Lecture 5
Numeric-Algebraic Computation with Curves

Chee Yap
Courant Institute of Mathematical Sciences
New York University

Overview

We introduce some basic concepts of algebraic curves and their computation. There is a general algebraic technique called cylindrical algebraic decomposition (cad). Such techniques are too slow even in the plane. We seek more adaptive techniques. We describe one such algorithm, for Bezier curves.

- 0. Review
- I. Cylindrical Algebraic Decomposition
- II. Bezier Curves
- III. Quadric Surfaces

0. REVIEW

QUESTIONS and DISCUSSIONS

- PROBLEM: You want to find all real solutions of the following "triangular system", $P(X) = 0, Q(X, Y) = 0$, numerically:
  - For each zero α of $P(X)$, find all β of $Q(X, Y)$.
  - REMARK: First figure out how to do this non-numerically
- PROBLEM: Suppose you want to plot a curve. Use resultants to compute points on the curve?
  - Can your approach resolve the topology of curves?
  - REMARK: This is implemented in CORE

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Fundamentals of Algebraic Computation

- Algebraic numbers form a (computational) field
  - Tradition algorithms (in computer algebra) use representation by minimal polynomials, or by isolating intervals
  - In contrast, we use numerical approach via Expressions
- Resultant is a main tool to derive basic properties of algebraic numbers, including zero bounds
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  - Use of bigfloats is essential
- In numerical computation, the local complexity of bigfloats computation is essentially $O(M(n)\log n)$, from Brent
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I. CYLINDRICAL ALGEBRAIC DECOMPOSITION

Skipped for time

II. Curves

Complete Subdivision Algorithm for Intersecting Bezier Curves

- There are two distinct approaches: algebraic and analytic
- In algebraic view, a curve is basically given by a bivariate polynomial $A(X, Y) \in K[X, Y]$.
- The analytic approach views curves as a parametrized curve $C(t)$. The emphasis is in differential properties and local properties of curves.
- One confusing aspect is that when we view curves in the complex setting, the curve is topologically a

surface! So the two terminology gets mixed.

- For this lecture, we will focus on a recent new algorithm for intersecting a very special class of curves: Bezier curves.
- Through this algorithm, we will expose many of the issues from our perspective of doing algebraic computation via numerical approximations.

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- Separation Bounds for Algebraic Curves
- Tangency Criterion for Elementary Curves
- Sub-Algorithms
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I. INTRODUCTION

Two Approaches to Curve Intersection

- Basic Problem: intersecting algebraic curves
- Two distinct approaches in literature:

	"Algebraic View"	"Geometric View"
1. Representation	polynomial equations	parametric form
	complete curves	curves segments
2. Techniques	symbolic/algebraic	numerical
	cell decomposition	homotopy, subdivision
3. Algorithms	exact, slow	inexact, fast
	theoretical	practical
	non-adaptive	adaptive

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Related Work

- Recent work:
  - Exacus Project, CGAL, etc
  - Arrangement of low-degree curves and surfaces
  - Devillers et al [SCG'00], Geissmann et al [SCG'01], Berberich et al [ESA'02], Wein [ESA'02], Eigenwillig et al [SCG'04], etc
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  - Make algorithms under the "Geometric View" robust
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  - $P(F) = (p_0, p_1, \ldots, p_n)$
  - De Casteljau's Algorithm to determine $F(1/2)$
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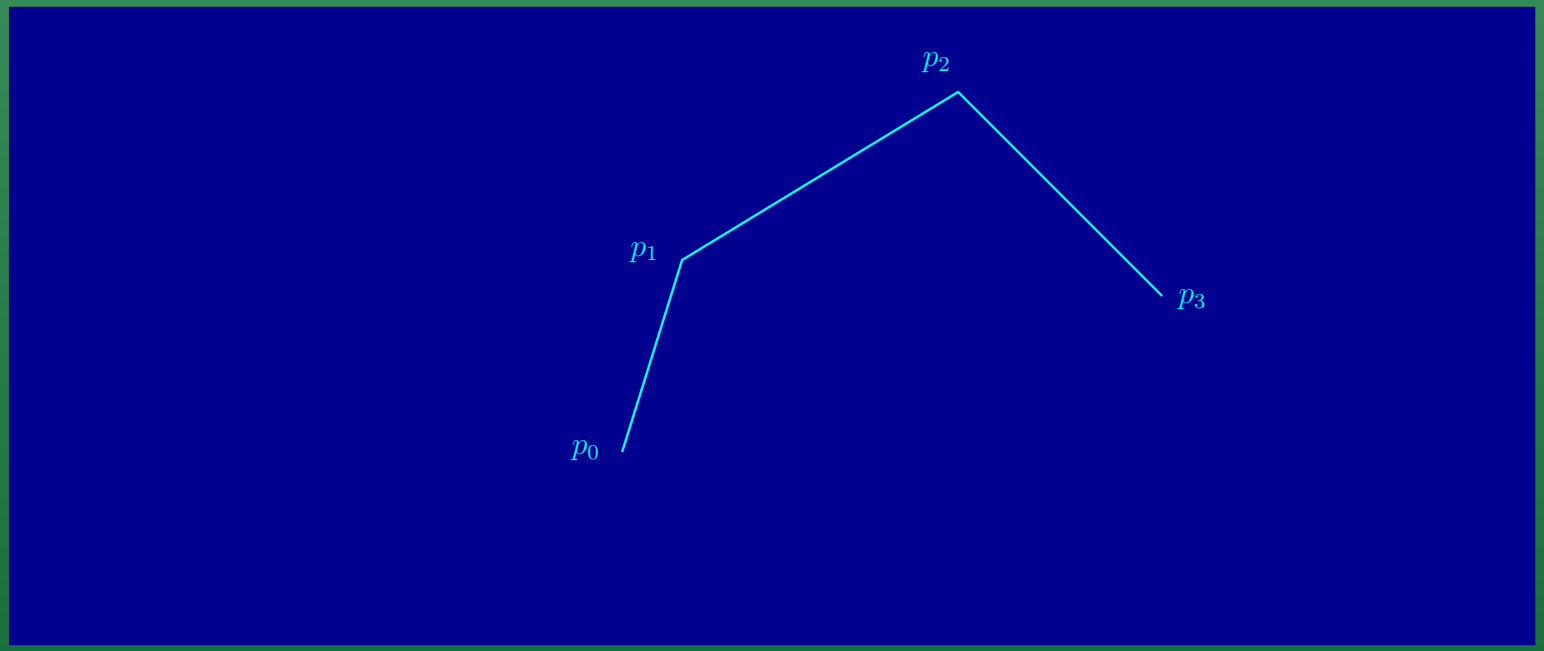

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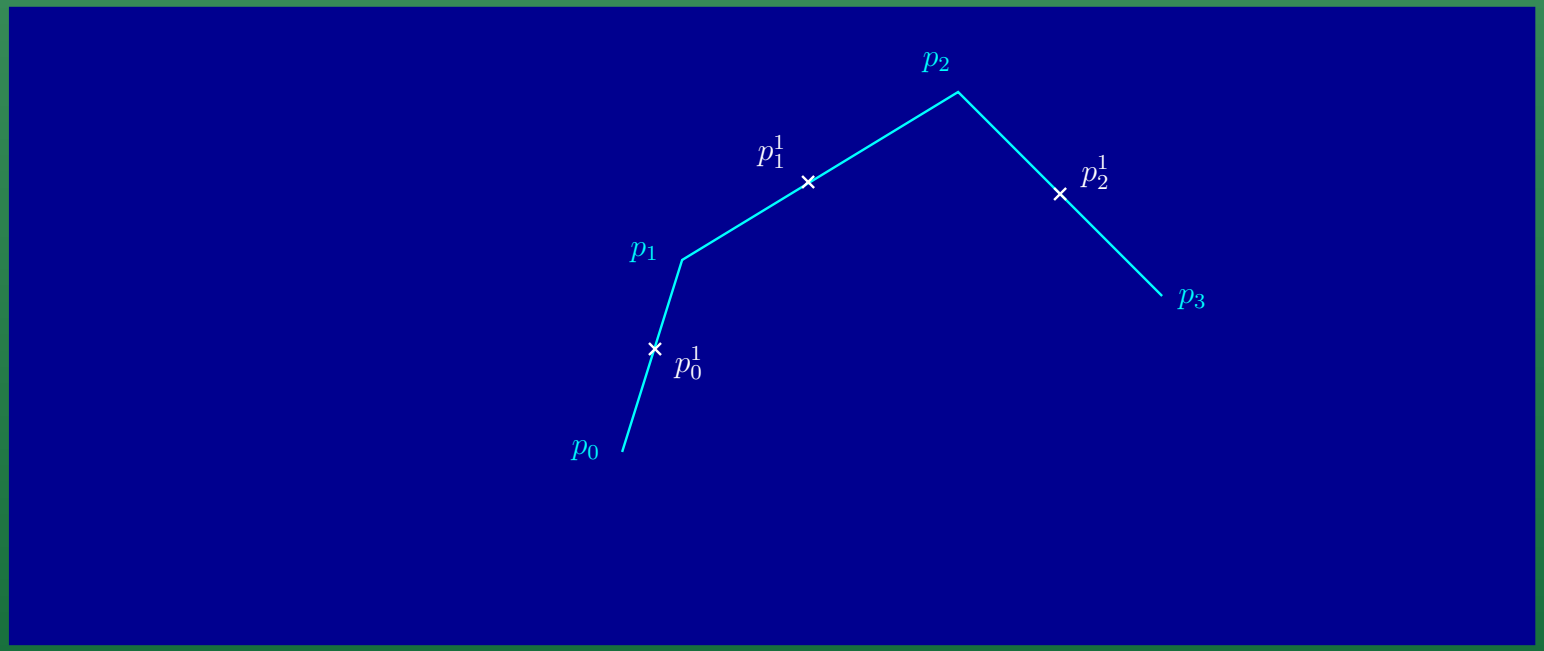

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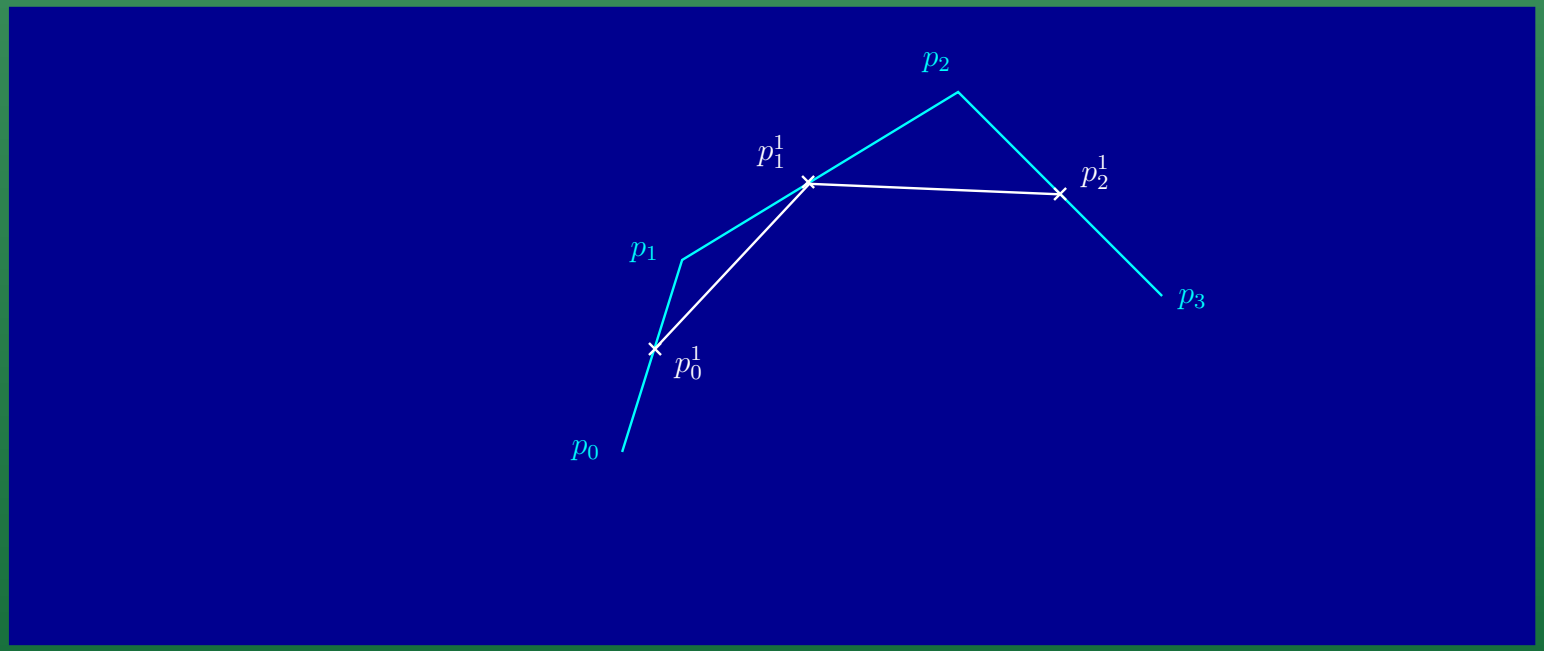

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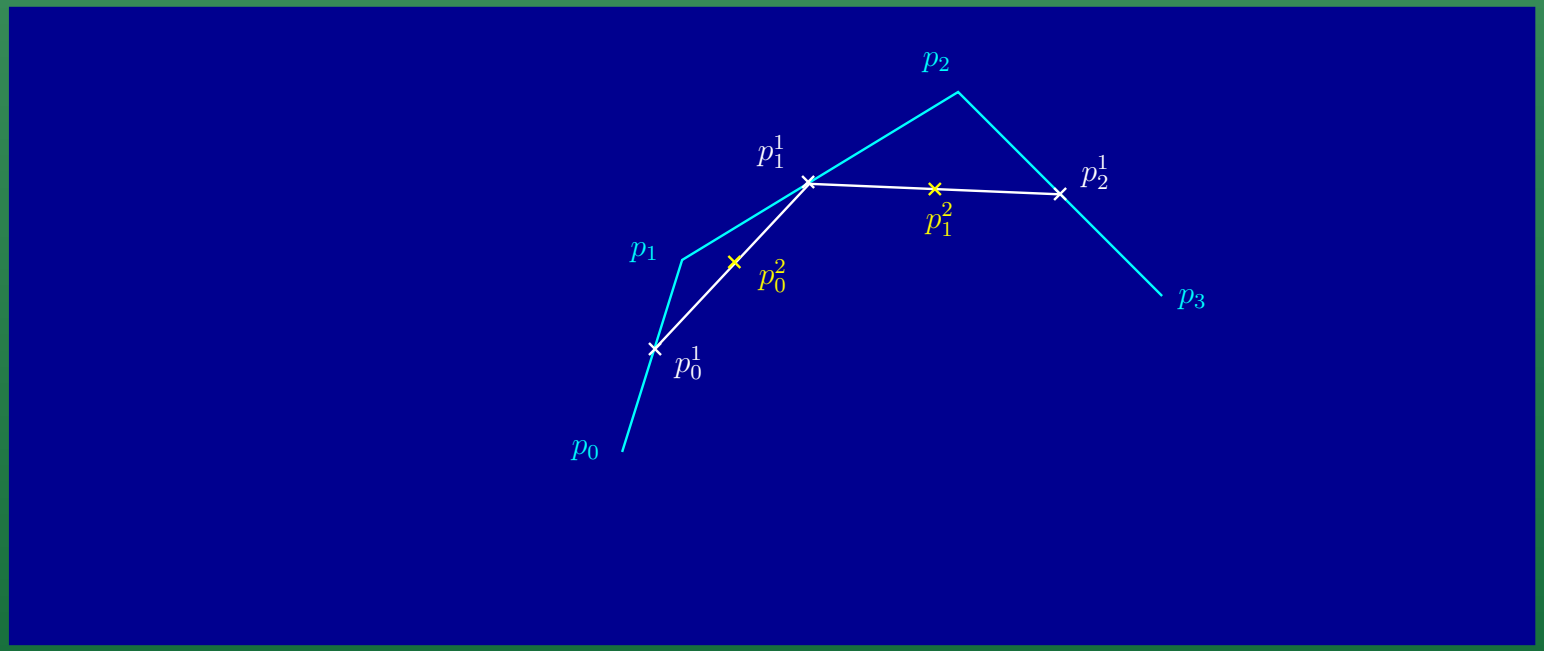

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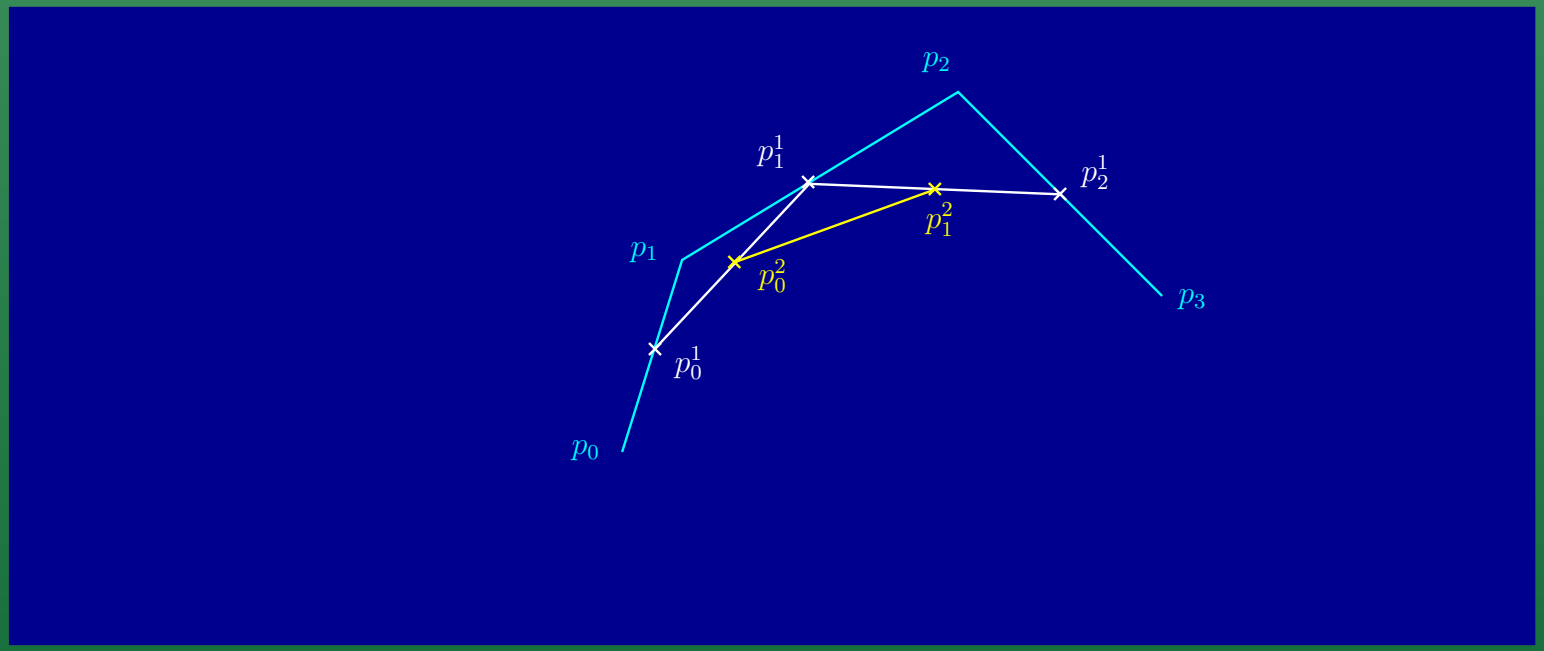

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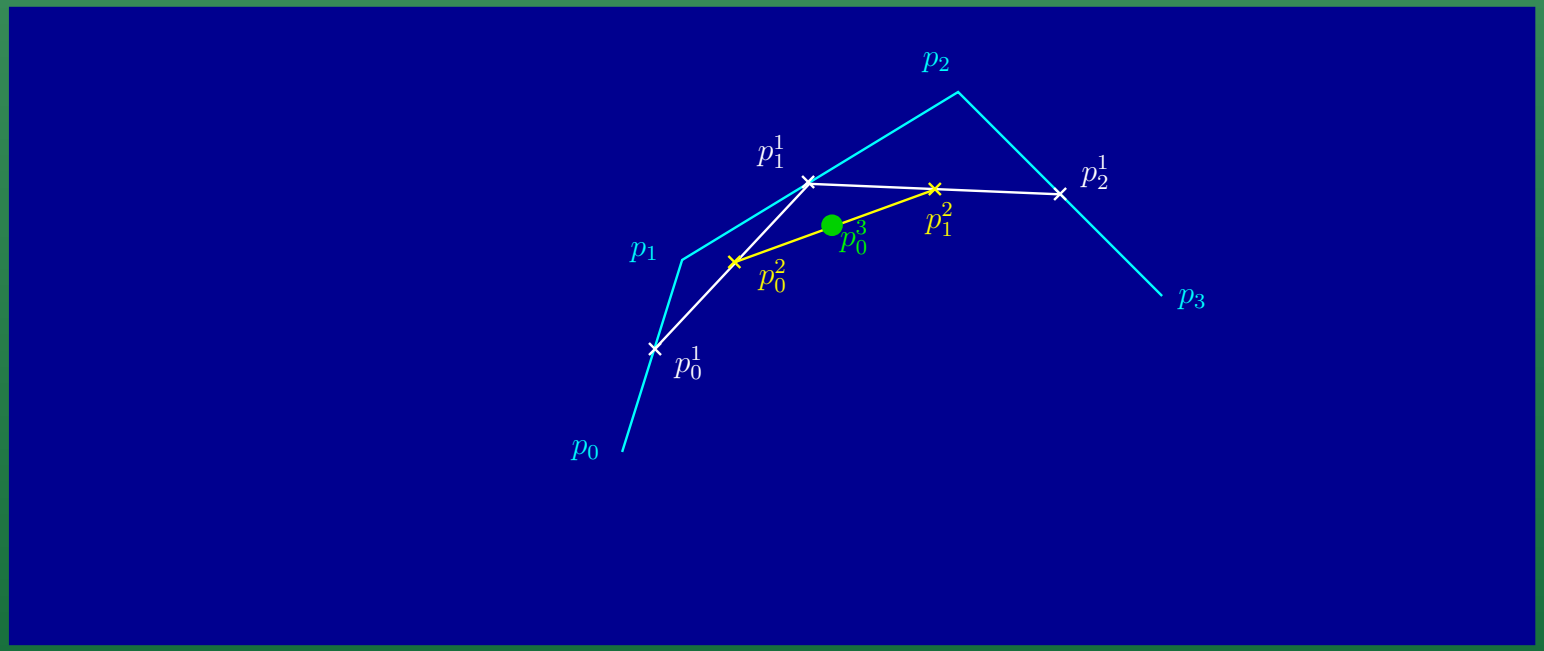

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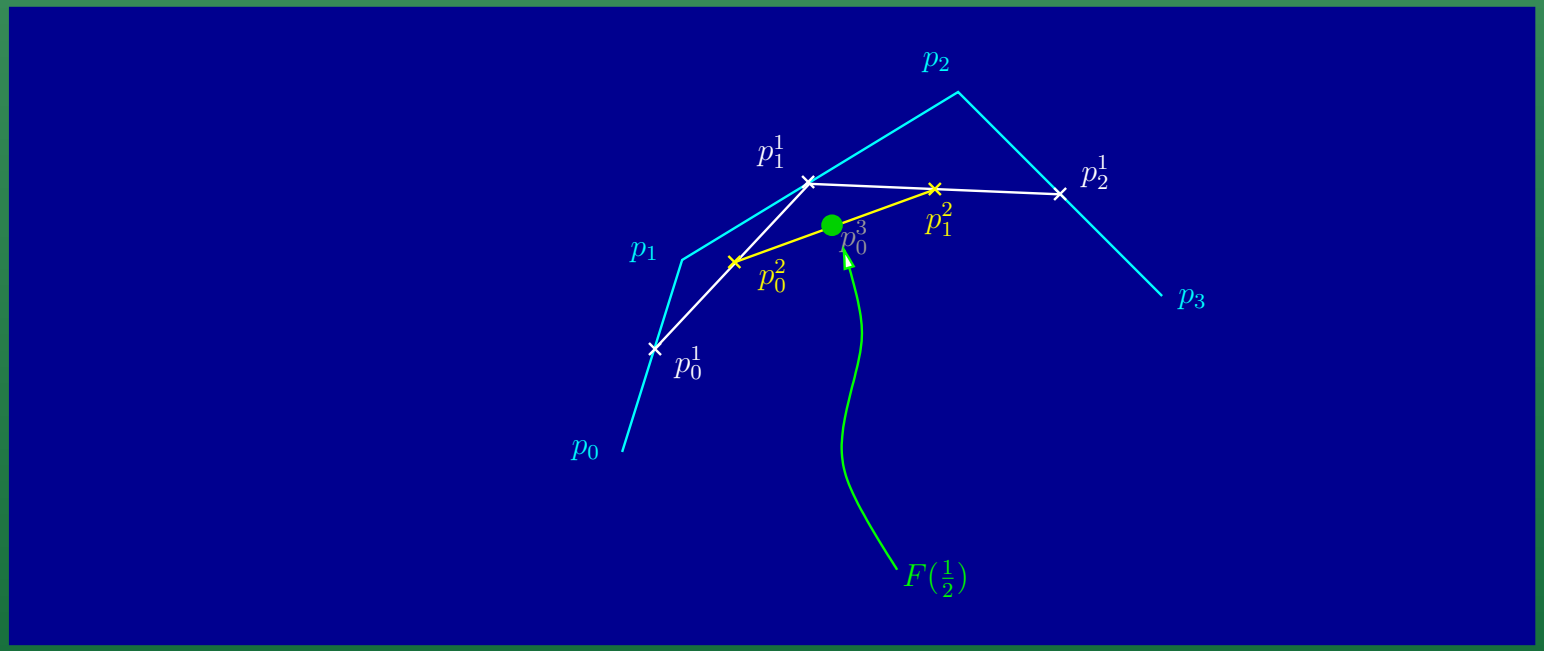

Intersection of Bezier Curves

- Generic Algorithm to intersect Bezier curves F, G:

  [1] If $CH(P(F)) \cap CH(P(G)) = \emptyset$, return(NO)

  [2] If diameter($P(F) \cup P(G)$) $< \varepsilon$, return(YES)

  [3] Split the larger curve (F) into subcurves (F_0, F_1)

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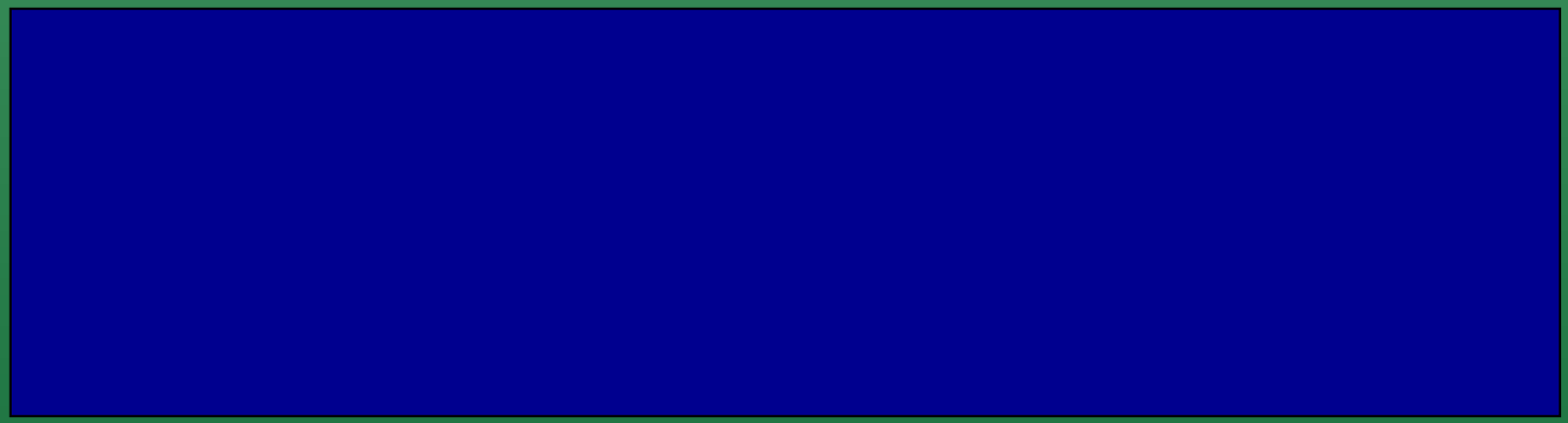

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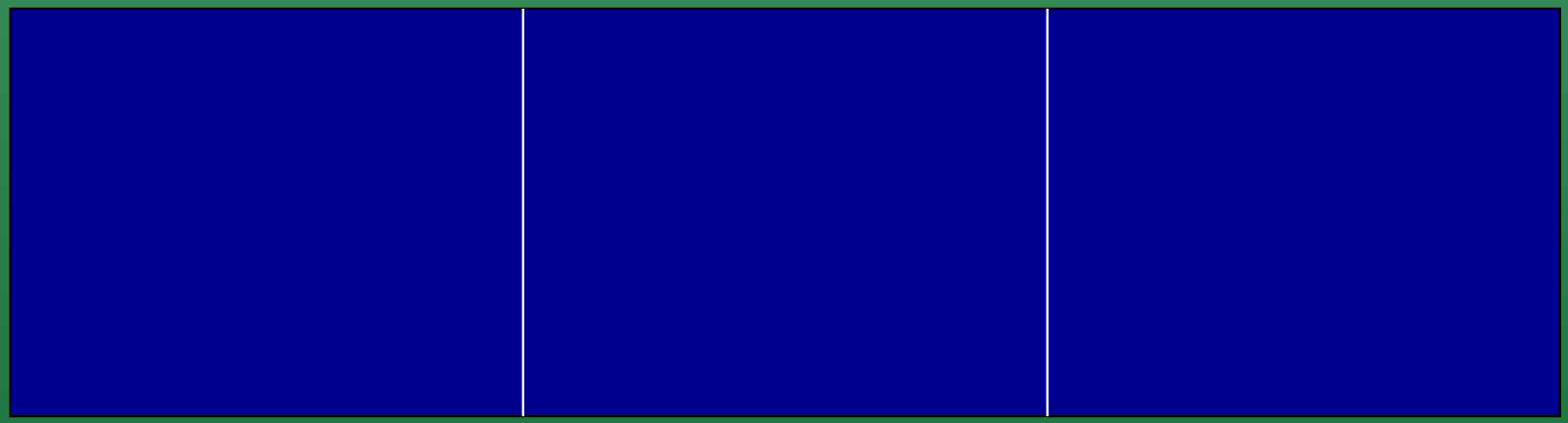

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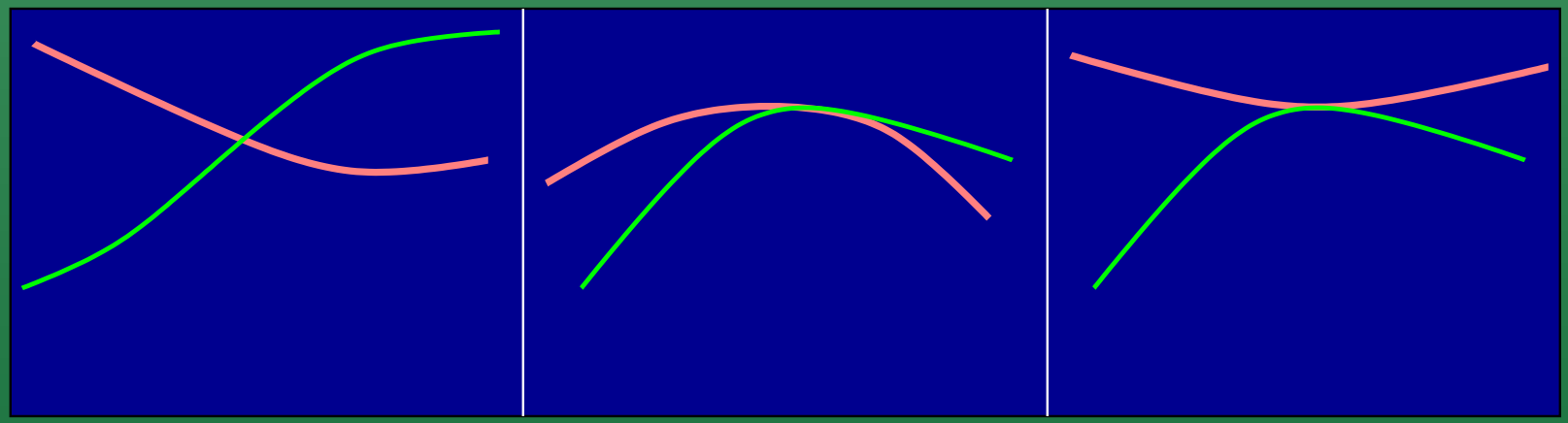

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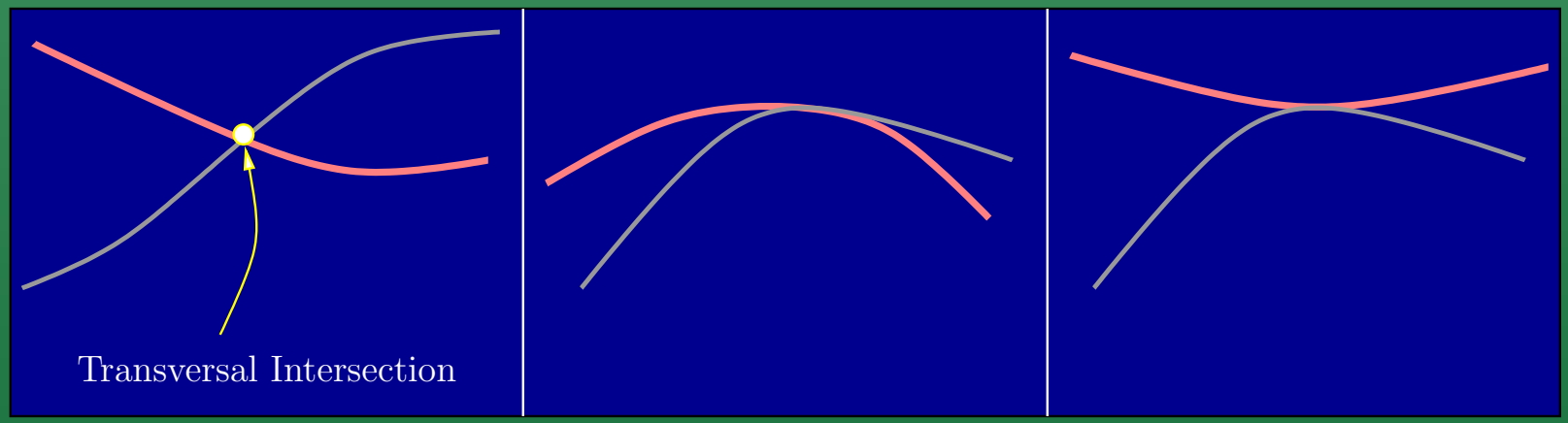

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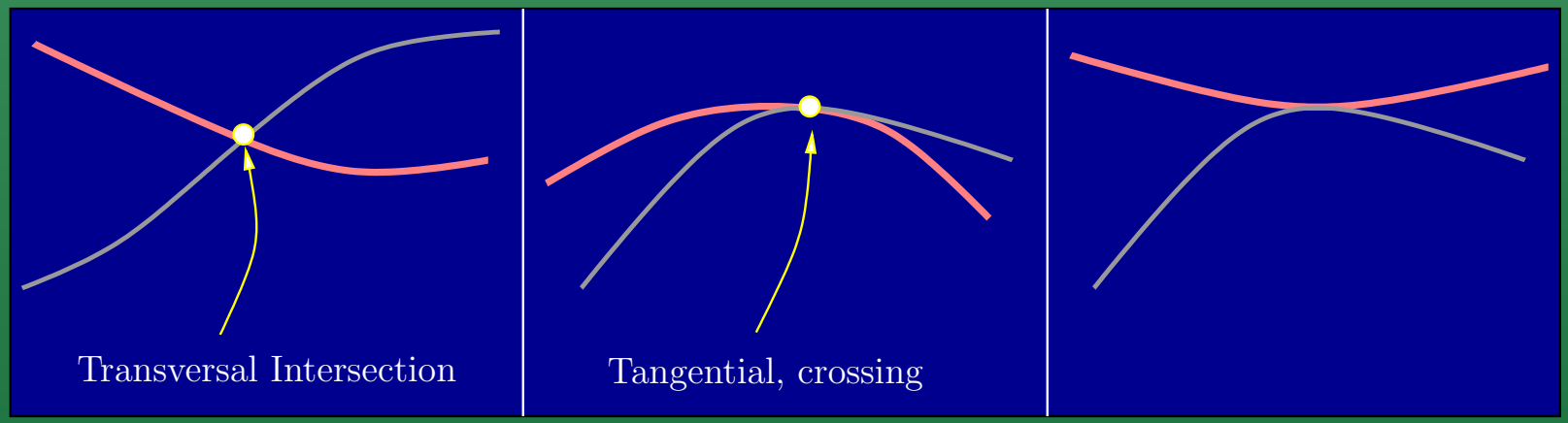

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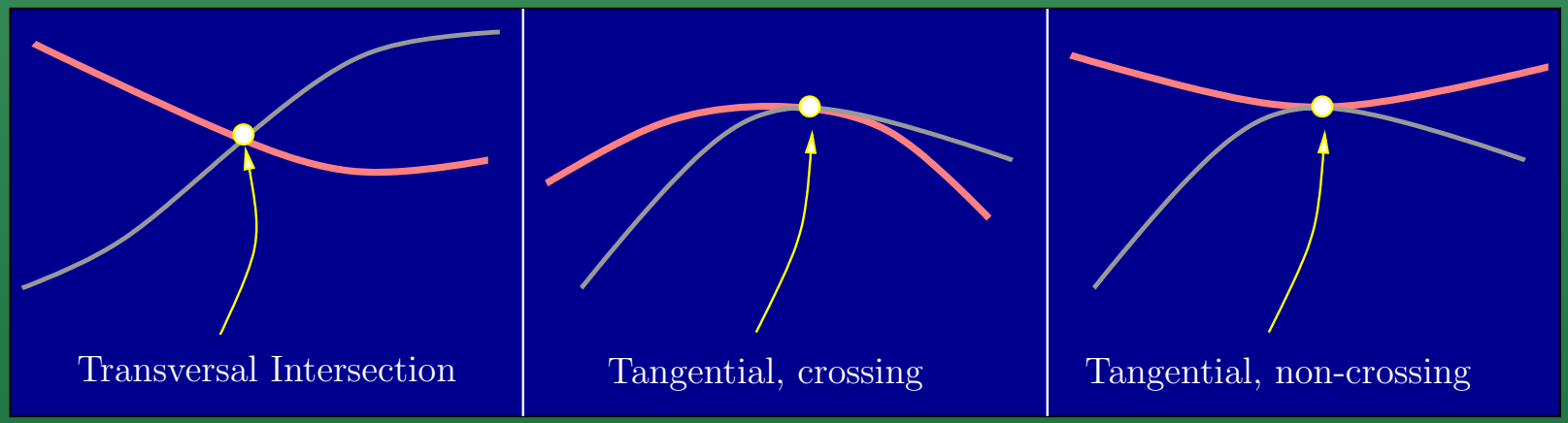

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- Intersection Criteria
  - Complete criterion: output YES/NO
  - Semi-criterion: output YES/NO/MAYBE
  - Semi-criteria are useful
- No complete criterion is known for noncrossing intersections

* How to ever affirm a noncrossing intersection?

Work of Nicola Wolpert

- If F, G are non-singular, how can we affirm a tangential intersection within a box?
  - Use Jacobi curves, $H_1 = F_x G_y - F_y G_x = 0$
  - Need generalized Jacobi curves, $H_1, H_2, \ldots$
- Comparison of Techniques:
  - Wolpert: Jacobi curves, Resultant computations
  - Ours: only subdivision

II. SEPARATION BOUNDS FOR CURVES

Main Algebraic Tool

Assumption for curves F, G

- F, G are the curves $A(x, y) = 0$ and $B(x, y) = 0$,
  - $m = \deg(A), \quad n = \deg(B)$
  - $a = \|A\|_2, \qquad b = \|B\|_2.$
- Definition of antipodal pair (p, q):
  - $p \in F$ and $q \in G$
  - The line $\overline{pq}$ is normal to F at p, and normal to G at q.
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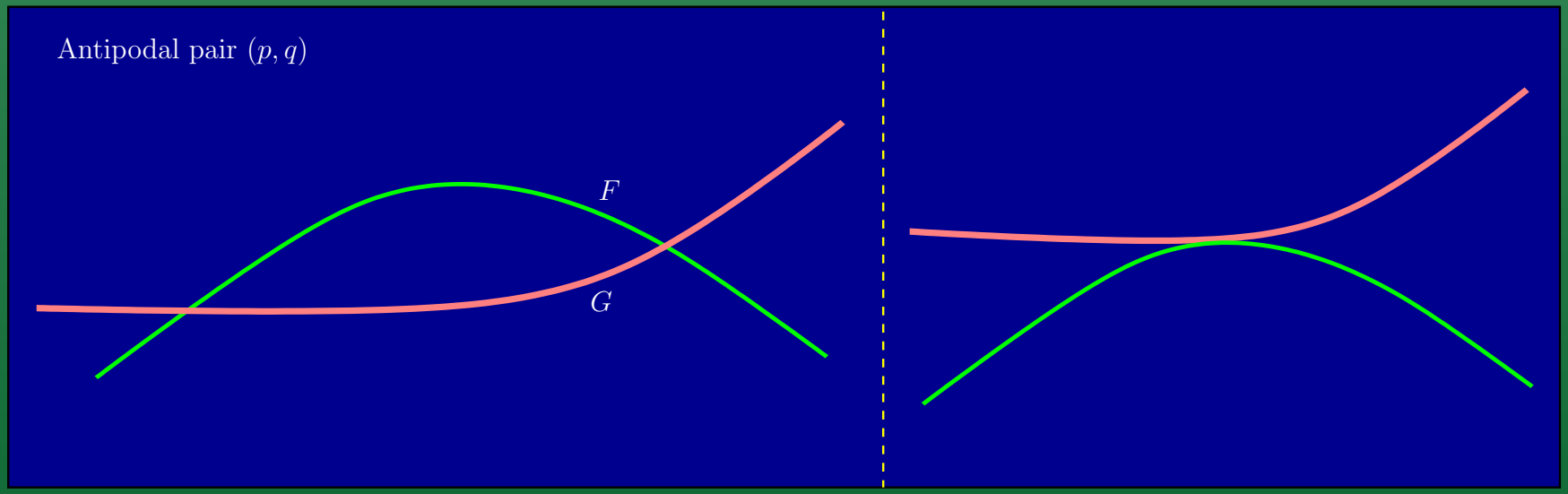
- 

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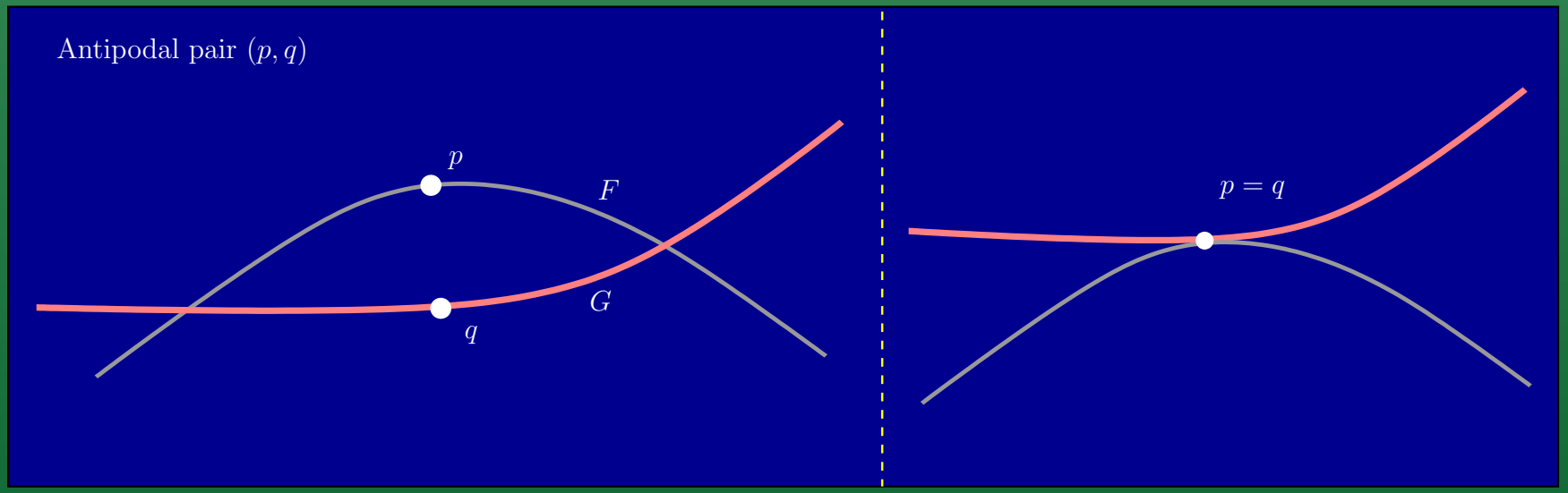
- 

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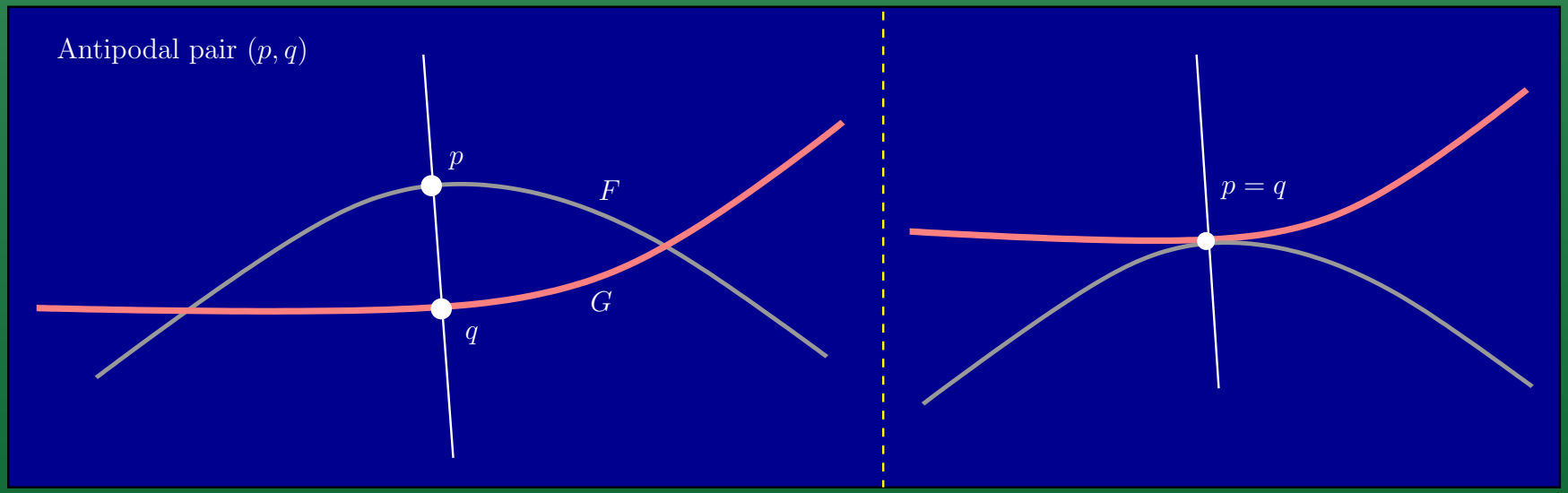
- 

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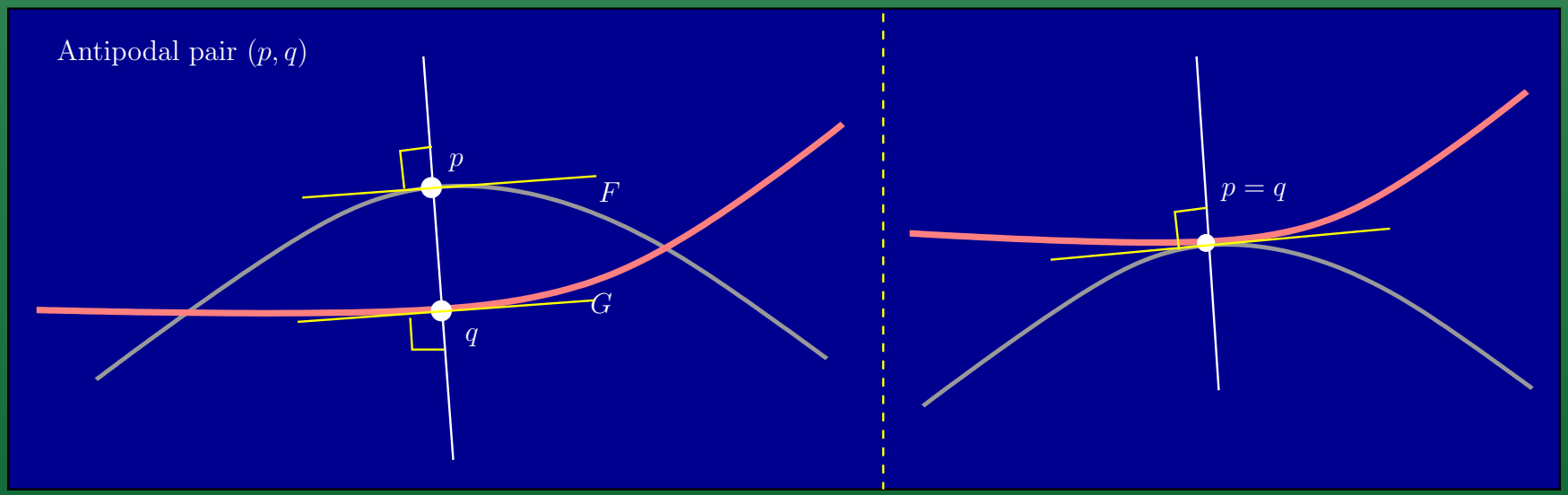
- 

- Assume (F, G) has finitely many anti-podal pairs.
  - This implies A, B are relatively prime
- If F contains an offset of G then there are infinitely many anti-podal pairs
  - Conjecture: converse holds
  - Proved by S.-W. Choi

Separation Bounds for Algebraic Roots

- Let $\Sigma = \{A_1, A_2, \ldots, A_n\}$, where
  - $A_i \in \mathbb{Z}[x_1, \ldots, x_n]$ and $\deg(A_i) = d_i$
  - Σ has finitely many complex zeros
  - $\|A\|_k$ is the k-norm (for $k = 1, 2, \infty$)

- THEOREM

  If $(x_1, \ldots, x_n)$ is a zero of Σ and $x_1 \neq 0$ then $|x_1| > (2^{3/2} N K)^{-D} 2^{-(n+1)d_1 \cdots d_n}$ where
  - $K = \max\{\sqrt{n+1}, \|A_1\|_2, \ldots, \|A_n\|_2\}$,
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- THEOREM 1: If (p, q) is an antipodal pair, then $p \neq q$ implies $\|p - q\| \geq \Delta_1(m, n, a, b)$ where
  - $\Delta_1 = (3NK)^{-D} 2^{-12m^2n^2}$,
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- THEOREM 2: If $p \in F \cap G$ and $q \in F \cap G$, then $p \neq q$ implies $\|p - q\| \geq \Delta_2(m, n, a, b)$ where
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  - with N, D as before.

How Close can a Point be to a Curve?

- Let q be a point not on the curve $F : A(x, y) = 0$.
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- THEOREM 3: If $p \in F$, and the curve F does not contain a circle centered at q, then $\|p - q\| \geq \Delta_3(m, a, L)$ where
  - $\Delta_3 = (3NK)^{-D}2^{-8m^2}$,
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Norm for Equation of Bezier Curve

- Apply the separation bounds to a Bezier curve F
  - Control points $(p_0, \ldots, p_m)$
  - Each coordinate of the p_i's are L-bit floats
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III. NONCROSSING INTERSECTION CRITERION (NIC)

How to affirm non-crossing intersection?

Elementary Curves

- $C^1[a,b]$: bounded, continuously differentiable real functions on interval $[a,b]$.
- $f \in C^1[a,b]$ defines a graph $F : [a,b] \longrightarrow \mathbb{R}^2$
  - $F(t) = (t, f(t))$.
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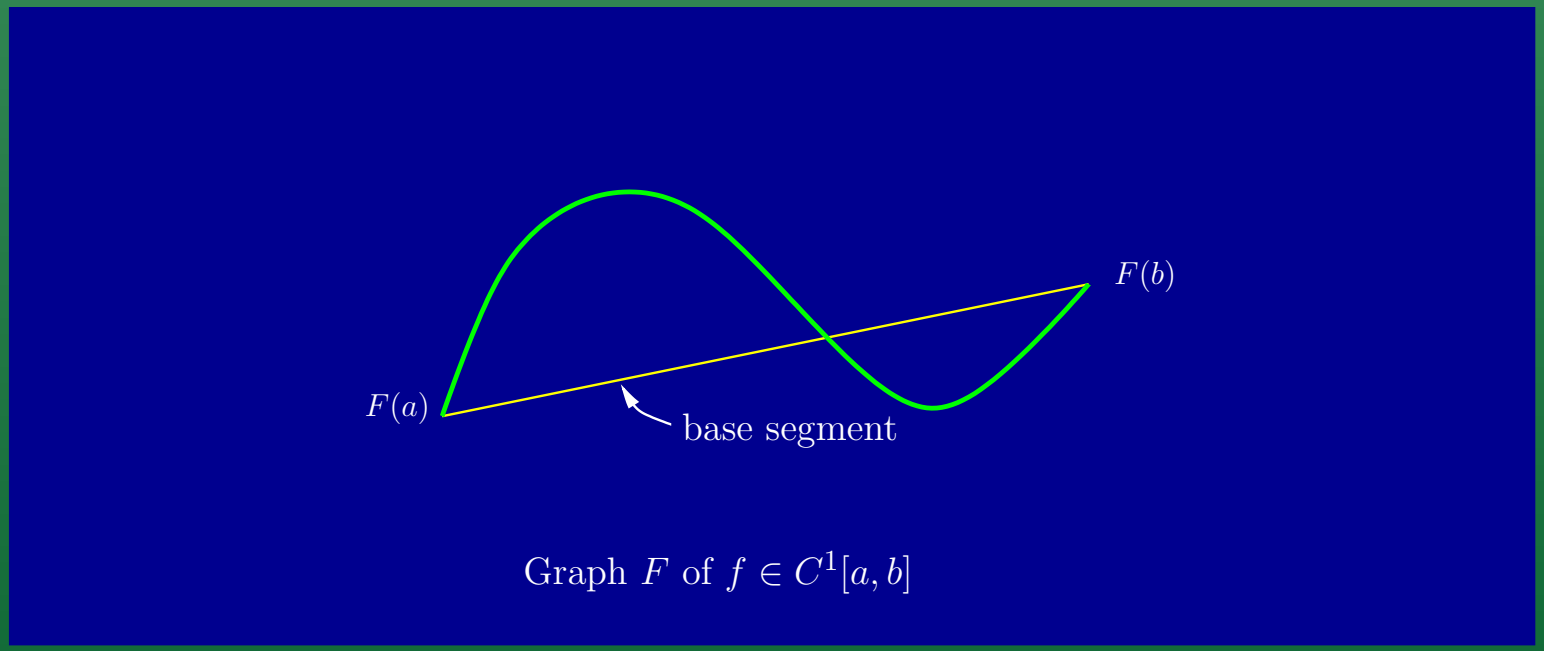
- 

Graph F of $f \in C^1[a,b]$

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  - F is A-elementary if it lies above the base segment
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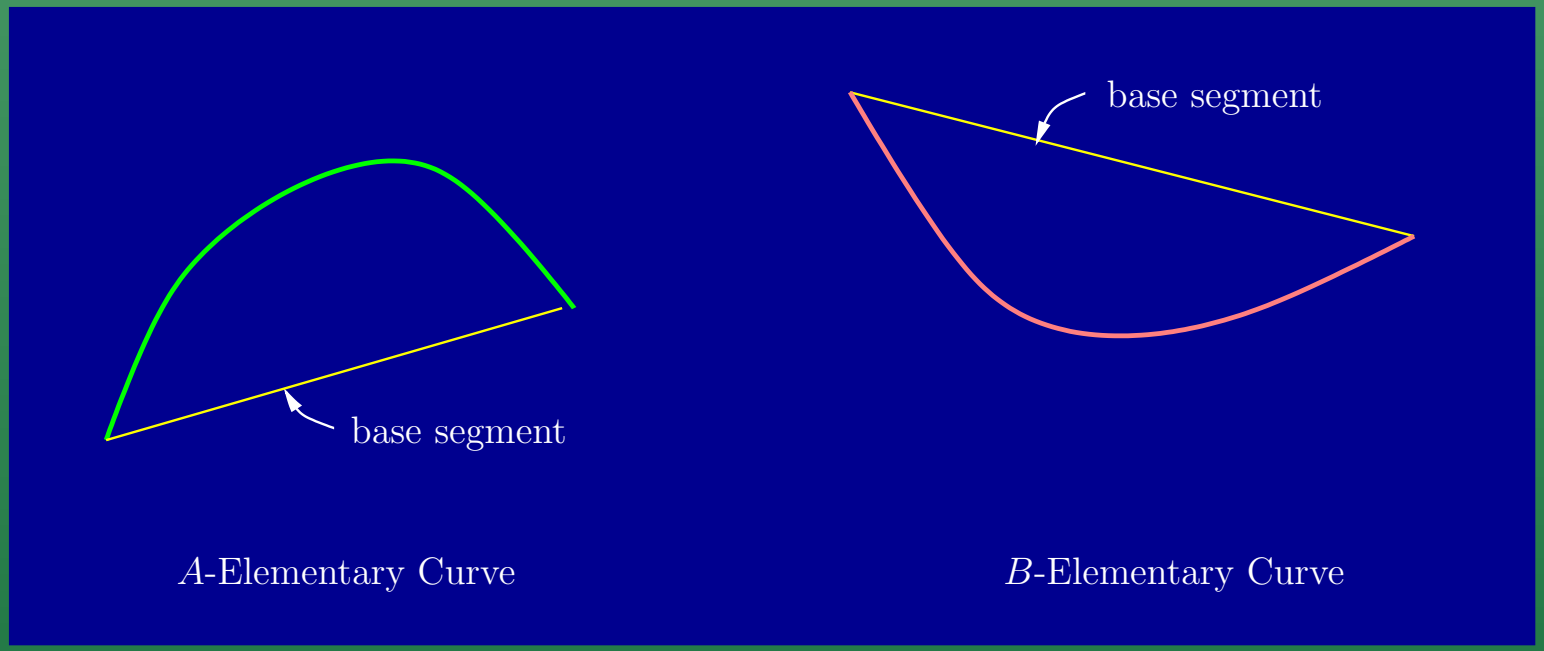

Elementary Couple

- Define (F, G) to be an elementary couple if
  - $F = F[0,1]$ and $G = G[a,b]$
  - $G(a) \in a_F(0)$ and $G(b) \in a_F(1)$
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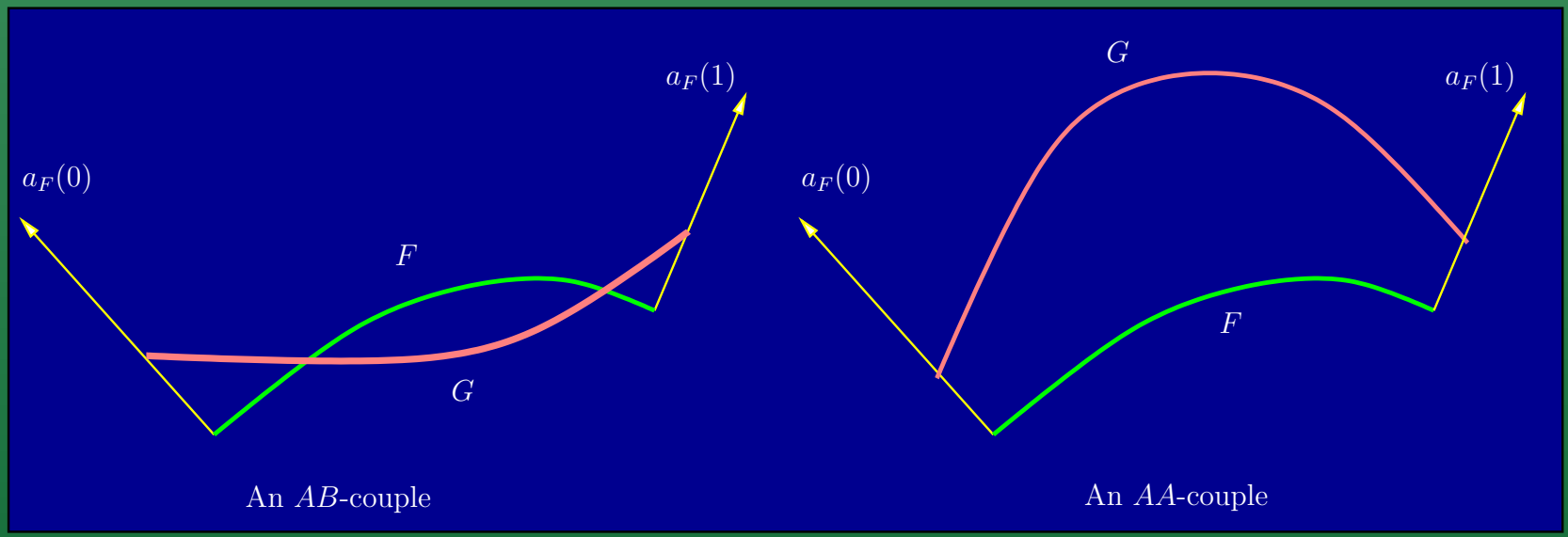

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  - $F = F[0,1]$ and $G = G[a,b]$
- LEMMA: If G never dips below F, then there is a continuous function $s : [0,1] \longrightarrow [a,b]$ such that for all t, the normal at $F(t)$ intersects G at a unique point $G(s(t))$.

- Define $\theta_F(t)$ to be the slope angle of the normal at $F(t)$
- Define $\alpha(t) = \theta_F(t) - \theta_G(s(t))$
  - Look at the sign of $\alpha_{F,G}(t)$
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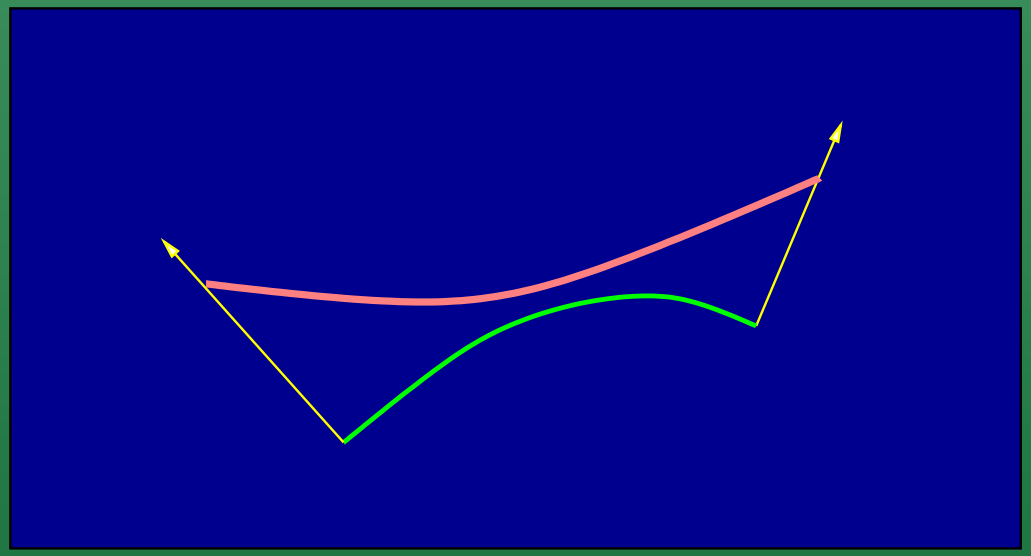
- 

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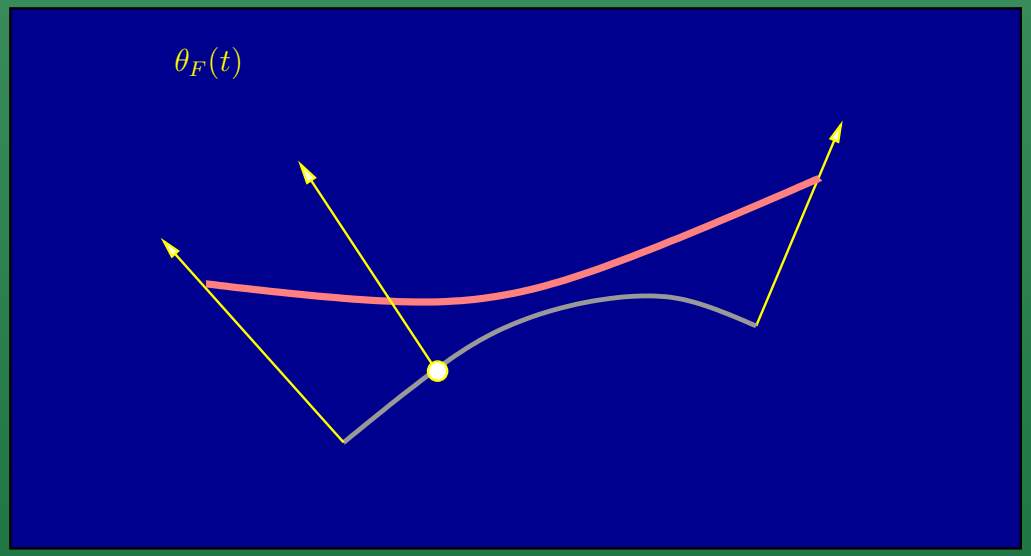
- 

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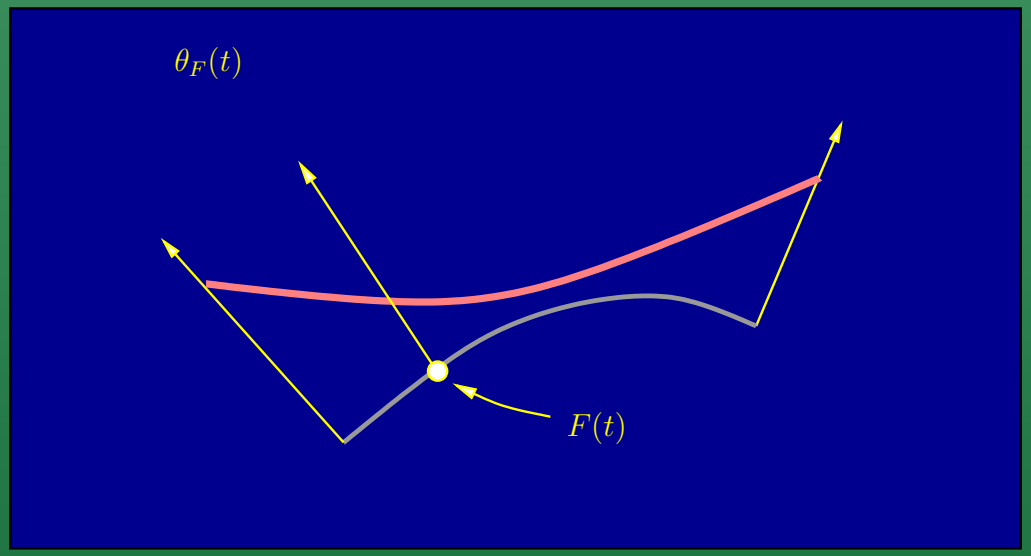
- 

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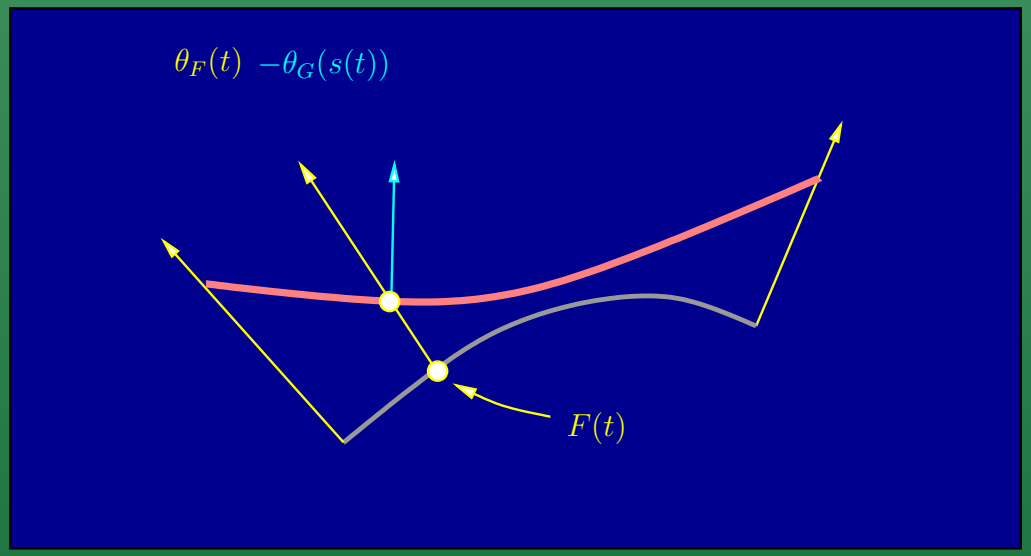
- 

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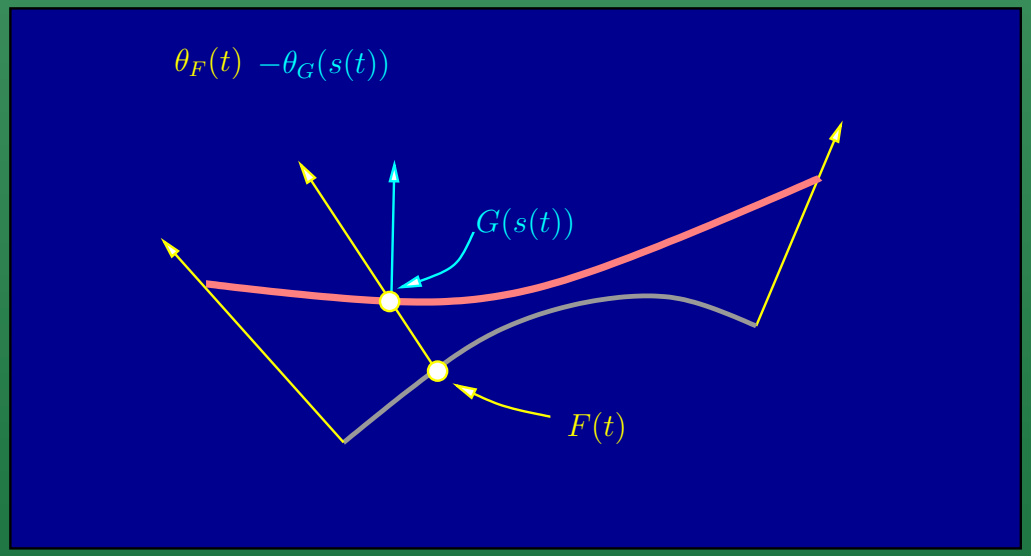
- 

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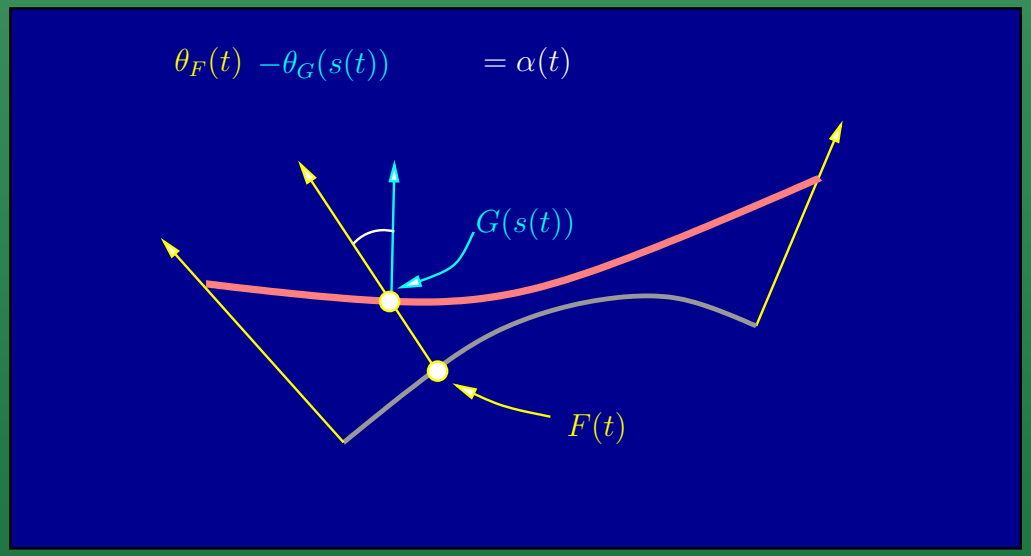
- 

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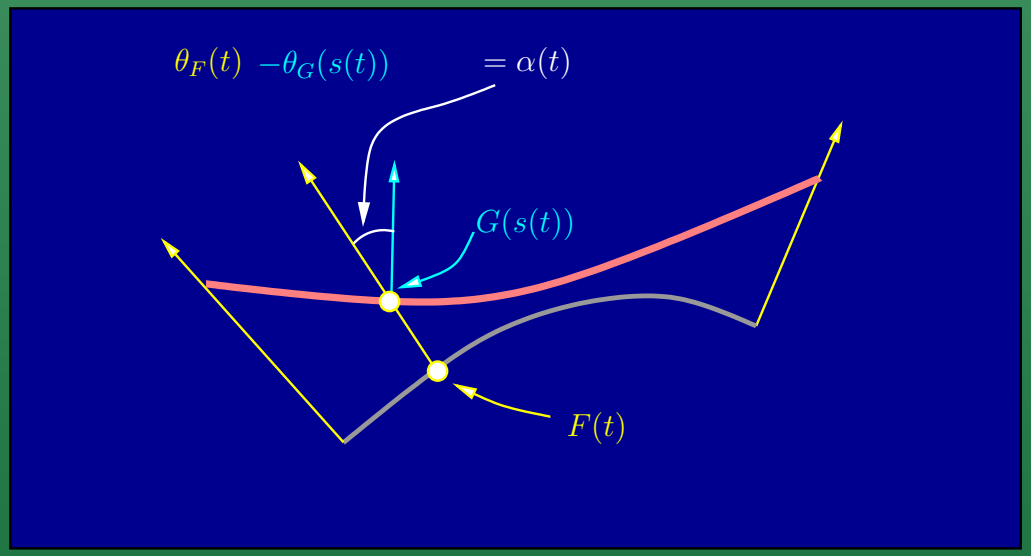
- 

Non-Crossing Intersection Criterion (NIC)

- F, G has Δ-separation property if for all $p \in F$ and $q \in G$,
  - if either (p, q) is an antipodal pair or $\{p, q\} \subseteq F \cap G$,
  - then $p \neq q$ implies $d(p, q) > \Delta$.
- THEOREM 5: Let (F, G) be an elementary couple with the Δ-separation property, and the diameter of $F \cup G$ is $\leq \Delta$.
  - (i) If $\alpha(0)\alpha(1) \leq 0$ then F and G intersect tangentially, in a unique point.
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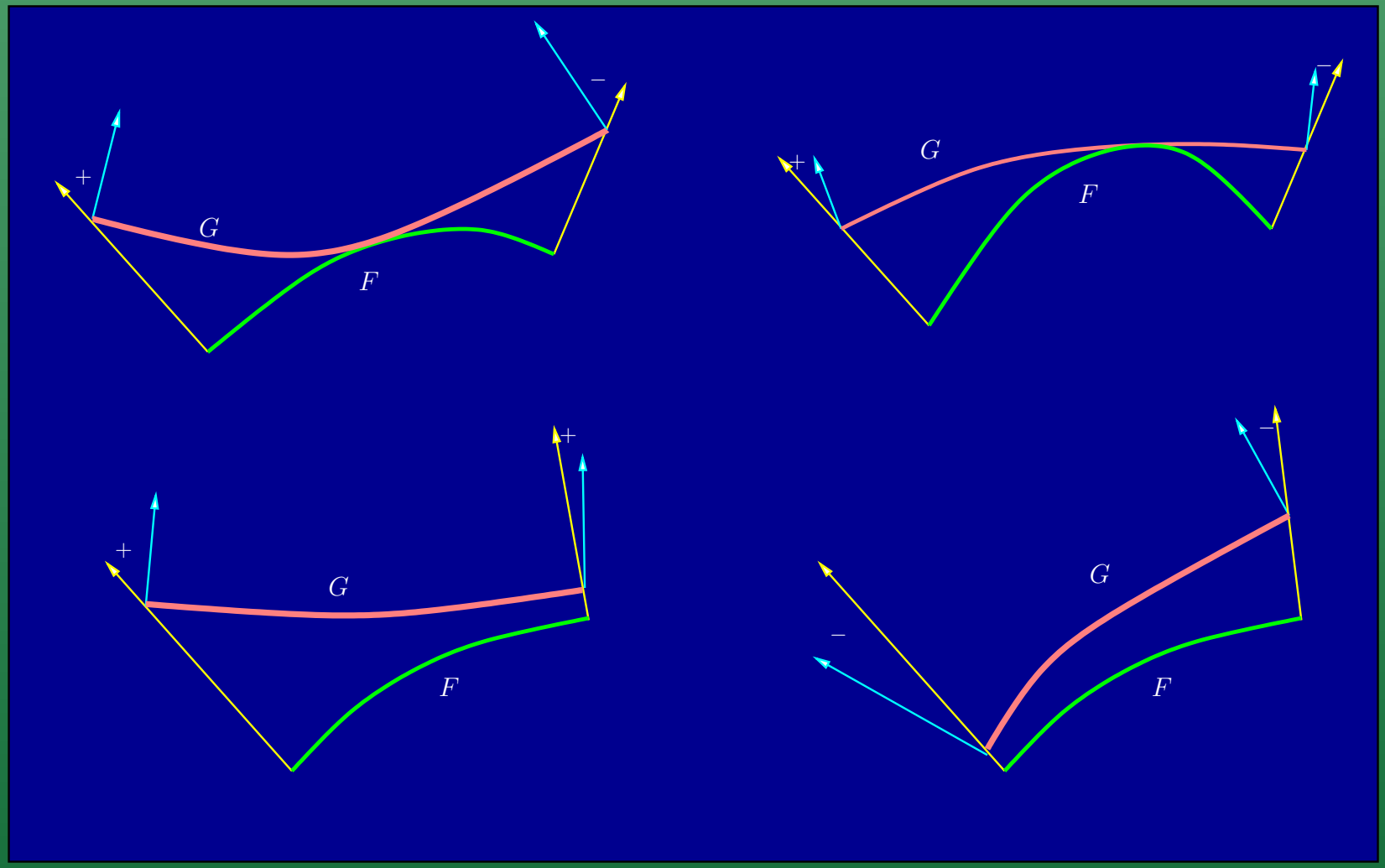

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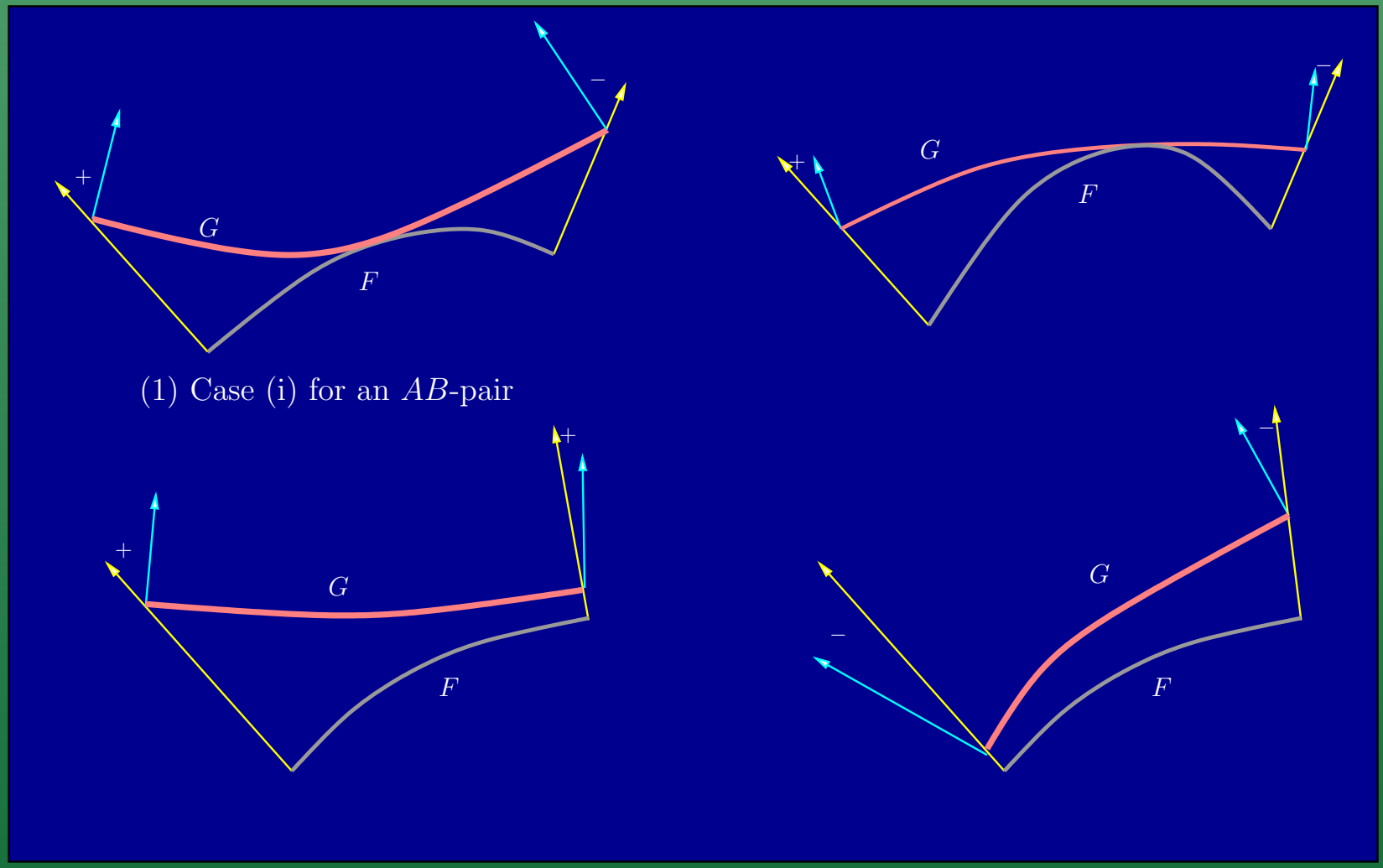

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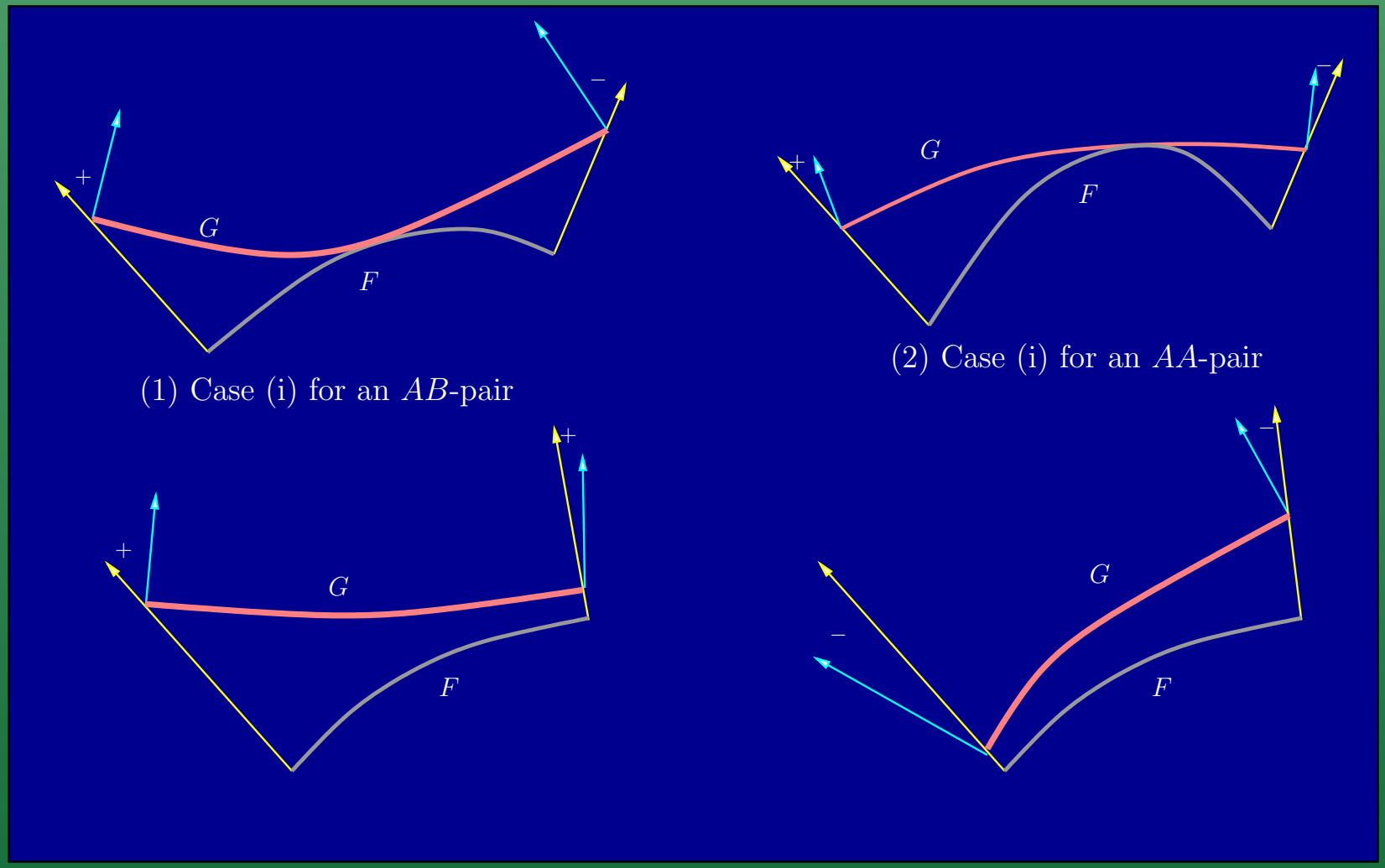

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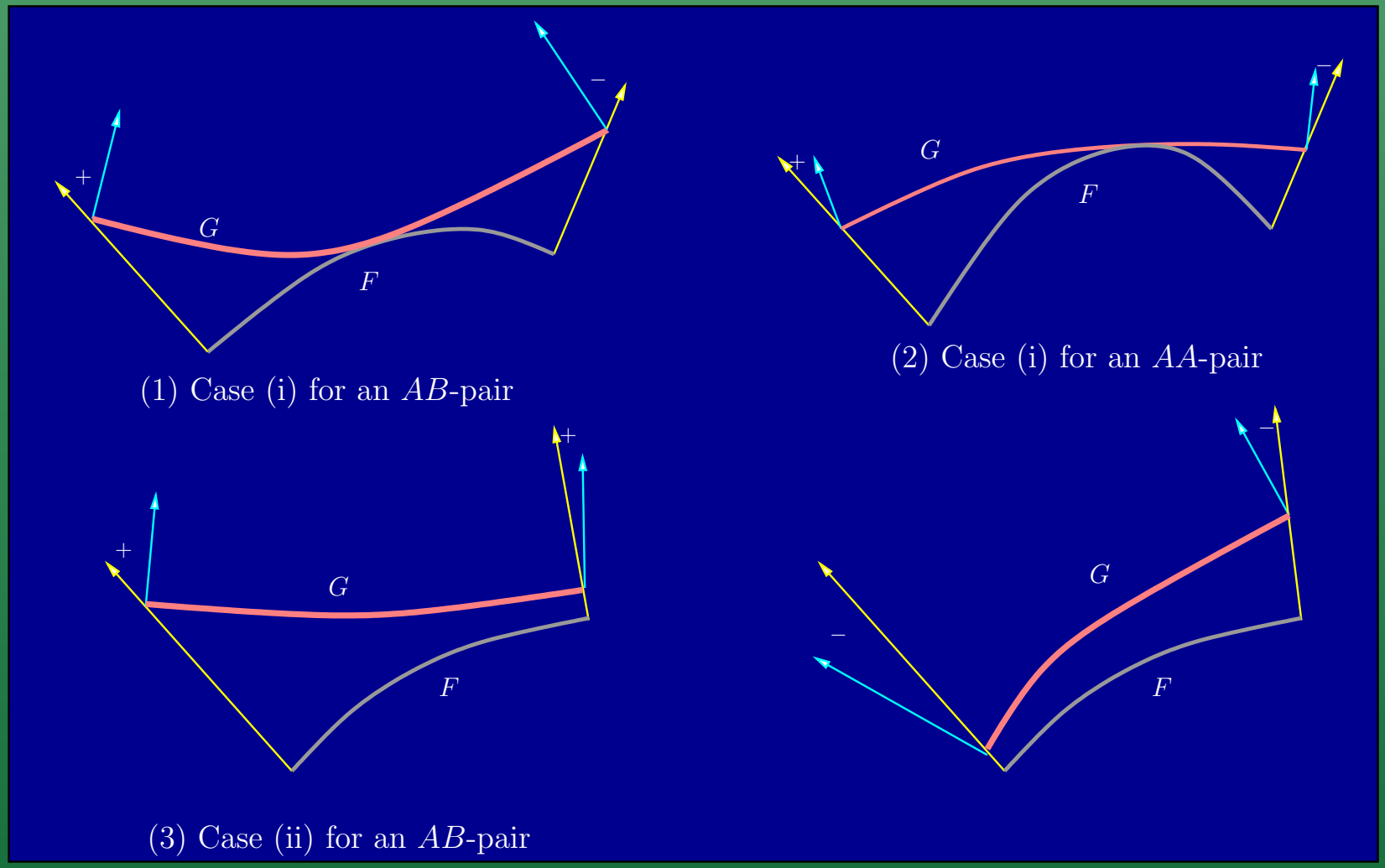

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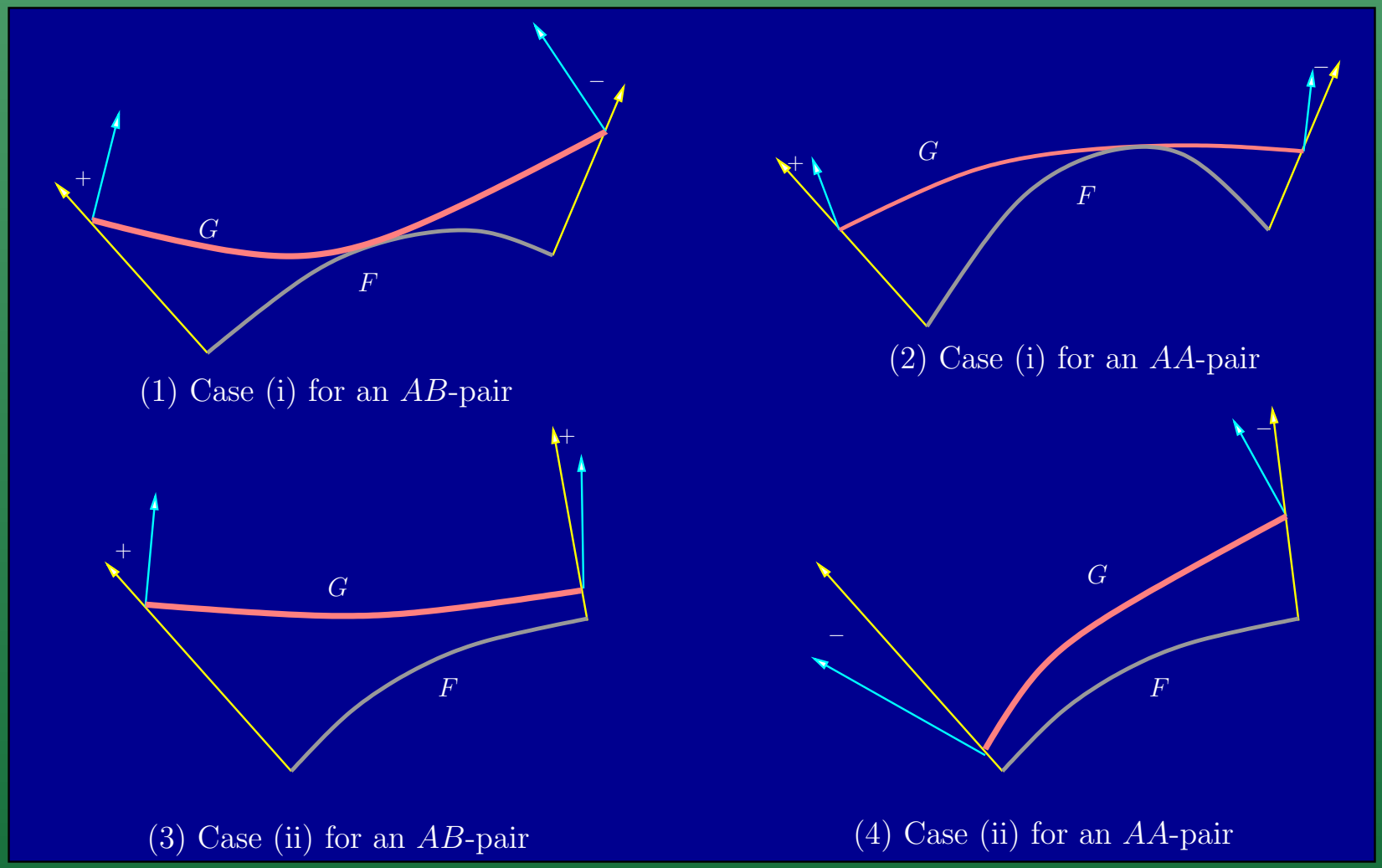

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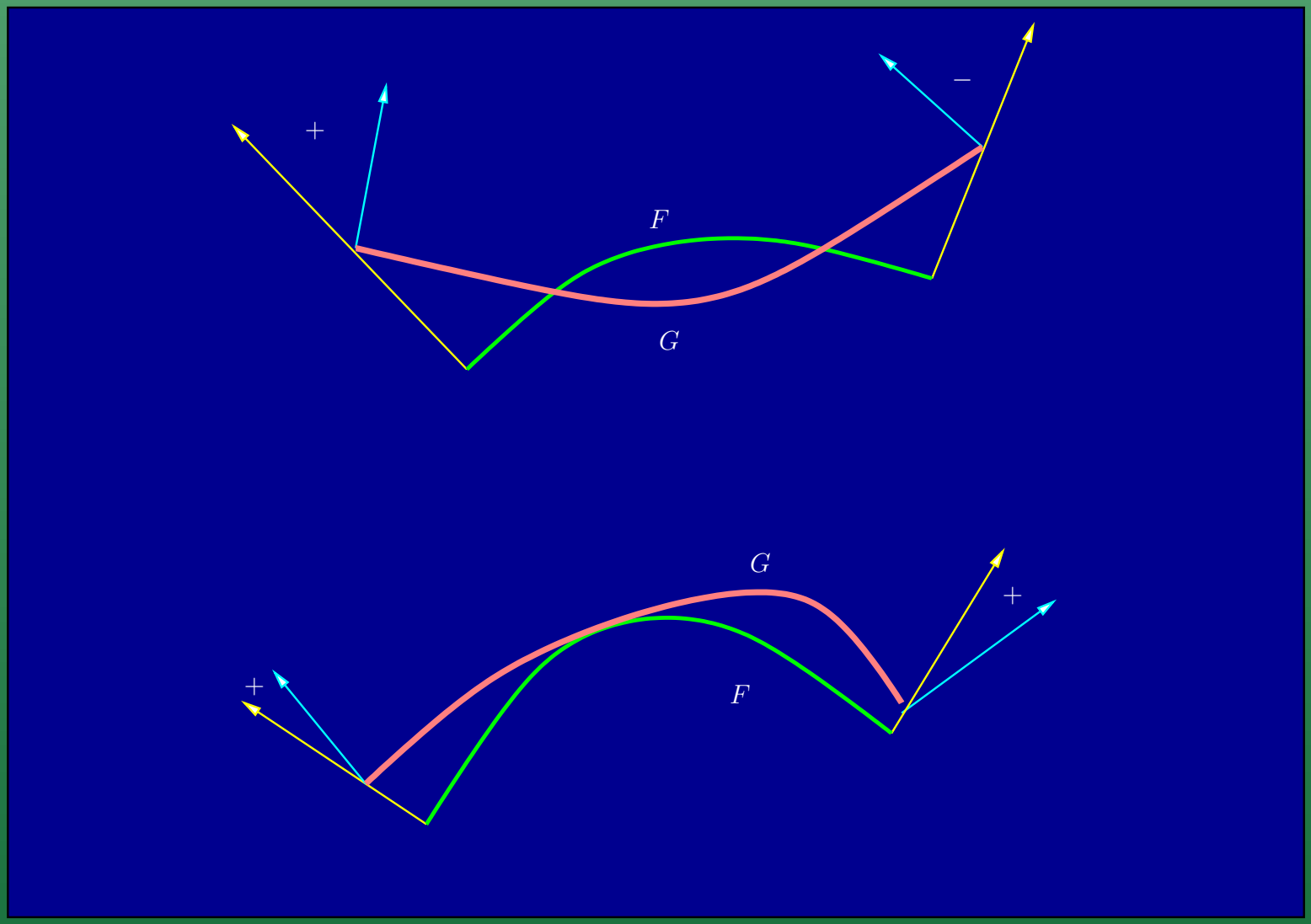

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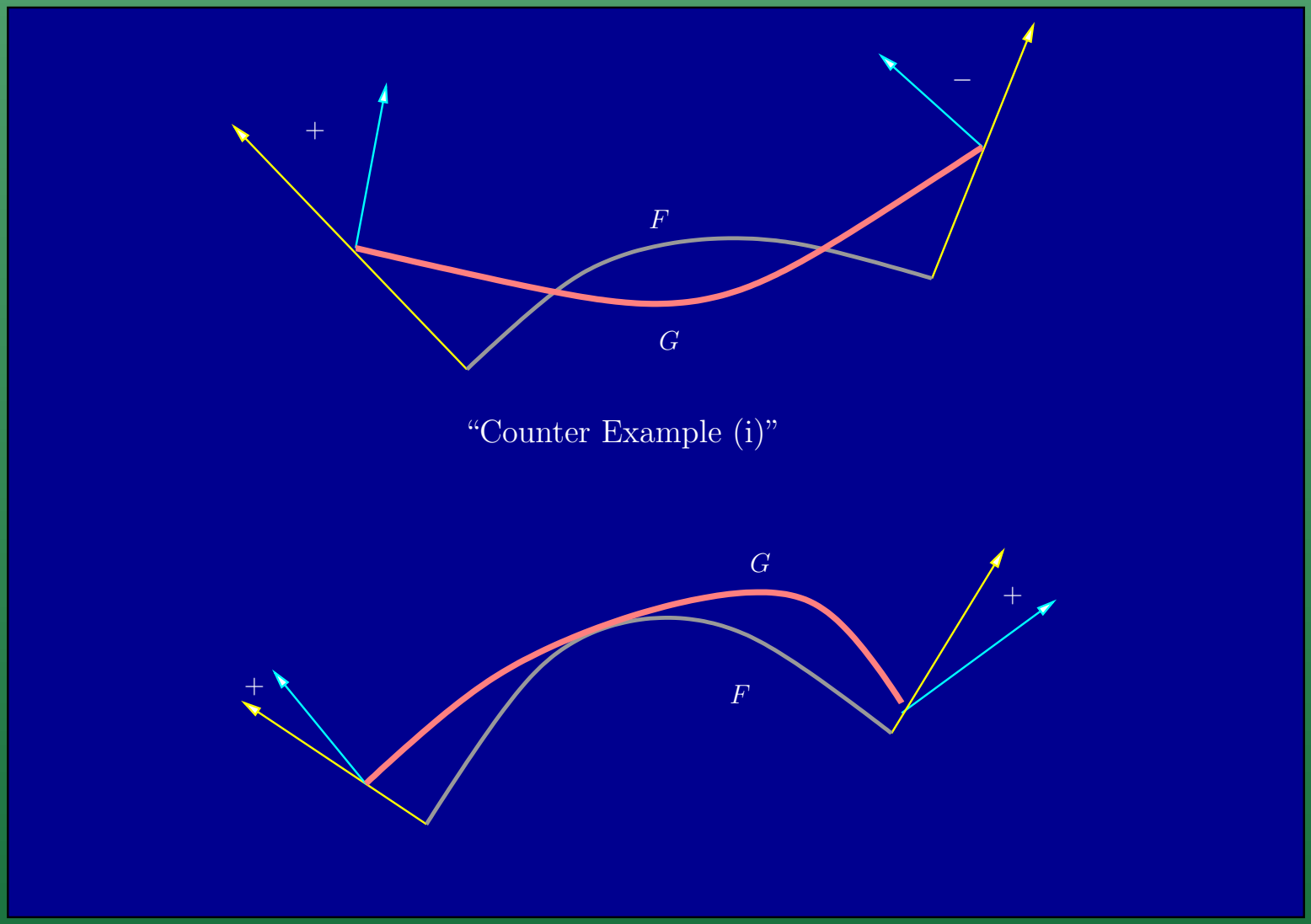

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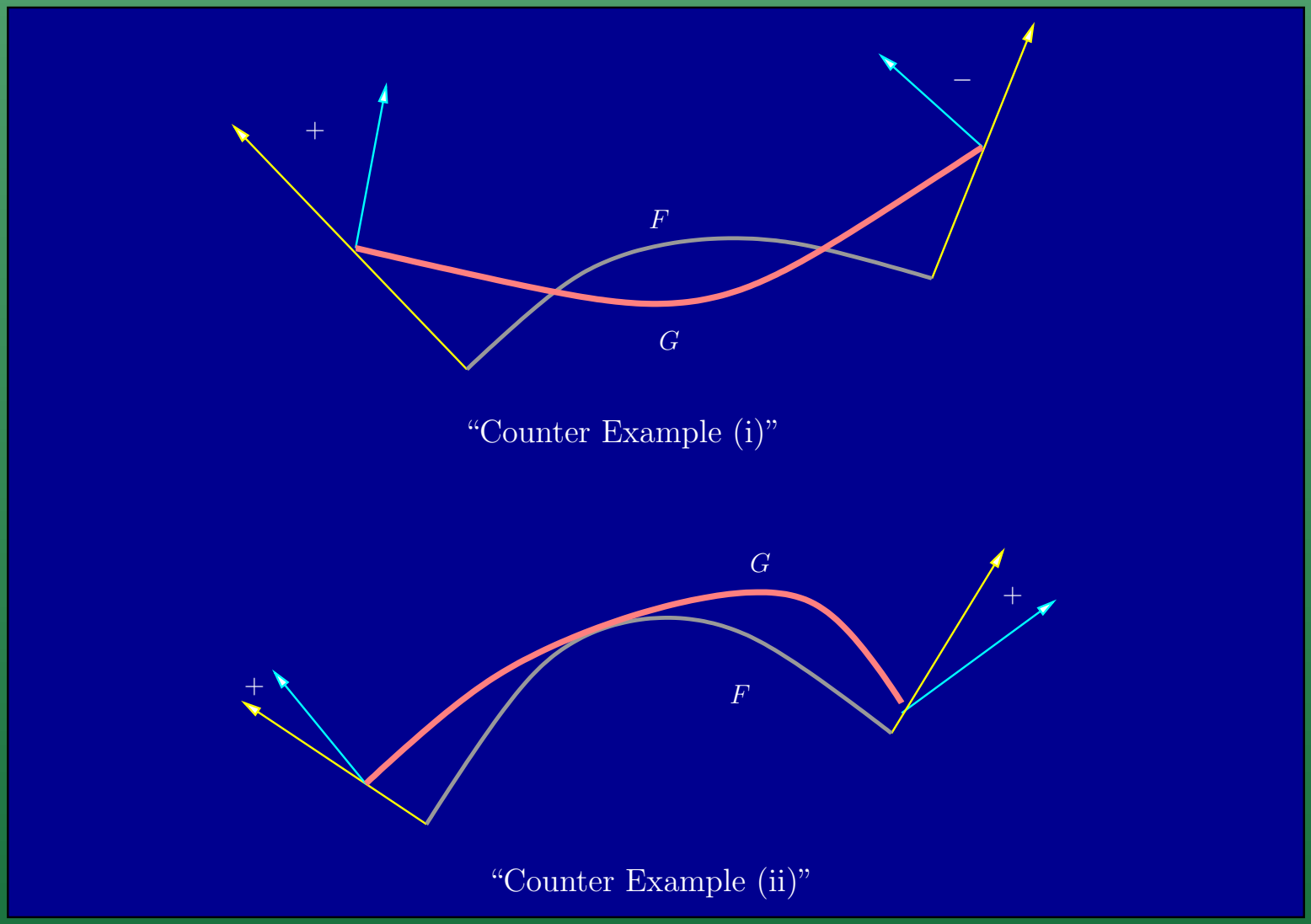

Proof

- By Δ-separation, $|F \cap G| \leq 1$. So by LEMMA, there is a continuous function $s : [0,1] \rightarrow [a,b]$.
- (i) Case $\alpha(0)\alpha(1) \leq 0$.
  - By continuity, there exists t such that $\alpha(t) = 0$.
  - This $(F(t), G(s(t))$ is an antipodal pair
  - If $F(t) \neq G(s(t))$, then $d(F(t), G(s(t)) > \Delta$, contradiction
  - So $F(t) = G(s(t))$, a tangential intersection

- (ii) Case $\alpha(0)\alpha(1) > 0$ (say $\alpha(0) > 0, \alpha(1) > 0$)
  - Assume F and G intersect at $F(t_0)$
  - Then F and G intersect tangentially at $F(t_0)$
  - Consider the antipodal pair $(F(t_0), G(s(t_0)))$
  - Since F is below G, $\alpha(t_0^-) > 0$ and $\alpha(t_0^+) < 0$
  - By continuity, there exists $t_1 \in (t_0, 1)$ s.t. $\alpha(t_1) = 0$
  - Then $F(t_1)$ must be another tangential intersection
  - This contradicts the Δ-separation property

Extended NIC

- Call (F, G) a half-couple if
  - $F = F[0,1]$ and $G = G[c,d]$
  - $G(c) \in a_F(0)$ or $G(d) \in a_F(1)$
  - The entire curve G lies inside the cone $C(F)$.

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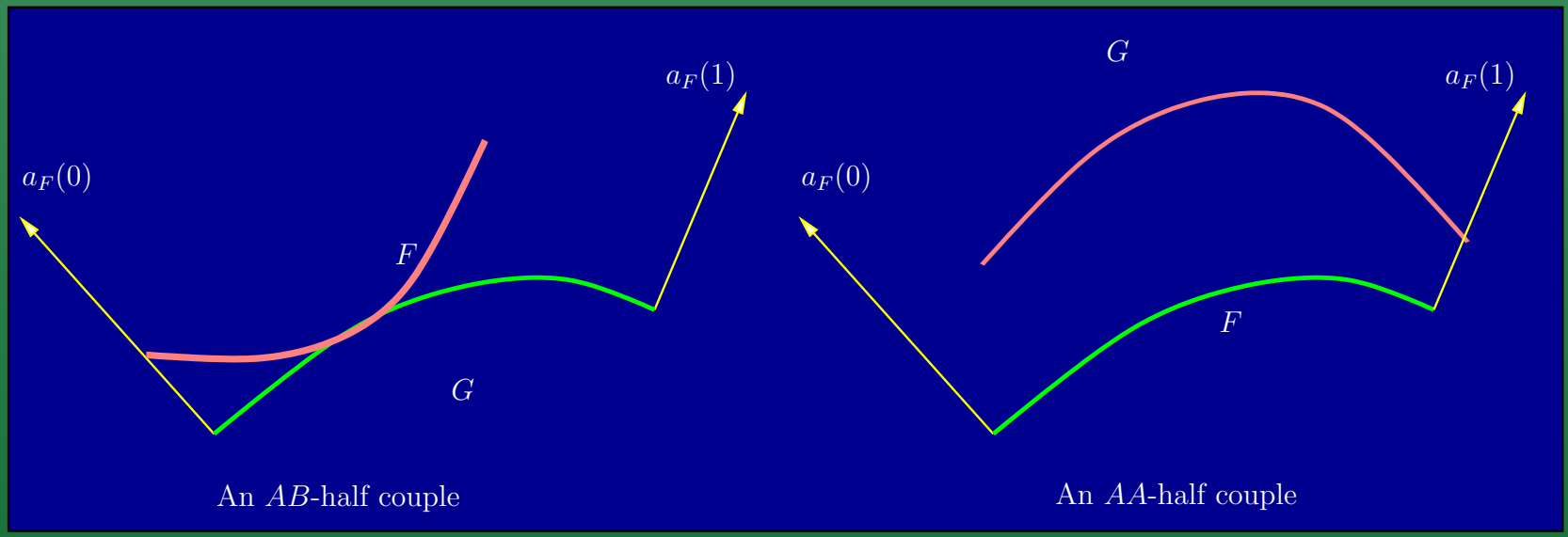

- The following theorem extends the NIC to half-

couples.

- THEOREM: Let $(F[0,1], G[c,d])$ be a half-couple where $a_F(0)$ passes through $G(c)$. Suppose the upper half-normal $a_G(d)$ makes the angle γ with the x-axis. Then the lower half-normal $b_G(d)$ satisfies exactly one of the following five cases:
(i) $b_G(d)$ intersects $a_F(0)$.
(ii) $b_G(d)$ intersects $a_F(1)$.
(iii) $b_G(d)$ intersects F at $F(t)$, and $\theta_F(t) - \gamma > 0$.
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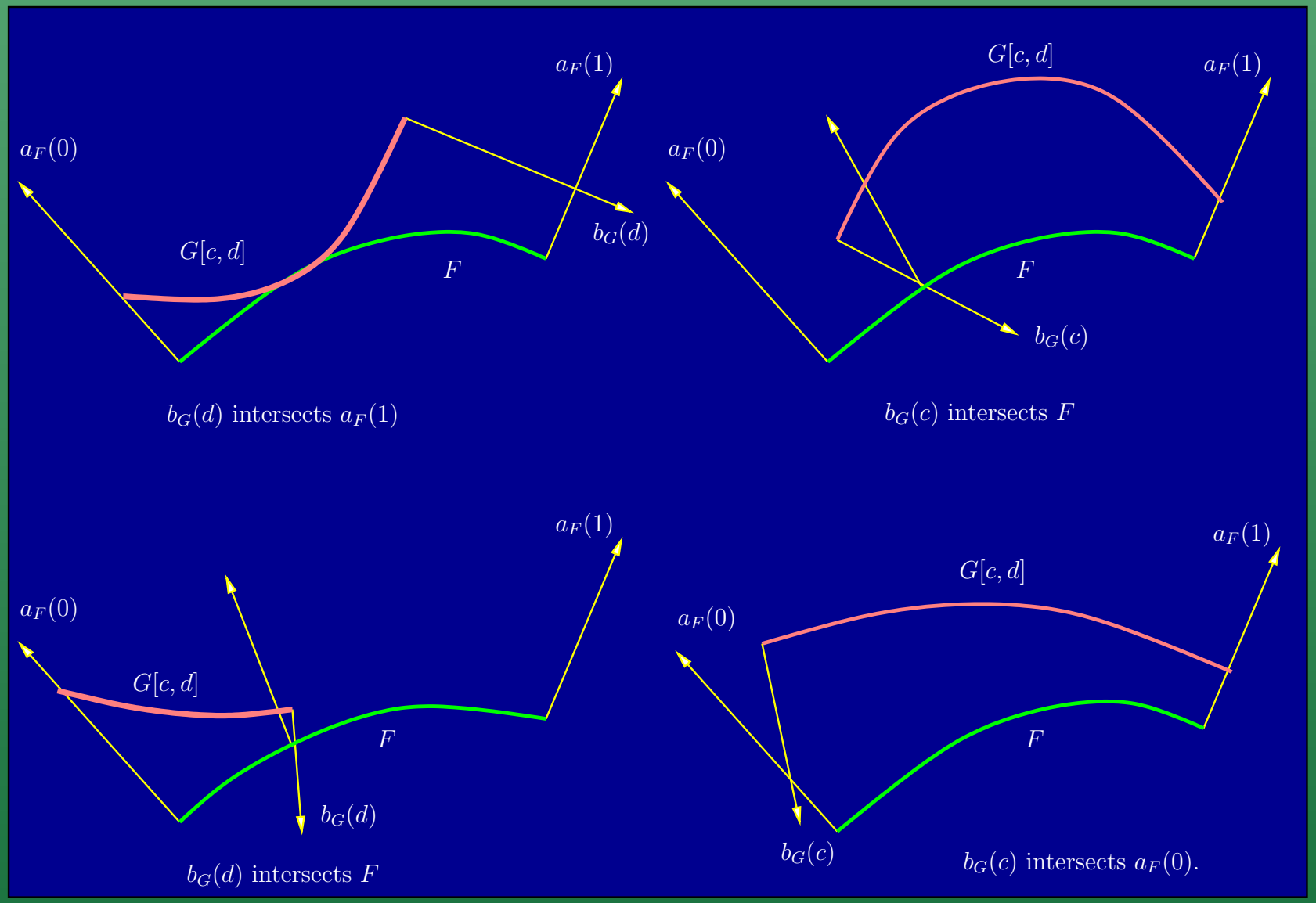
as follows:

(A) In cases (i) or (iii), $\alpha(t_0) > 0$.

(B) In cases (ii) or (iv), $\alpha(t_0) < 0$.

(C) In case (v), $\alpha(t_0) = 0$.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IV. SUB-ALGORITHMS

How can we apply the noncrossing criterion?

Delayed versus Immediate Objects

- Geometric constructors for objects:
  - E.g., $p = \cap[\ell, \ell']$ is a point expression
- Expressions: represents an object as a DAG
  - Internal nodes are constructors
  - Leaves are primitive objects
  - Similar to Expressions in Core Library

- Motivation: bigFloats (immediate) vs. algebraic numbers (delayed)
  - Queries on immediate objects are $O(1)$.
- Some "Immediate Objects":
  - number: A floating point number
  - point: coordinates are all immediate numbers
  - line: defining equations with only immediate numbers
  - Bezier curve: points in control polygon are immediate
  - Apply "transparent" constructors on immedite objects
- "Delayed Objects": These are all other objects
  - e.g., irrational numbers
  - e.g., points whose coordinates are delayed numbers

- 2 Bezier curve constructors:
  - "Transparent": $F \sim [F^*, s_0, t_0]$
  - "Opaque": $F \sim [F^*, \ell_0, \ell_1]$

Curve-Line Intersection Reps

- When is a pair (F, ℓ) a rep?
  - Transversal rep: ℓ intersects base of F
  - Tangential rep: ℓ misses base of F,
    and $diameter(P(F)) \leq \Delta$

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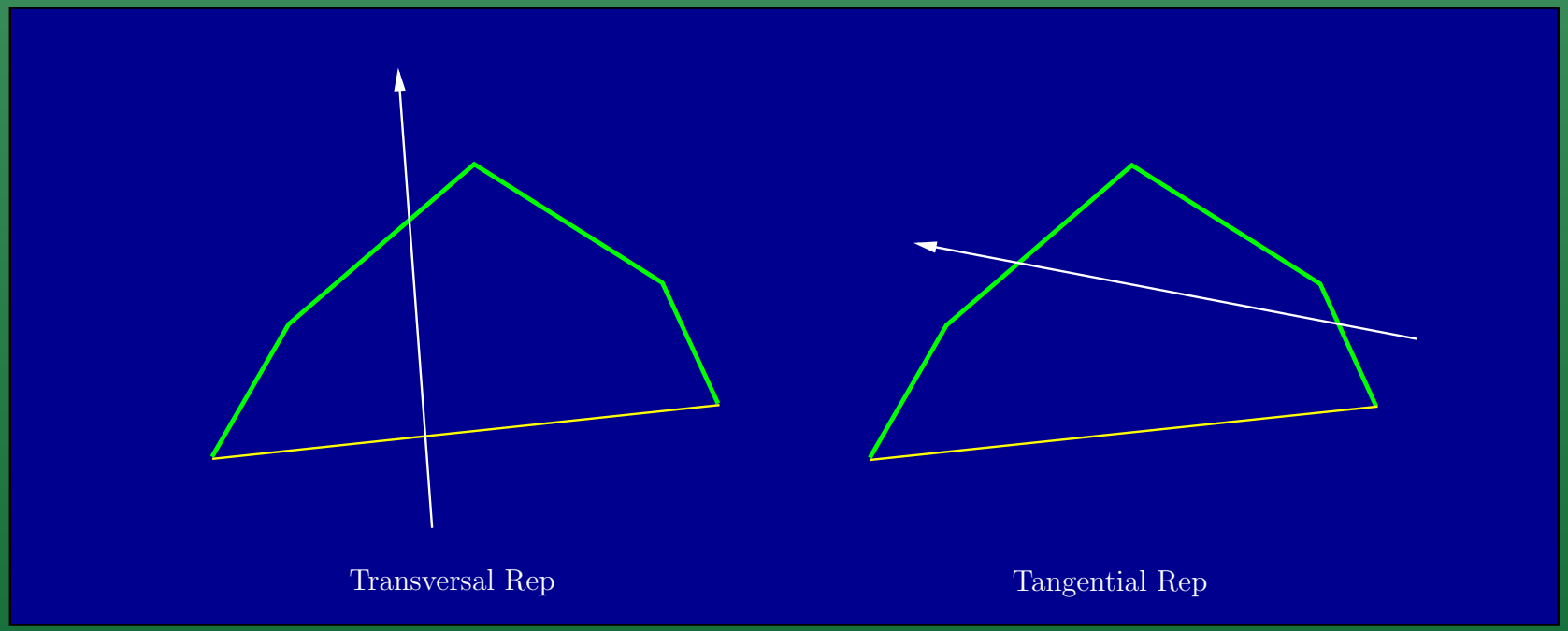

Curve-Line Intersection Algorithm

- Input: Elementary curve F and line ℓ
- Output: list of intersection points or reps

  [1] If ℓ misses $P(F)$ return(NULL)

  [2] If ℓ intersects endpoint(s) p, return(p)

  [3] If ℓ intersects base segment, return(F, ℓ)

  [4] If $diam(P(F)) < \Delta$, return(F, ℓ)

  [5] Subdivide F into (F_1, F_2) at $F(1/2)$

  [5.1] If $F(1/2) \in \ell$, recursively call (F_i, ℓ) for $i = 0$ or 1

  [5.2] Recursively call (F_i, ℓ) for both $i = 0, 1$

Determining Sign of Alpha Angle

- Bezier curve $F(t) = (F_1(t), F_2(t))$, and Line $\ell(t) = (ct + d, et + f)$
  - c,d,e,f are L-bit floats
  - Let $\alpha^* = \theta_F(t^*) - slope(\ell)$ where $F(t^*) \in \ell$

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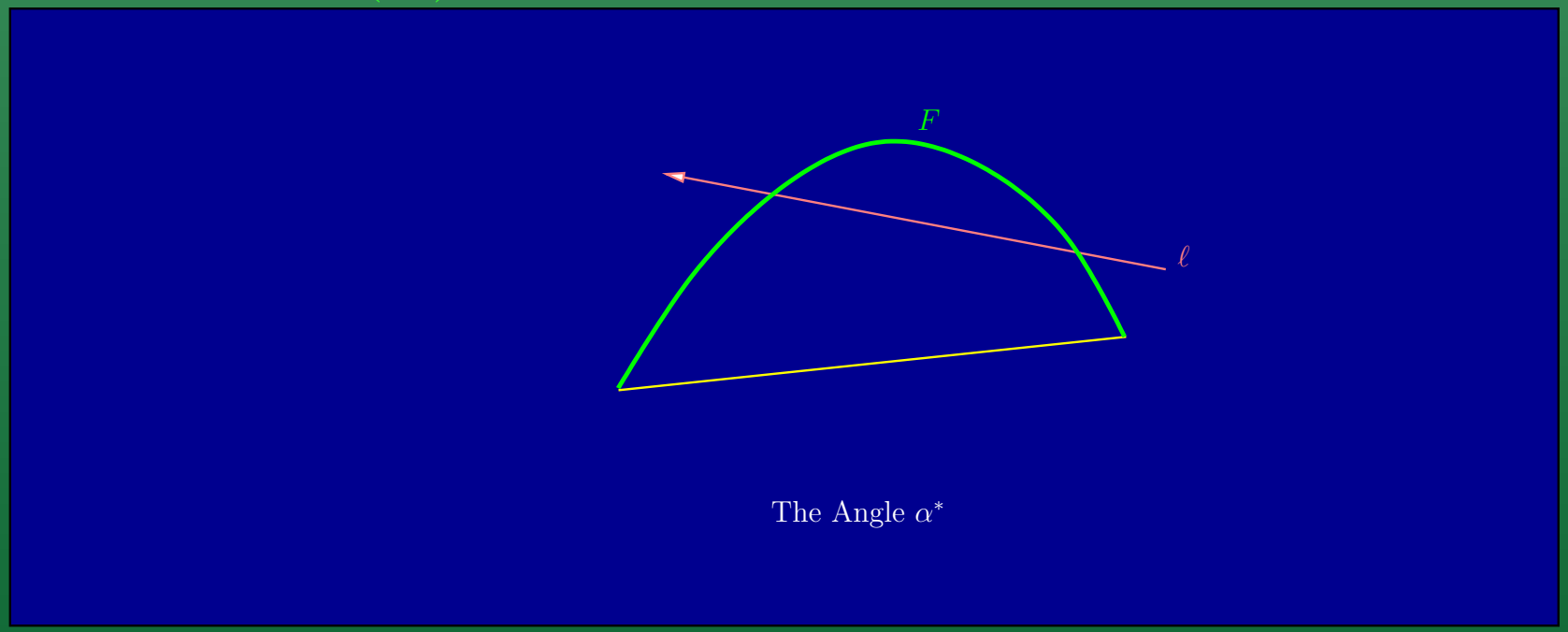

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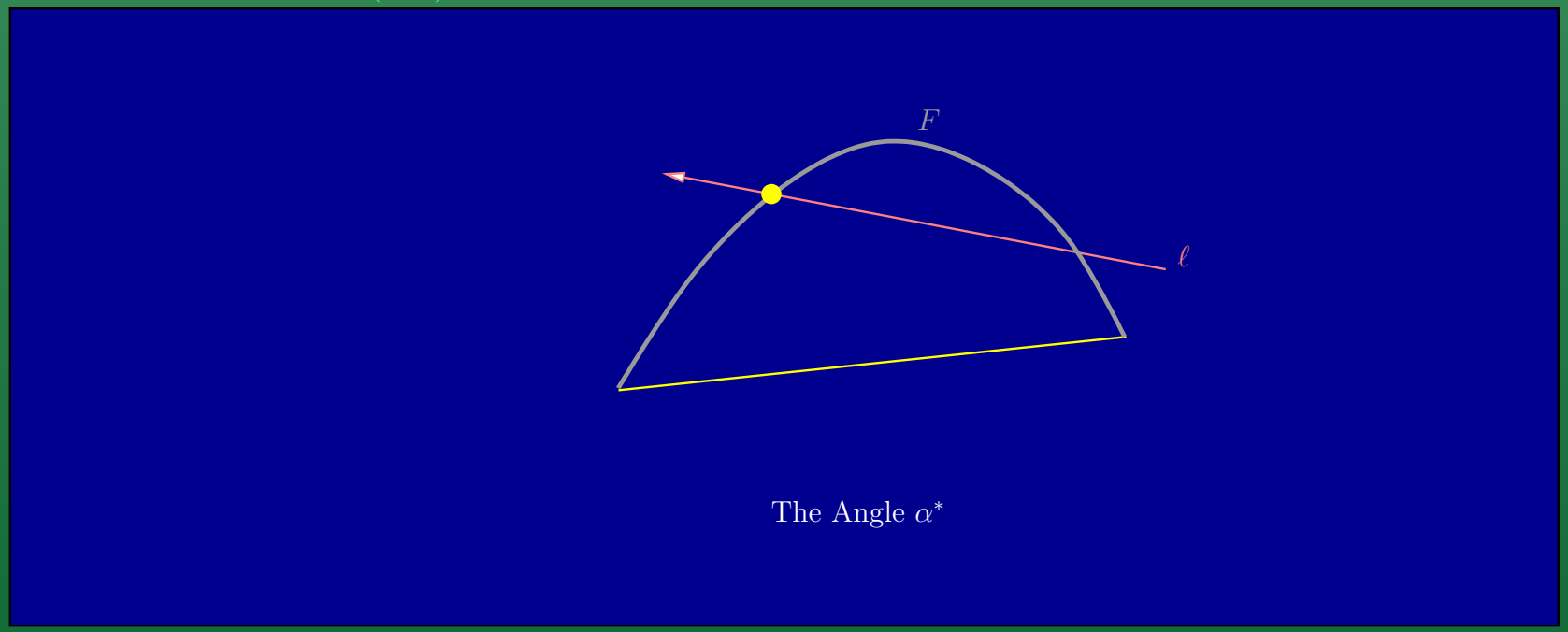

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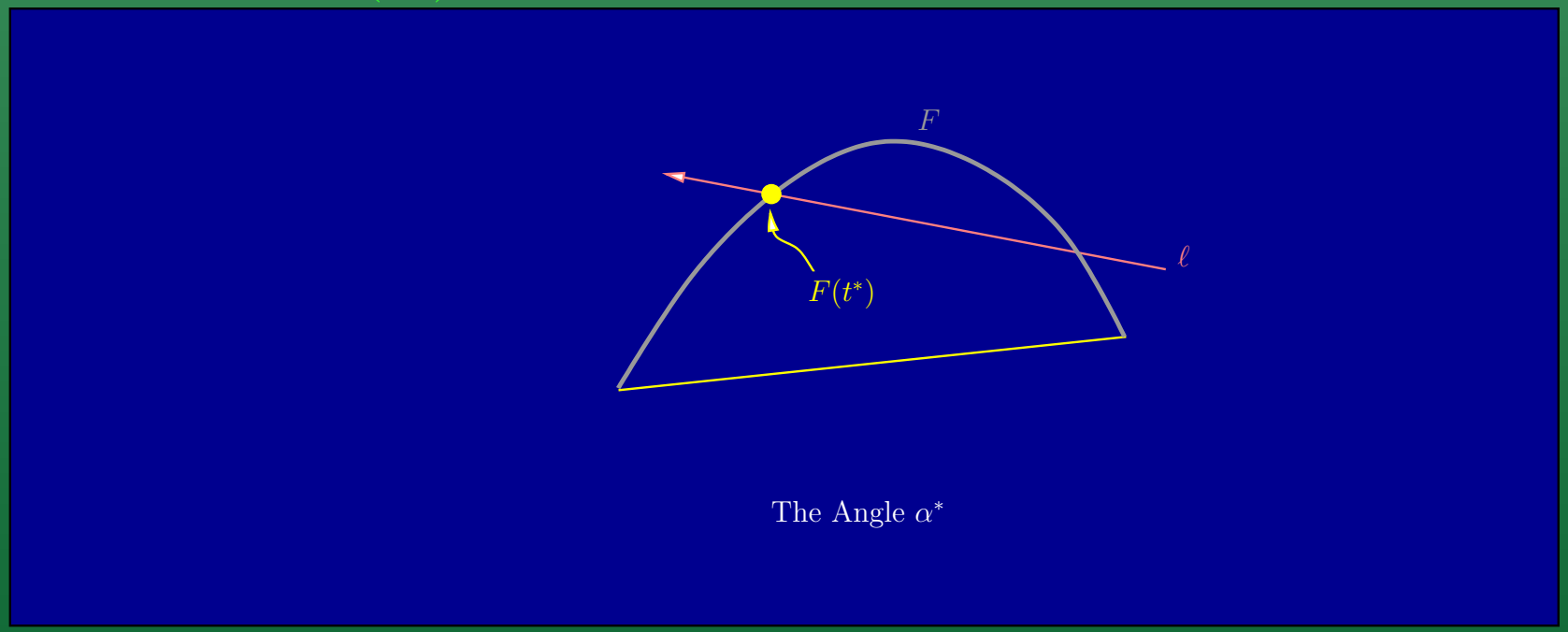

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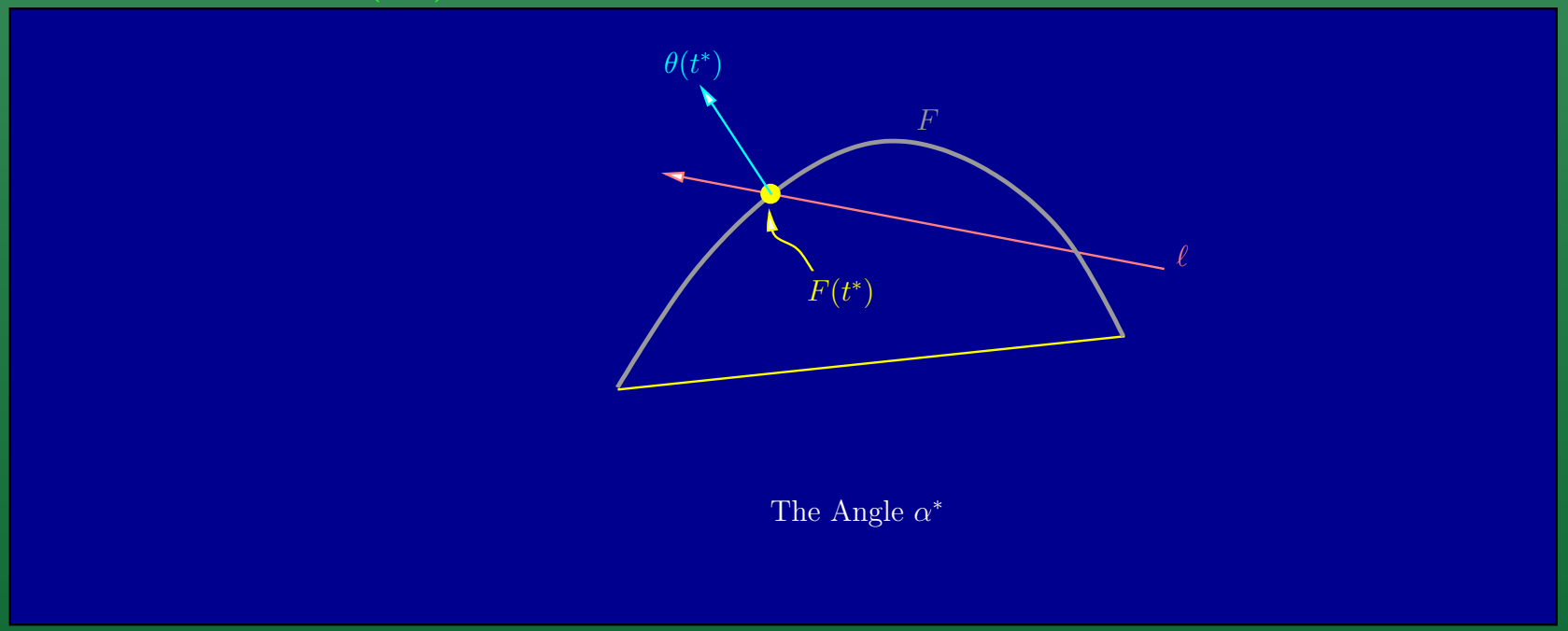

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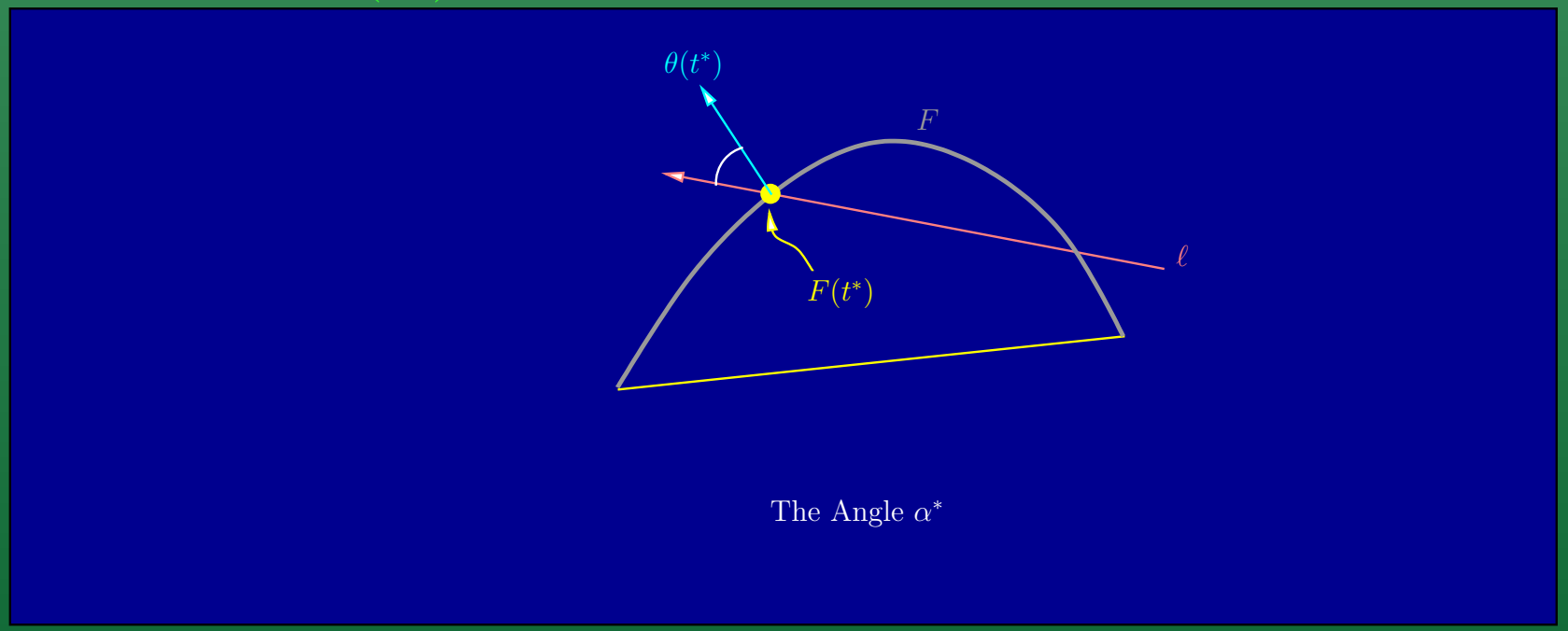

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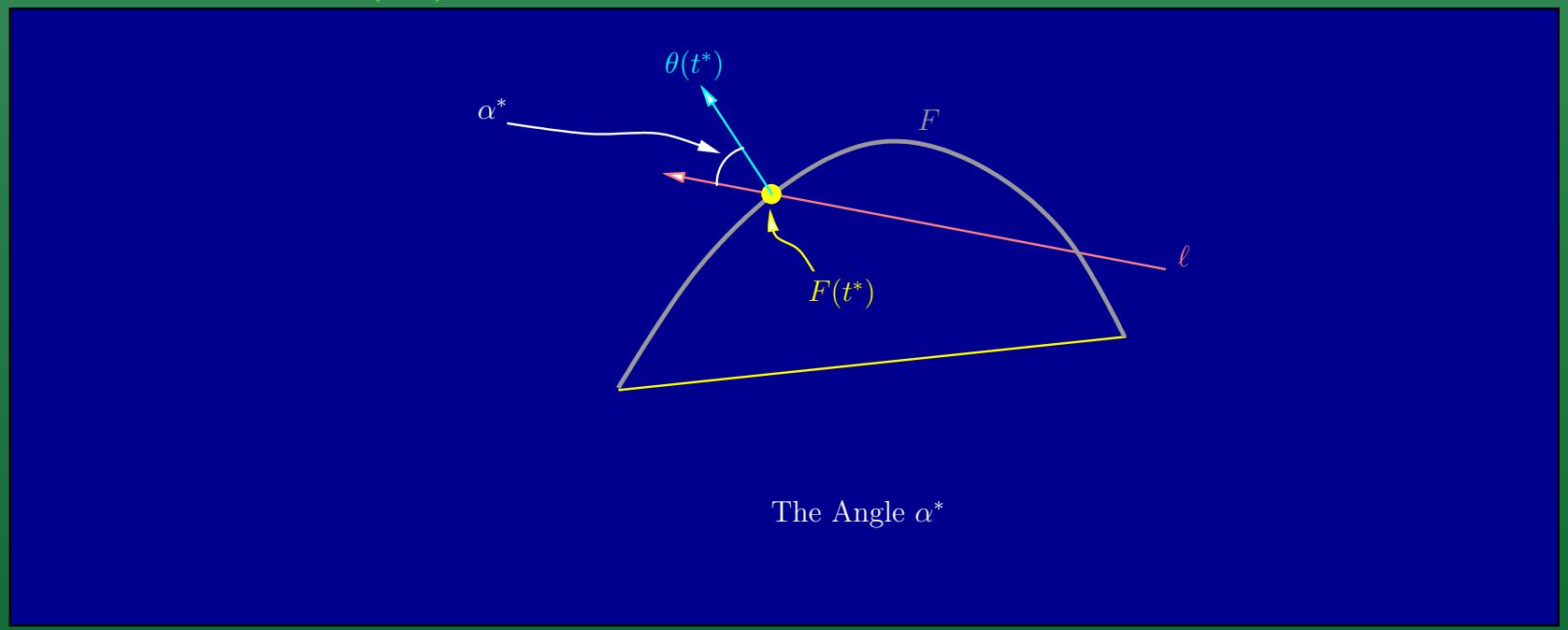

- Define $g(t) = cF_1'(t) + eF_2'(t)$
- THEOREM 6: we have $\mathtt{sign}(\alpha^*) = \mathtt{sign}(g(t^*))$. If the control polygon of F uses L-bit floats, and $g(t^*) \neq 0$ then $|g(t^*)| \geq (6m128^L 9^m)^{-m} = B(m, L)$.
- Problem: t^* is not immediate.

Sign of Alpha Angle Algorithm

- Input: curve $F \sim [F^*, s_0, t_0]$ and line ℓ
  - (F, ℓ) is a transversal rep
- Output: sign of α^*
  - [1] Evaluate $g^{(i)}(t)$ of $g(t)$ at $t = s_0$ (all $i \geq 0$)
  - [2] Compute bound ε on $|g(t^*) - g(t_0)|$ via Taylor
  - [3] If $|g(s_0)| > \varepsilon$, return($\mathtt{sign}(g(s_0))$)
  - [4] If $\lg(\varepsilon) \leq -1 - m(\lg 6 + \lg m + 7L + m \lg 9)$, return$(0)$
  - [5] Refine F to $[F^*, s_1, t_1]$ and go back to step 1.
- Correctness: $|g(s^*)| \leq |g(s_0)| + \varepsilon \leq 2\varepsilon < B(m, L)$.

Coupling Process

- Let (F, G) be an elementary pair.
  - They are a micro pair, i.e., their union has diameter less than Δ. So $|F \cap G| \leq 1$.
- First we detect if they have crossing intersections.
  - This is easy to do by checking the intersection of their vertical spans $S(F)$ and $S(G)$.
  - This uses at most two line intersection probes.
- Assuming no crossing intersections, we now try to apply NIC or its extension:
  - Wlog, assume F is below G in the strip $S(F) \cap S(G)$.
  - Let $F = F[0, 1]$ and $G = G[c, d]$. Check if $a_F(0)$ and

$a_F(1)$ intersects G. If so, we are done

* Otherwise, we conduct a binary search for a $t_0 \in [0, 1]$ such that $a_F(t_0)$ intersects G.

* It is not hard to see that we can now reduce the problem to check non-crossing intersections for two half-couples.

V. INTERSECTION ALGORITHM

Putting the pieces together

What about Non-Elementary Curves?

- $F(t)$ is critical iff
  - stationary: $F'_x(t) = F'_y(t) = 0$
  - x-extreme: $F'_x(t) = 0, F'_y(t) \neq 0$
  - inflection: $F'_x(t)F''_y(t) = F''_x(t)F'_y(t)$
- Approach 1: Cut curves at critical points
- Approach 2: New types of elementary curves
  - S-, X- and I-elementary
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Separation Bound for Critical Points

- E.g., singular cubic Bezier.
- Prove separation bound $\Delta_4 > 0$ such that:
  - Distinct critical points are $\geq \Delta_4$ apart
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  - (Inflexion) $orient(p_0, p_1, p_2)orient(p_{m-2}, p_{m-1}, p_m) < 0$

- COROLLARY: If $diam(F) < \Delta_4$, and $CH(F) \cap CH(G) \neq \emptyset$, then we can detect any intersection involving critical points.

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Overall Algorithm Curves

- Generic subdivision algorithm: has queue Q_0 containing pairs of curves (F_i, G_i), $i \geq 0$.
- We now use 2 Queues, Q_0 and Q_1 for macro and micro pairs
  - (F, G) is a micro pair iff $diam(F, G) \leq \Delta$
- 2 Stages: macro stage and micro stage.
  - Initially, $Q_0 = ((F, G))$ and $Q_1 = \emptyset$
  - First do macro stage, then micro stage

- Macro Stage: acts like the generic subdivision algorithm.
  - But put pairs into macro or micro queue
- Micro Stage: extract (F', G') from Q_1 and apply "micro process".

Micro Process

- Input: micro pair (F, G)
- Output: intersection reps
  - 2 cases: base segments intersect or not
  - Basic principle: do easy tests first
  - Critical Point intersections can be directly detected
  - Either output intersection rep, or call "tangential process"

Open Problems

- Remove requirement on antipodal pairs for (F, G).
- Prove conjecture about antipodal pairs
- Better Separation Bounds: exploit Bezier form
- Implementation and comparison
- Complexity Analysis
- Extensions to other curves and surfaces

III. QUADRIC SURFACES

Skipped for time

Conclusions

- First complete adaptive intersection algorithm
- Complicated, but most of cases are unlikely
- Adaptive complexity
- Micro stage may be fast
  - Q_1 is most likely small
- Arithmetic on algebraic numbers are possible via resultant methods, but such methods are inefficient
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  - HINT: Easiest to just adapt my algorithm above!

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REFERENCE

- Chapter on curves in [Mehlhorn-Yap]
- Paper on Bezier Curves by Chee

"A rapacious monster lurks within every computer, and it dines exclusively on accurate digits."
– B.D. McCullough (2000)

THE END