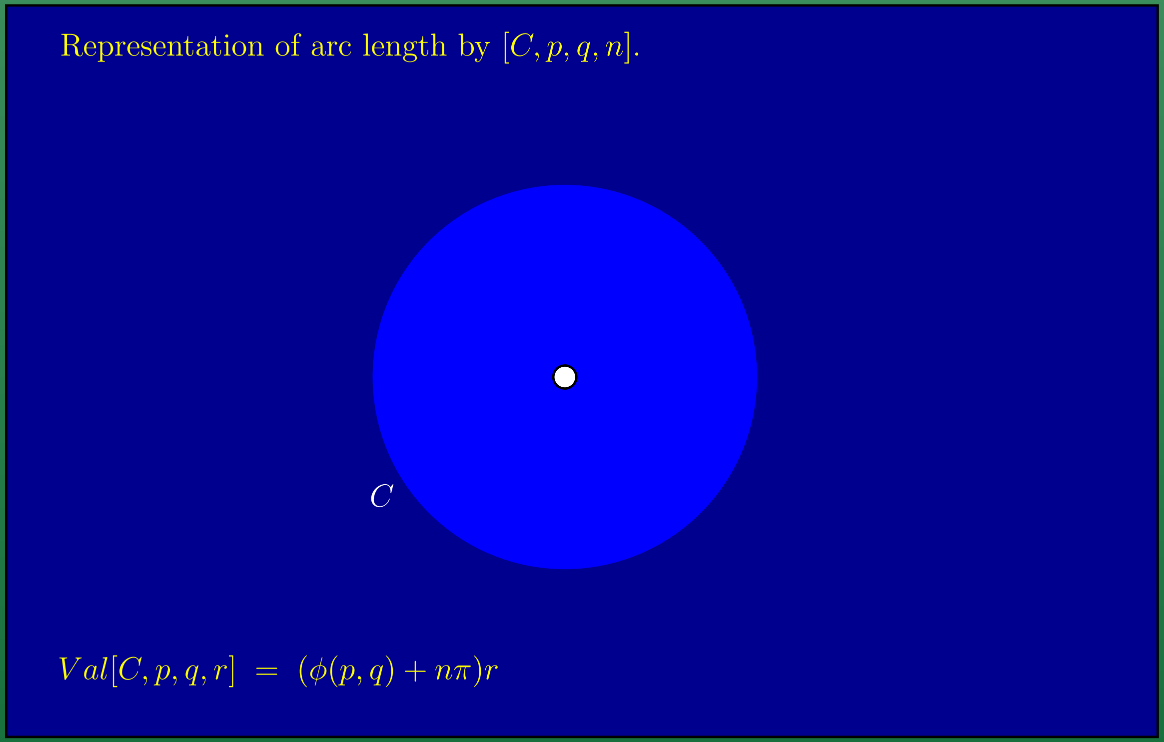

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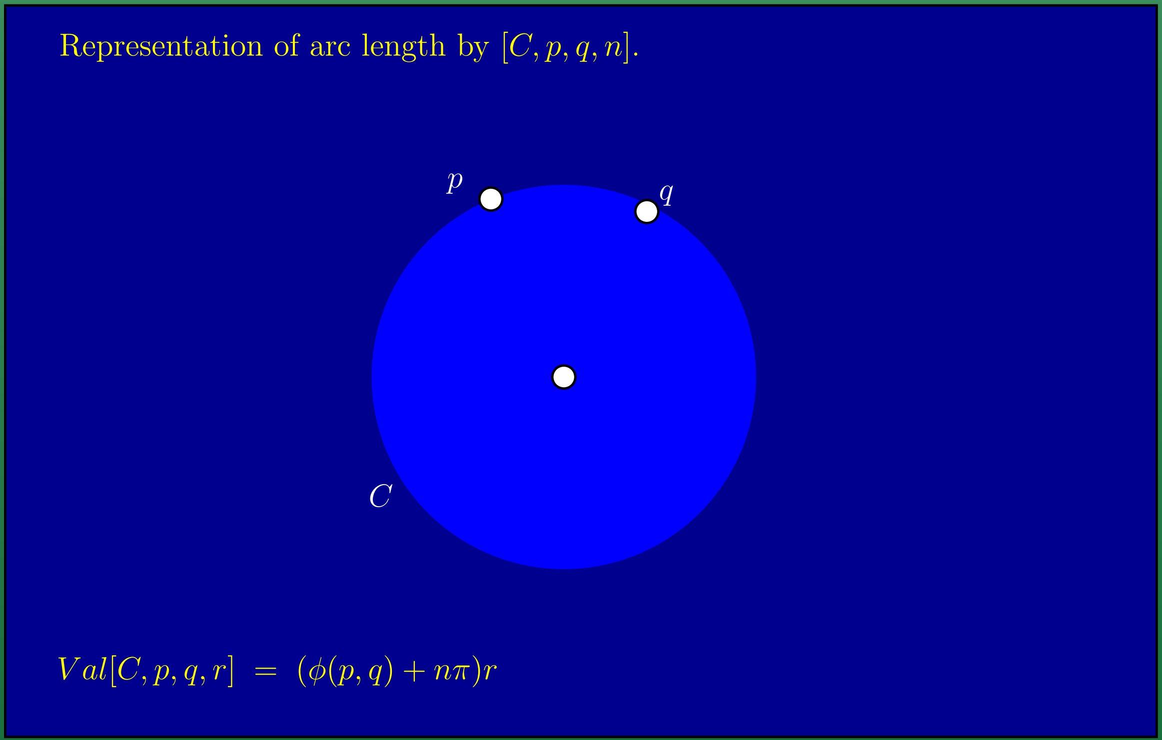

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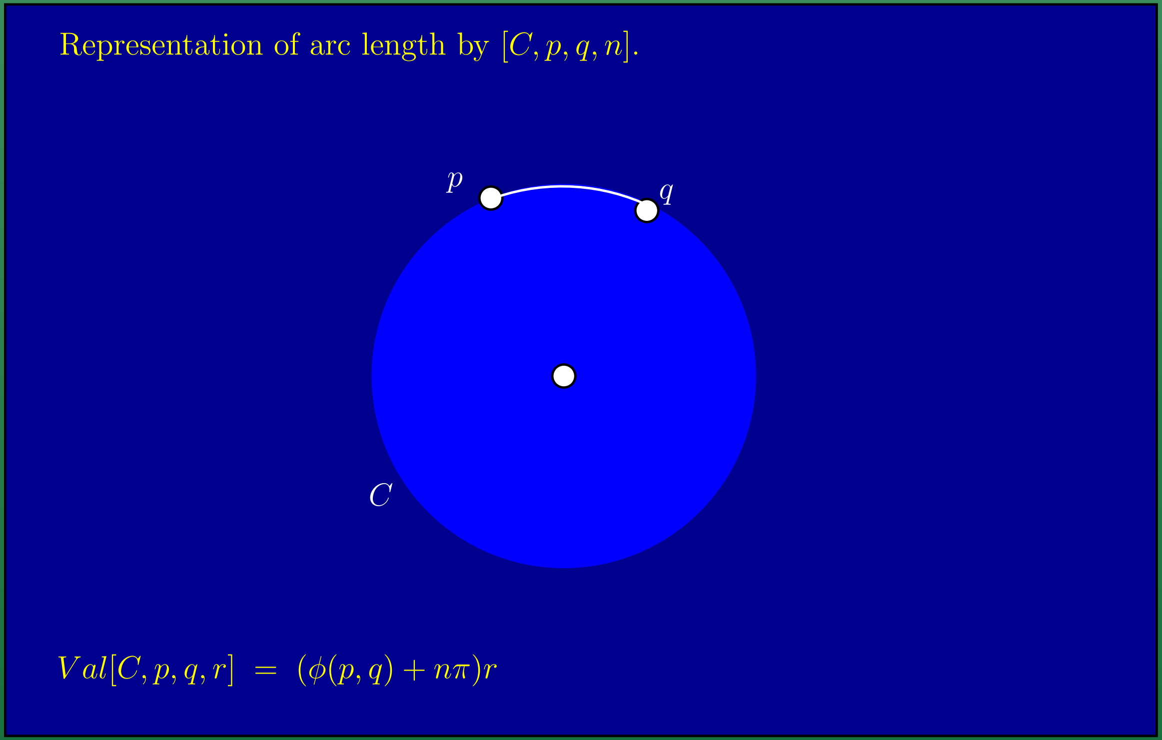

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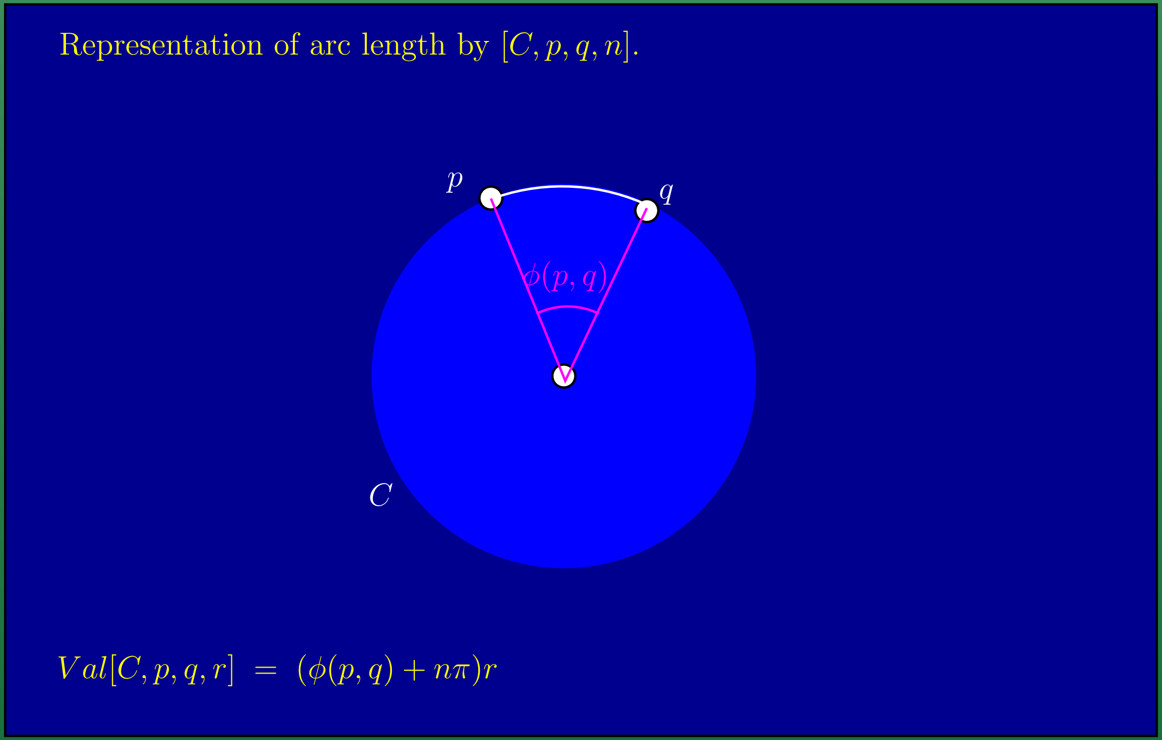

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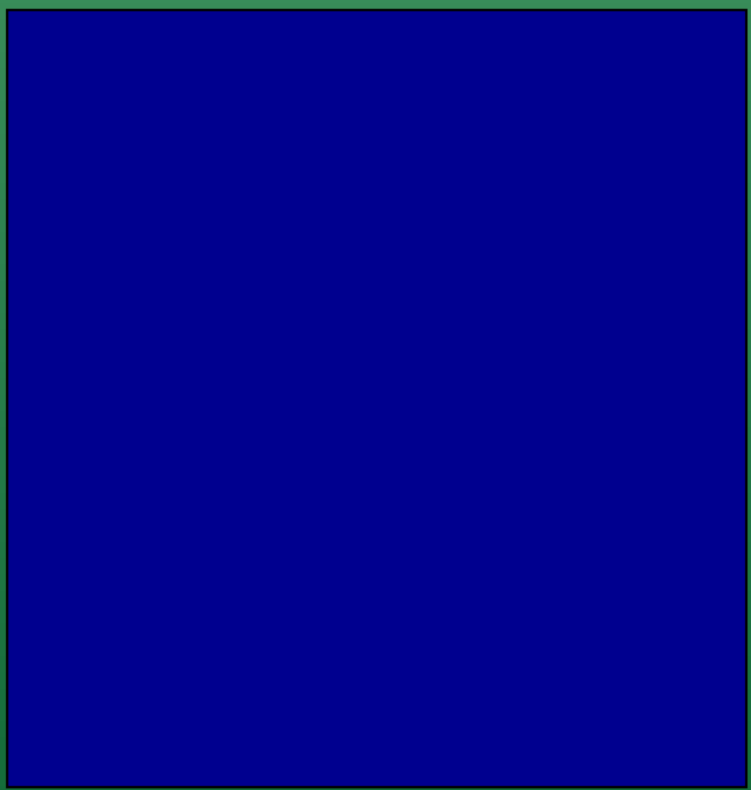
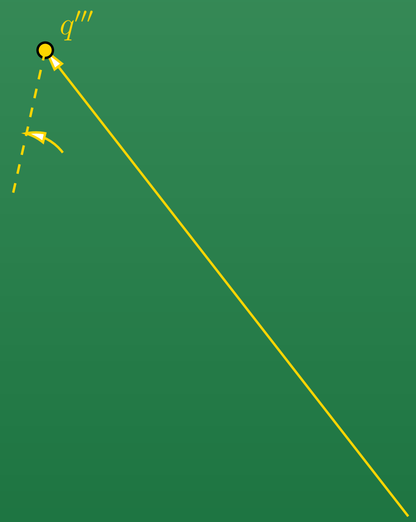
Addition of Arc Lengths

- Let $A = [C, p, q, n]$ and $A' = [C', p', q', n']$
  - * Say A and A' are compatible if $r(C) = r(C')$ and $q - o(C) = \pm(p' - o(C'))$
  - * Special case: line qp' is common tangent





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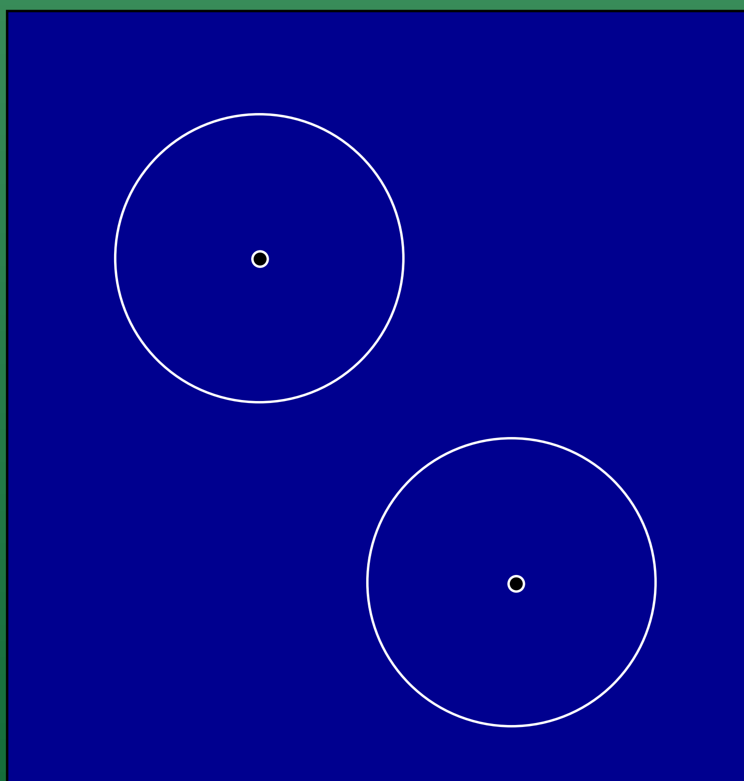

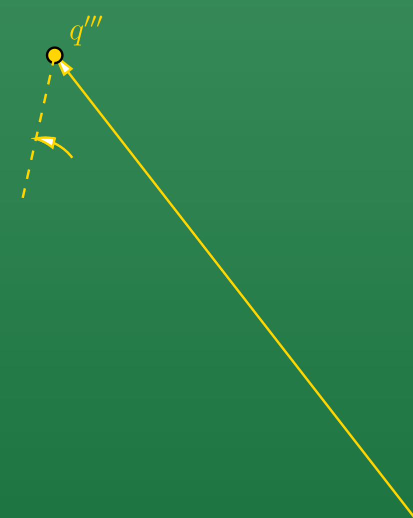

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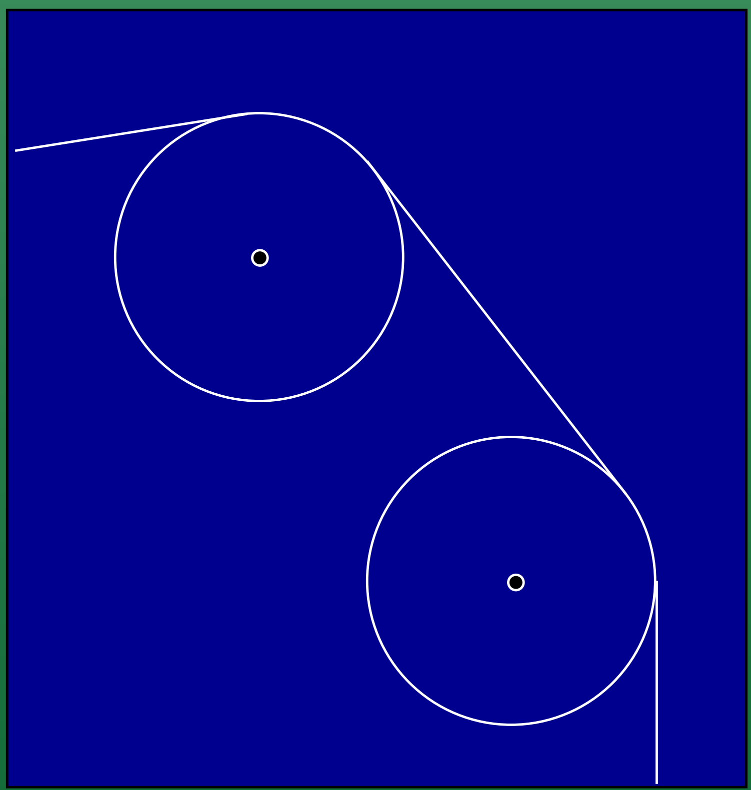

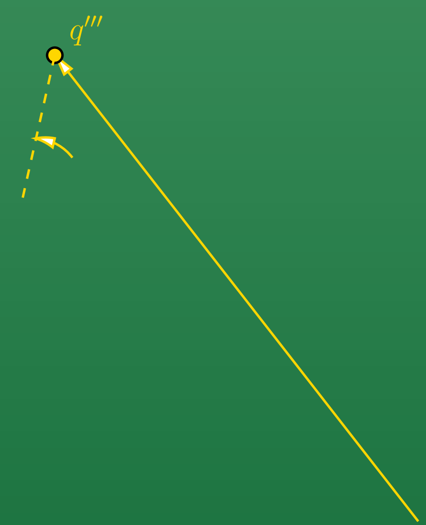

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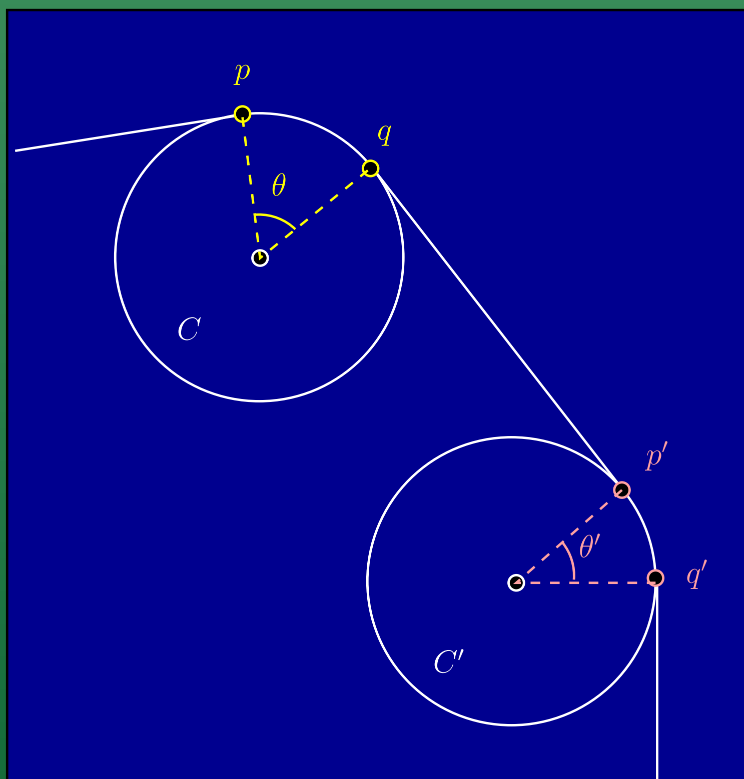

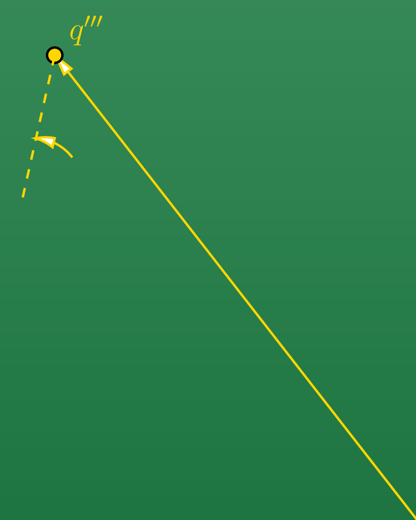

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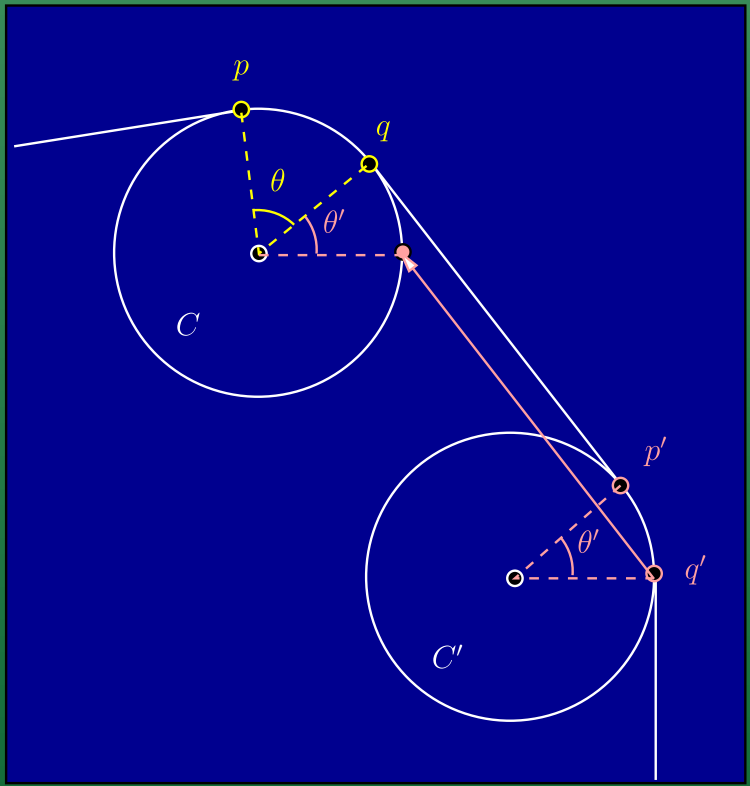

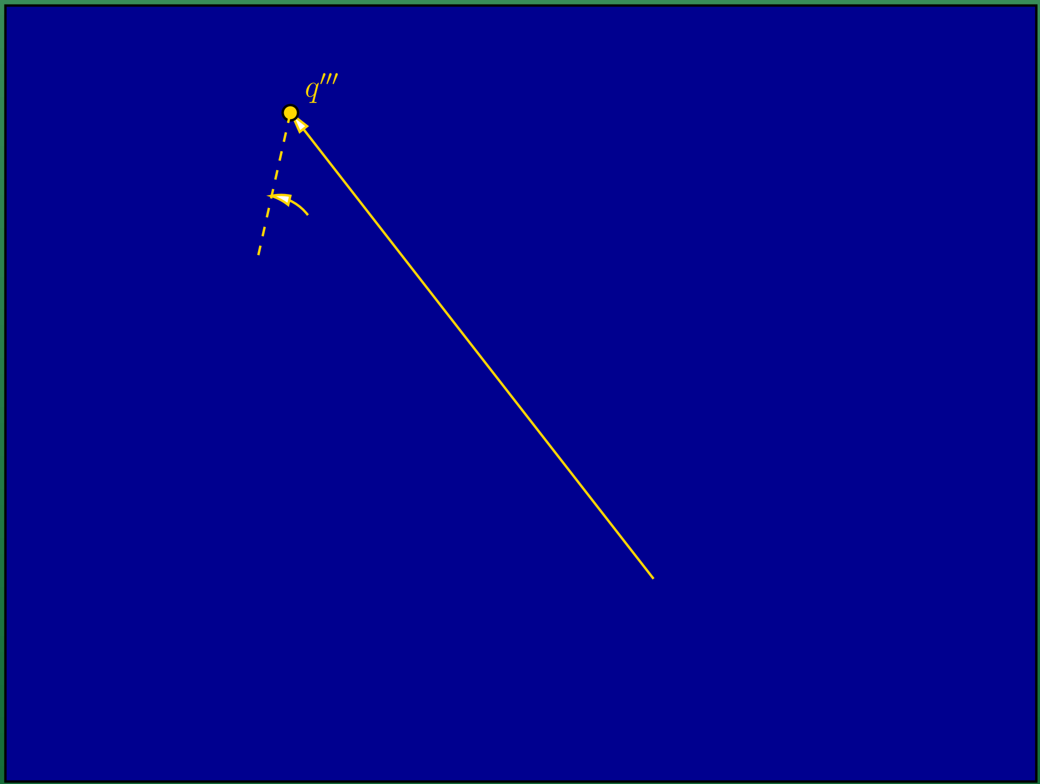

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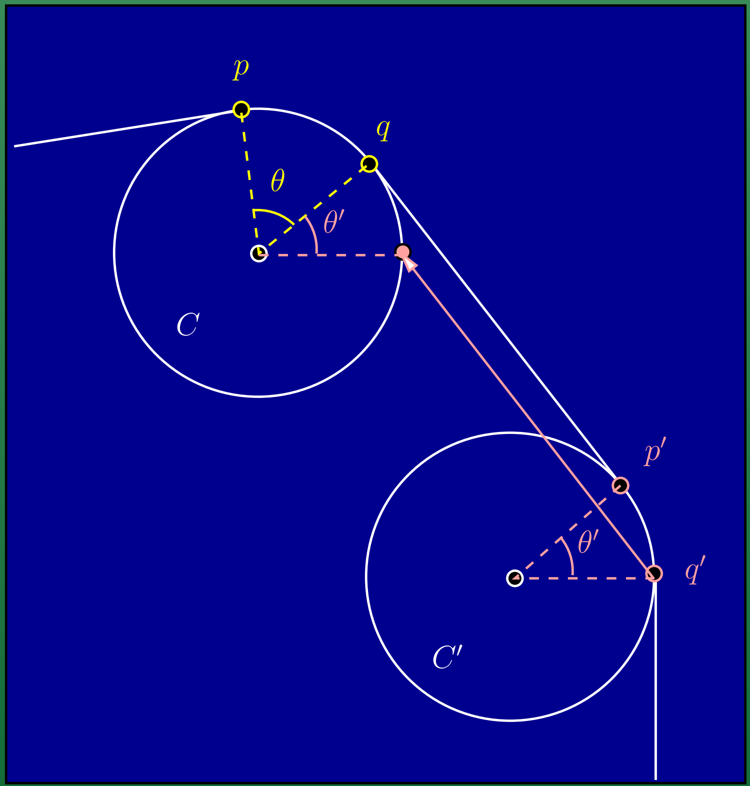

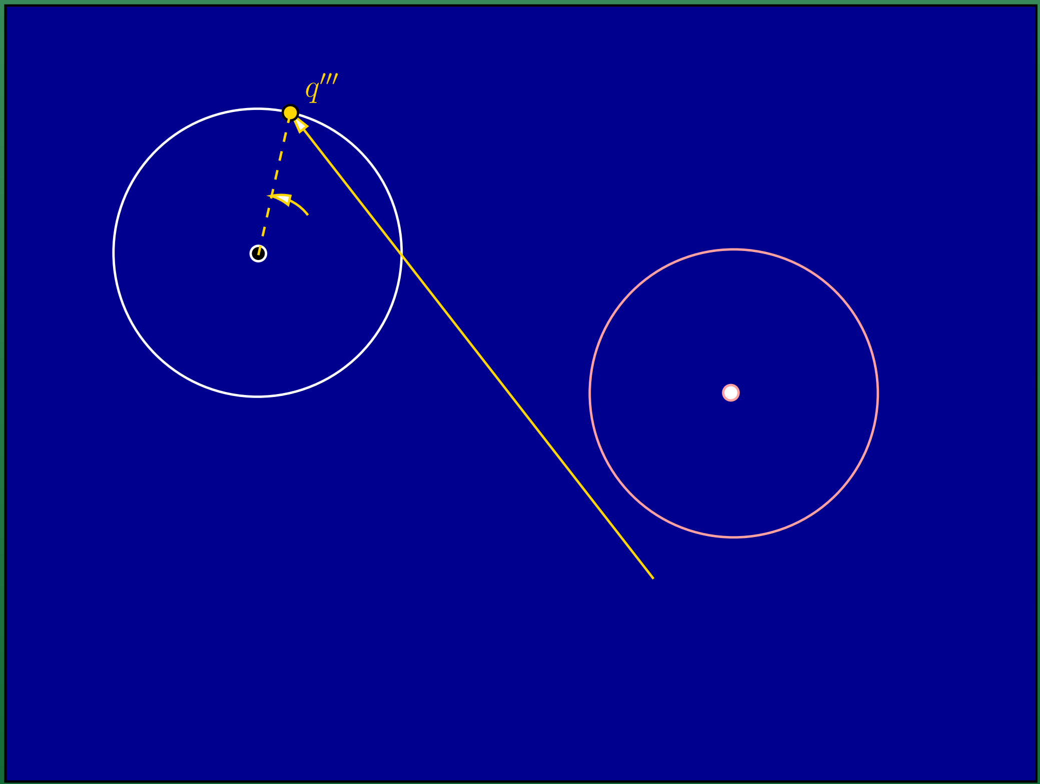

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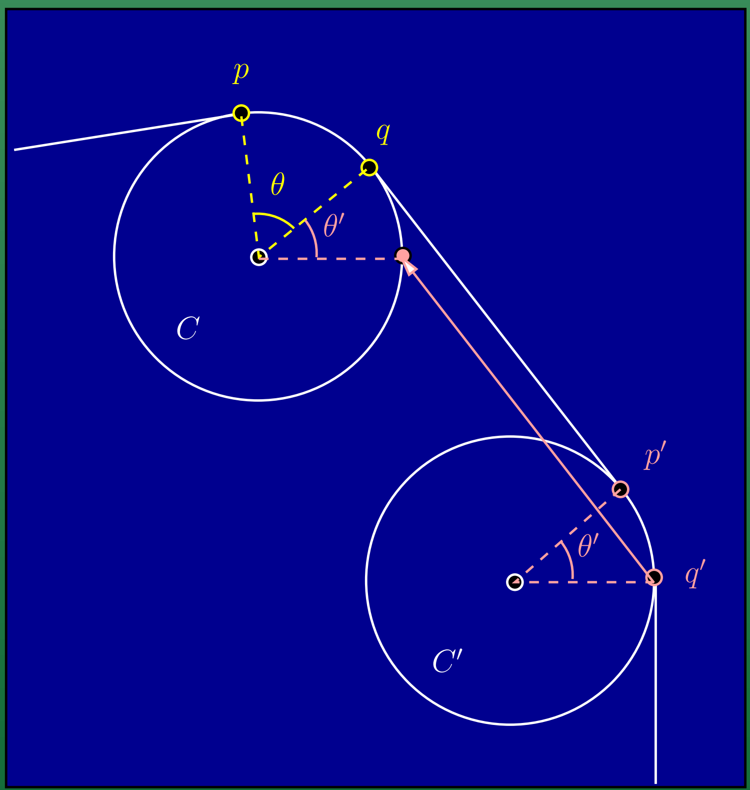

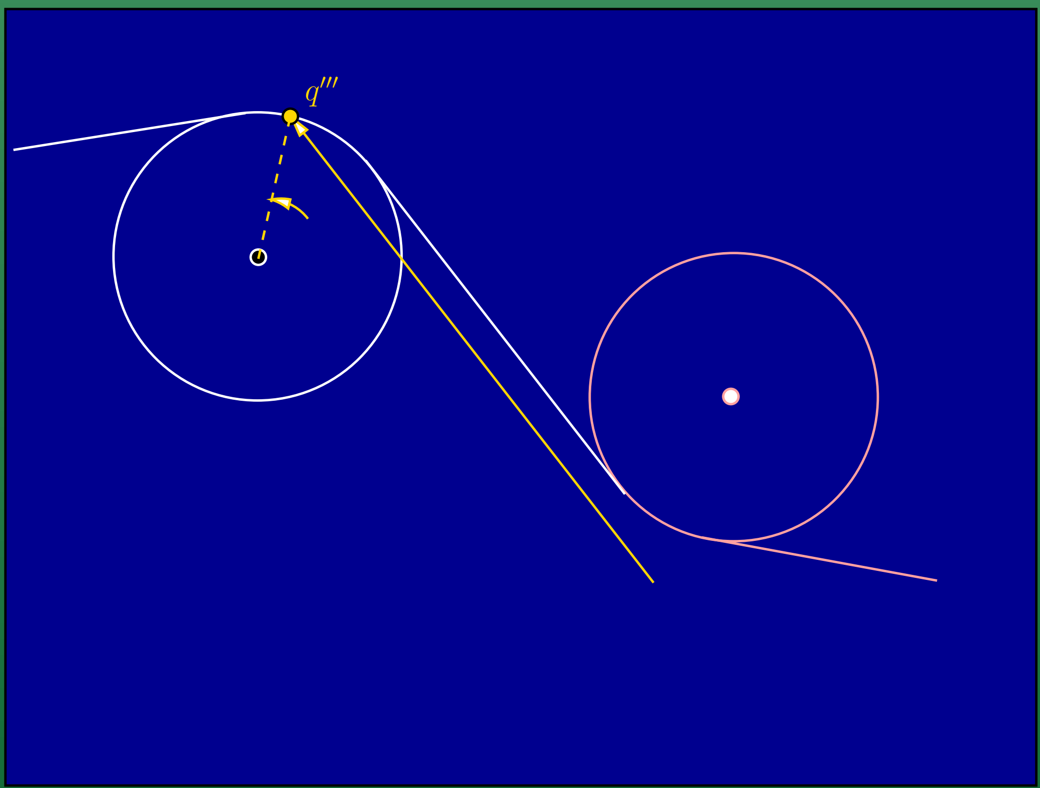

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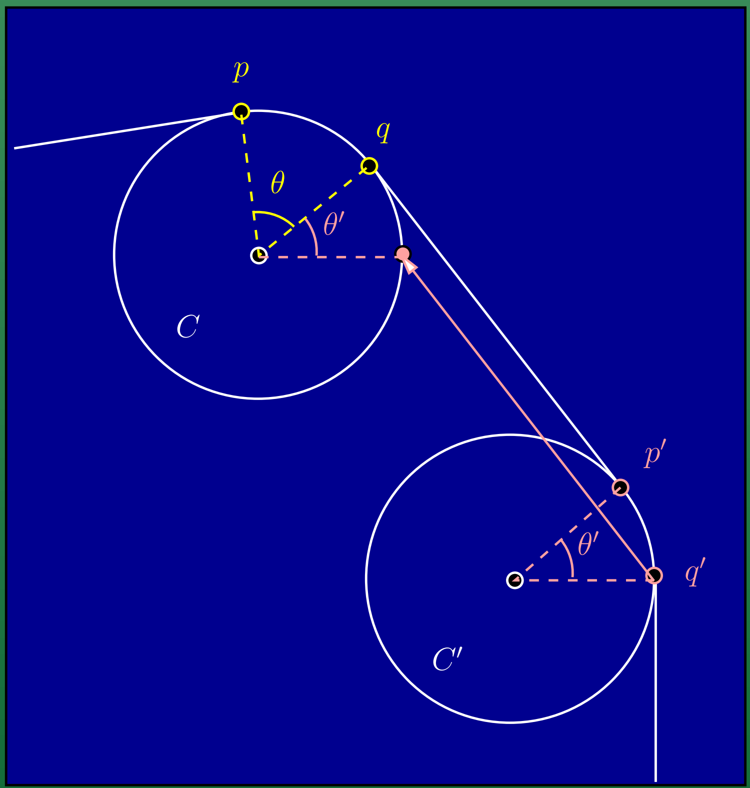

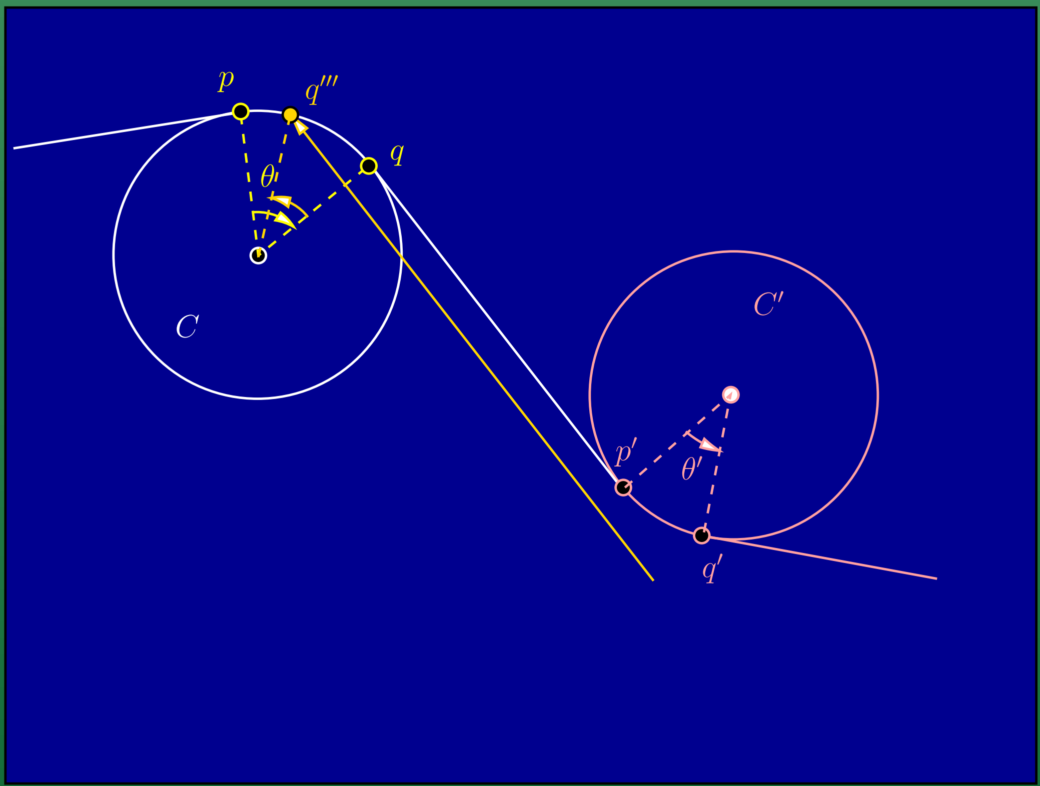

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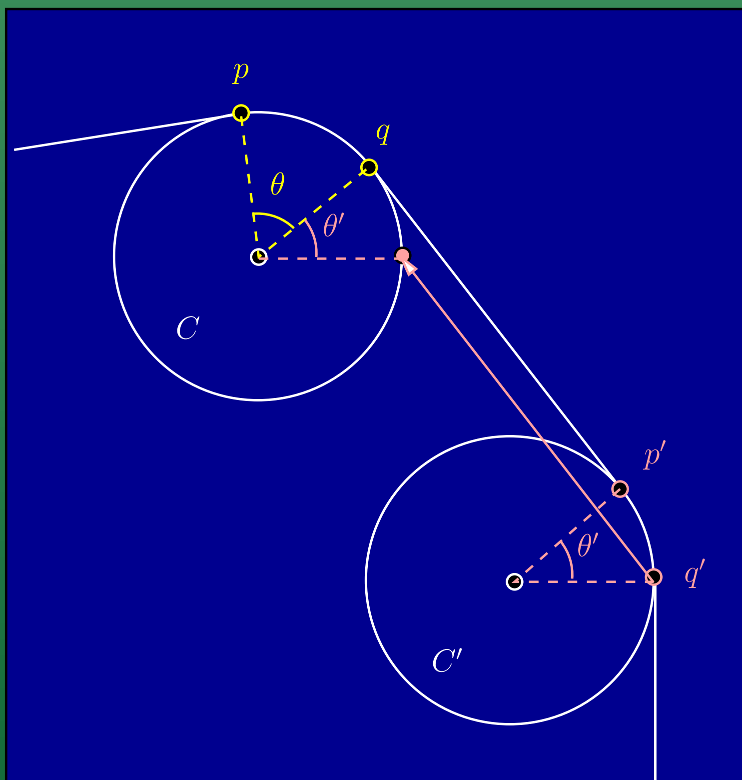

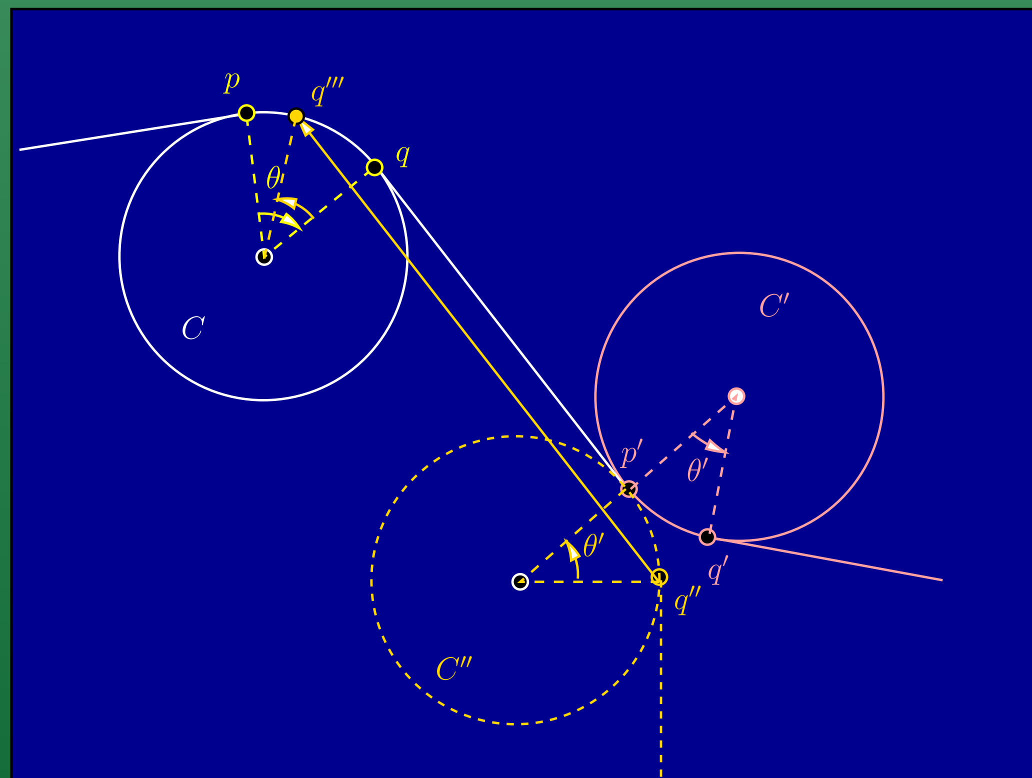

Decidability

- THEOREM: Shortest Path for unit disc obstacles is computable.

- Extensions:
  * When Radii of discs are "commensurable"
  * Complexity Bound?
  * Baker's Linear Form in Logarithms:

$$\left|\alpha_0 + \sum_{i=1}^{n} \alpha_i \log \beta_i\right| > B$$

- THEOREM: Shortest Paths for algebraic discs is computable.

- THEOREM: Shortest Paths for rational discs is in single exponential time.

Conclusions

- First computability result for a (combinatorially non-trivial) transcendental computational problem
- Positive Result from Transcendental Number Theory!
  - Also: Lyapunov (1955)
- Open Problems:
  - Extend to ellipse obstacles
  - Extend to sphere obstacles
- Other examples of transcendental problems
  - Helical motion in robot motion planning

EXERCISES

- Assume n is not a square. Generalize the usual proof for $n = 2$ to show $\sqrt{n}$ is irrational when n is even
  - $*$ Try to extend to odd n

- Locate the zero problem for the following:

  $*$ There is a point p that is rotating with constant angular velocity about the origin O.

  $*$ A unit disc D is translating with known constant velocity.

  $*$ You want to decide whether p collides with D

REFERENCE

- "Shortest Paths for Disc Obstacles is Computable"
  - E.Chang, S.Choi, D.Kwon, H.Park, C.Yap. 21st SoCG, 2005.

"A rapacious monster lurks within every computer, and it dines exclusively on accurate digits."
– B.D. McCullough (2000)

THE END