I-TANGO for CAGD
(Intersections –
Topology, Accuracy & Numerics for
Geometric Objects)

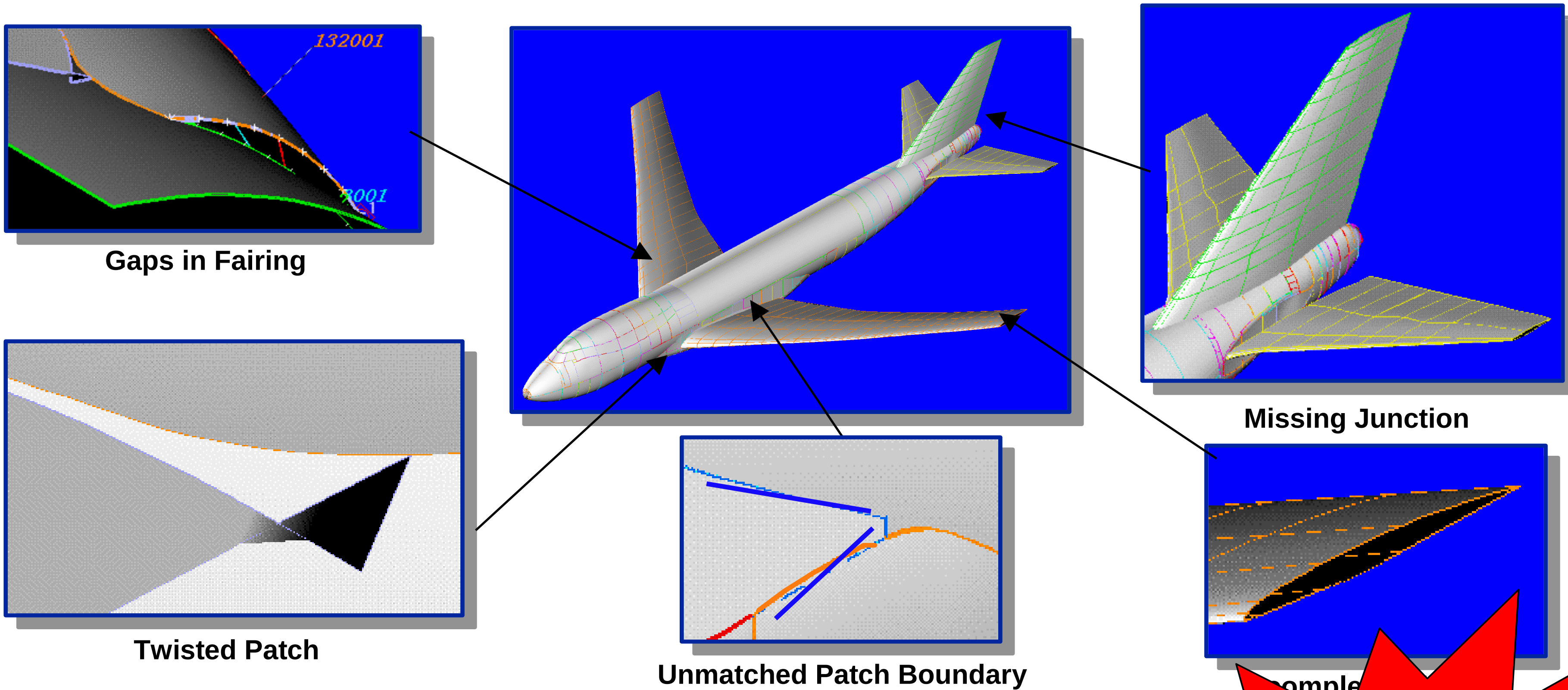
Thomas J. Peters (computational topology)
N. F. Stewart (numerical analysis & topology)
C. M. Hoffmann (robust computing)
N. M. Patrikalakis (shape modeling)
T. Maekawa (interval arithmetic)
T. Sakkalis (geometric theory)
D. R. Ferguson (approximation theory)

Talk Overview & Goals

Geometric modeling & spline surface intersectors:

- Analytic error bounds & topological fidelity,
- Robust & efficient solutions of nonlinear polynomial systems of equations, especially in the presence of multiple roots,
- Neighborhoods and rigorous interval spline enclosures of intersection sets.

Common Geometry Flaws - Examples



- Complex geometries come from CAD models
- Critical flaws in geometry are detected visually and repaired in ad h manner
- Improved/automated geometry diagnostics and repair tools will sig reduce cycle time

Accuracy, Robustness Problems Persist

‘I’ for Intersections

- intersectors for CAGD
- spline surfaces as input
- need for approximation
- two parametric pre-images

Davis Workshop

R. Farouki & D. R. Ferguson

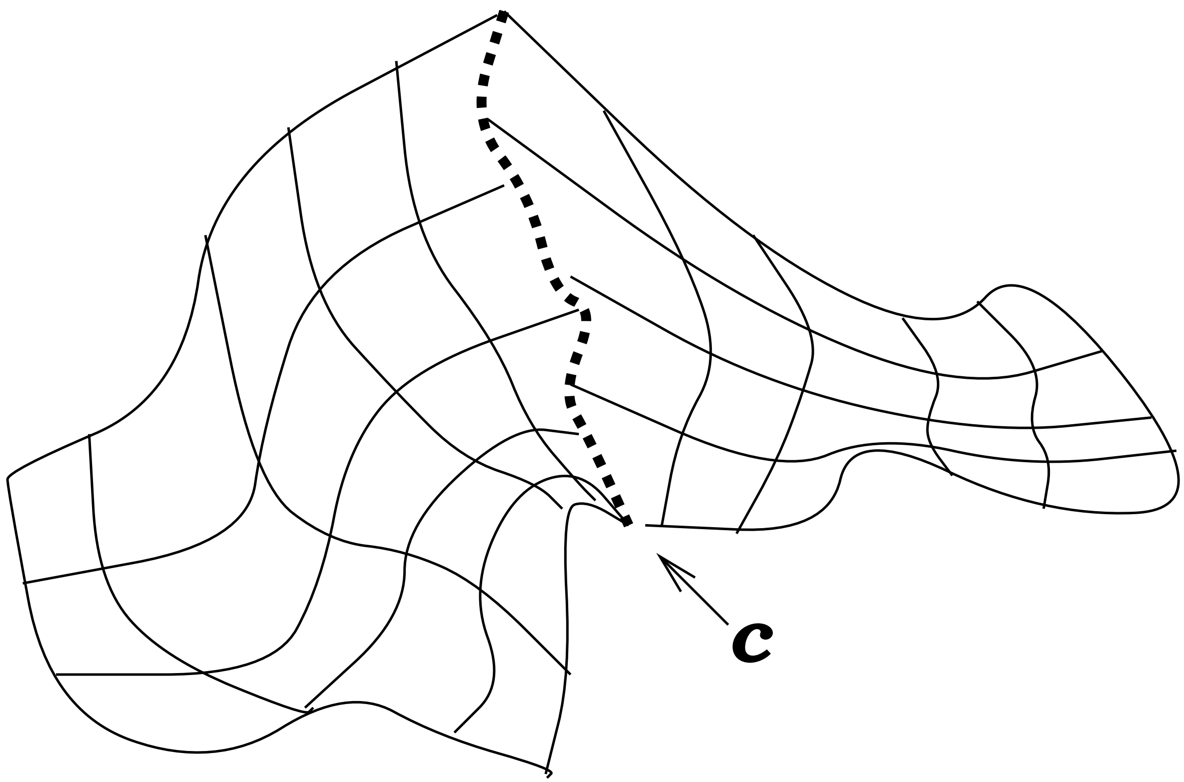
SIAM News, **6/99**

"Although modern CAD systems have attained a certain degree of maturity, their efficiency, reliability, and compatibility with subsequent analysis tools fall far short of what was envisaged at their inception, some 25 years ago. At the heart of this problem lie some deep mathematical issues, concerned with the computation, representation and manipulation of complex geometries, ..."

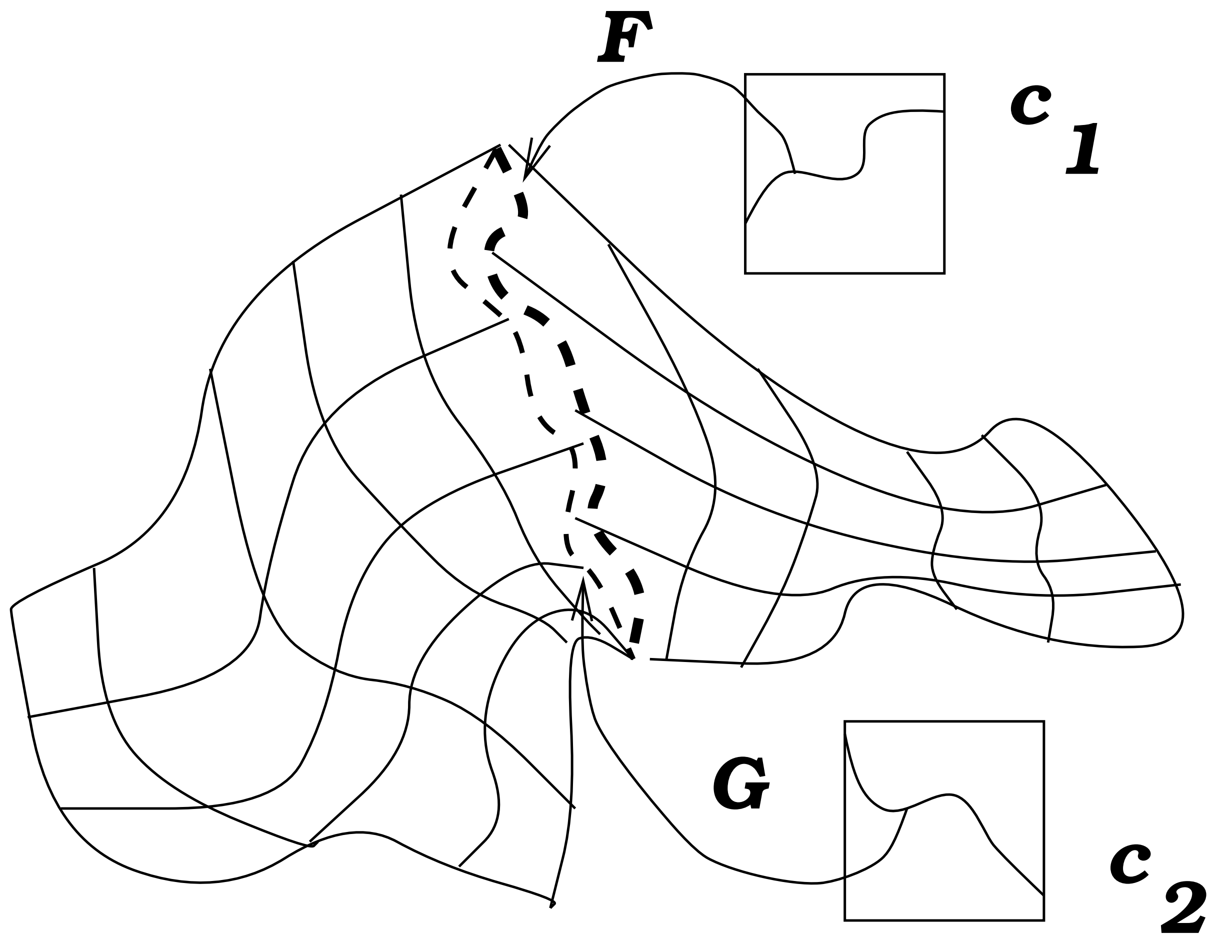
Tight integration remains elusive.

Intersecting surfaces are central.

Exact Conceptual View

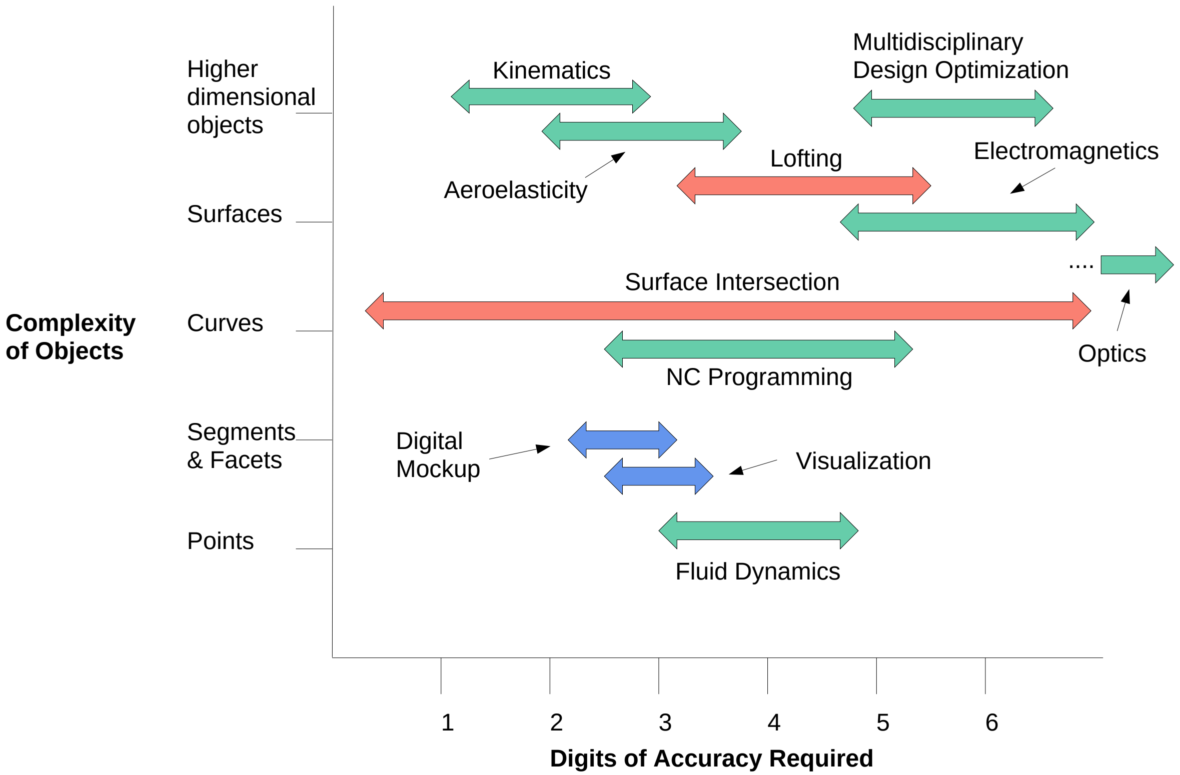


Approximations in Implementations



Geometry Flaws in Aerospace Design

Need to Satisfy Many Users but Requirements Vary



Demonstration

‘T’ for Topology

- Need for model-space bounds on errors
- The demo illustrates such errors
- Need for topological well-formedness
- Well-formed objects: definitions and sufficient conditions
- Perturbations don’t destroy well-formedness

Model-space Bounds

$$\|F(u_1, v_1) - F(u_0, v_0)\| \leq \epsilon M_1 + \epsilon^2 M_2,$$

where

$[u_0, v_0] = [\tilde{u}(\tau), \tilde{v}(\tau)]$, (approximate p-curve),

$[u_1, v_1] = [u(\tau), v(\tau)]$ (exact p-curve),

ϵ is an error bound (parametric domain) provided by a numerical method [GK]

M_i are (realistic, computable) bounds on the i-th order derivatives.

(Taylor's theorem & bounds on derivatives of basis functions.)

Auxiliary Slide

We have

$$\|\frac{\partial F}{\partial u}(u_0, v_0)\| + \|\frac{\partial F}{\partial v}(u_0, v_0)\| \leq M_1$$

and

$$\frac{1}{2}\|\frac{\partial^2 F}{\partial u^2}(u*, v*)\| + \|\frac{\partial^2 F}{\partial u \partial v}(u*, v*)\|$$
$$+\frac{1}{2}\|\frac{\partial^2 F}{\partial v^2}(u*, v*)\| \leq M_2.$$

Model-space Bounds (continued)

Let $\gamma(F)$ be such that

$$\epsilon M_1 + \epsilon^2 M_2 \leq \gamma(F),$$

and similarly for $\gamma(G)$.

Then,

$$\|F(\tilde{u}(\tau), \tilde{v}(\tau)) - G(\tilde{s}(\tau), \tilde{t}(\tau))\| \leq \gamma(F) + \gamma(G)$$

Gaps & Overlaps

Occur at the boundary between two trimmed patches, with non-zero error:

$$F(\tilde{u}(\tau), \tilde{v}(\tau)) - G(\tilde{s}(\tau), \tilde{t}(\tau))$$

. . . .

Further complications:

- An explicit representation of the boundary between these two patches;
- An explicit representation of the corner points between patches;
- Possibly mutually inconsistent & inconsistent with (given) topological information.

The Problem of Well-formed Representations

It would be useful, therefore, to give reasonable conditions for well-formedness of a representation:

- Under what conditions does the inconsistent data represent a homogeneously three-dimensional subset of R^3 with boundary close to the given data?
- And what exactly is this subset?

Perturbations should not destroy well-formedness

- Sufficient conditions that perturbations of polyhedral objects will produce a new object that is linked to the original by an ambient isotopy [Andersson, Dorney, Peters, Stewart, CAGD, 95];
- Sufficient conditions precluding self-intersection of faces of a curvilinear Bézier complex, and conditions precluding unwanted intersections of neighboring faces of curvilinear Bézier patches [Andersson, Peters, Stewart, CAGD, 98];
- A theorem for general curvilinear complexes showing that if there are no self-intersections, or other unwanted intersections, then the complex has the same topological form (ambient isotopy) [Andersson, Peters, Stewart, IJCGA, 00].

‘A’ for accuracy issues

Solving a System of Nonlinear Polynomial Equations f(x)=0

Solve the system

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \ldots = f_n(\mathbf{x}) = 0 \ .$$

in the domain

$[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_l, b_l]$

- $n > l$: Overdetermined system
- $n = l$: Balanced system
- $n < l$: Underdetermined system

Projected Polyhedron (PP) Algorithm

- Use affine parameter transformation such that

$$f_1(\mathbf{u}) = f_2(\mathbf{u}) = \ldots = f_n(\mathbf{u}) = 0 \ ,$$

where $\mathbf{u} \in [0,1]^l$.

- Change the basis to Bernstein basis, and restate the problem as the intersection of the graphs of the f_k and the hyperplane $u_{l+1} = 0$.

$$\begin{aligned} \mathbf{f}_k(\mathbf{u}) &= (u_1, u_2, \ldots, u_l, f_k(\mathbf{u})) \\ &= (\mathbf{u}, \sum_I^{M^{(k)}} \mathbf{v}_I^{(k)} B_{I,M^{(k)}}(\mathbf{u})) \ . \end{aligned}$$

- Use the convex hull property of multivariate Bernstein basis.

Rounded Interval Arithmetic (RIA)

Definition of interval number $[a, b] = \{x | a \leq x \leq b\}$

$$[a, b] + [c, d] = [\nabla(a + c), \triangle(b + d)]$$

$$[a, b] - [c, d] = [\nabla(a - d), \triangle(b - c)]$$

$$[a, b] \cdot [c, d] = [\nabla(ac), \triangle(bd)] \text{ or...}$$

$$[a, b]/[c, d] = [\nabla(a/c), \triangle(b/d)] \text{ or...}$$

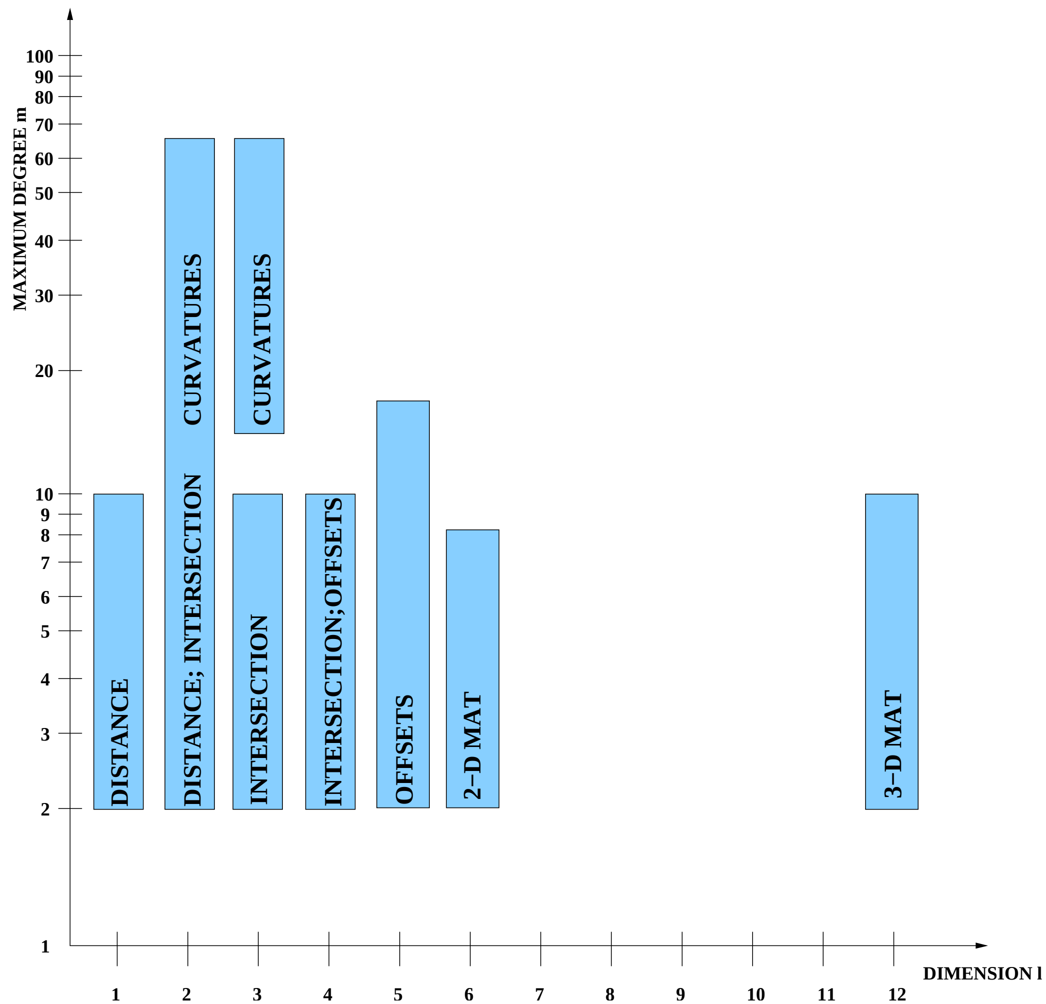
$$where \quad 0 \notin [c, d]$$

The lower (upper) bound is extended to include its previous (next) consecutive FP number.

Interval Projected Polyhedron (IPP) Algorithm

- There exist software rounding and hardware rounding methods.
- Software rounding is computationally more expensive than hardware rounding.
- Hardware rounding produces tighter interval bounds.

Applications of IPP Algorithm



Experience with the IPP Algorithm

- Loss of achievable accuracy from e.g. 10^{-12} for simple roots to 10^{-7} for double roots, etc.
- Slow for underdetermined systems with transversal or non-transversal intersections.
- Slow for balanced systems with a continuum of roots.

CAD Model Defects

Consequences of defects

- Failure of modeling operations
- Useless analysis results
- Defective products
- Tremendous rework in data exchange

Objective for Model Validity

Given a B-rep model in a certain data structure or file format, we want to answer the following questions:

- Is this model valid?
- If it is not valid, can it be rectified, and how?

Discussion

Prior approaches:

Local "identify-and-rectify" methods

Question:

What do we trust if there are inconsistencies between the topological structure and the geometric representation?

Hypothesis:

A given subset of surfaces is trustworthy.

Goal:

Given a B-rep model, if it contains defects, find the boundary representation intended by the designer.

Approach

Rectify-by-reconstruction approach:
Given a B-rep model, if its topological structure is valid, reconstruct a valid boundary using the surfaces, such that the new boundary is topologically equivalent to the original model and has the minimum geometric change.

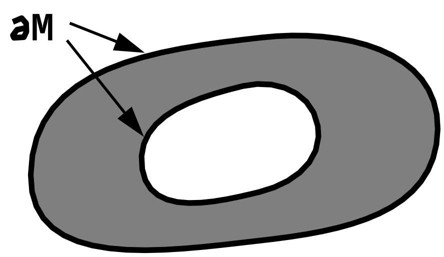
Definition

Let M be an ideal model, and $\mathcal{B}$ be a finite collection of boxes such that the following conditions are satisfied:

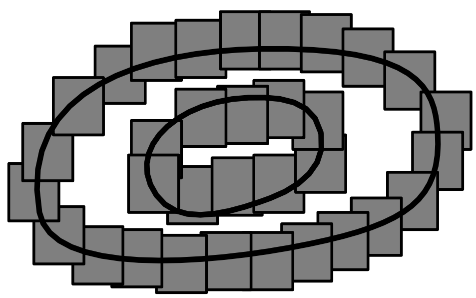
1. Let $b_i, b_j \in \mathcal{B}$. Whenever $b_i \cap b_j \neq \emptyset$, then $b_{ij} = b_i \cap b_j$ is a box.
2. $\partial M \subset \mathbf{B}$, where $\mathbf{B} = \cup\{b|b \in \mathcal{B}\}$, that is, $\mathcal{B}$ covers ∂M, and
3. $b \cap \partial M \neq \emptyset$, for every $b \in \mathcal{B}$.

We call $M^{\mathcal{B}} = M \cup \mathbf{B}$ the *interval solid* generated by M and $\mathcal{B}$.

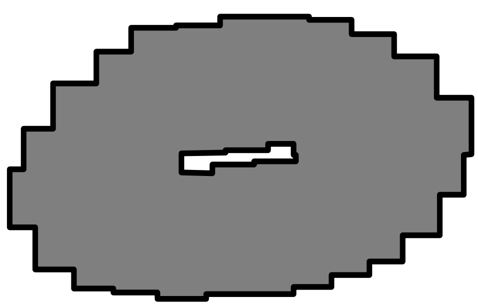
A 2d Example of an Interval Solid



(a) ideal solid M



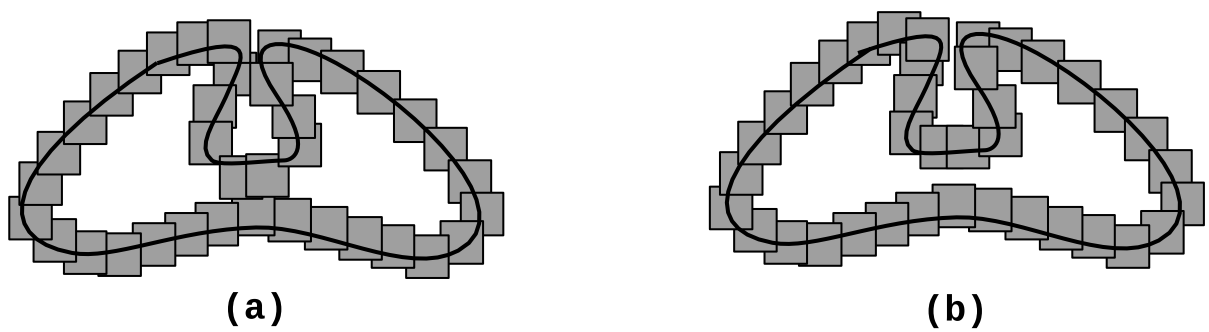
(b) boxes B



(c) interval solid M^B

Approximate equality

Motivation



Definition

Let M be a solid, and $\mathcal{B}$ be a finite collection of boxes. We say that the interval solid $M^{\mathcal{B}}$ is *approximately equal* to M if $M^{\mathcal{B}}$, as well as $M - \mathbf{B}$, are homeomorphic to M.

Theorem

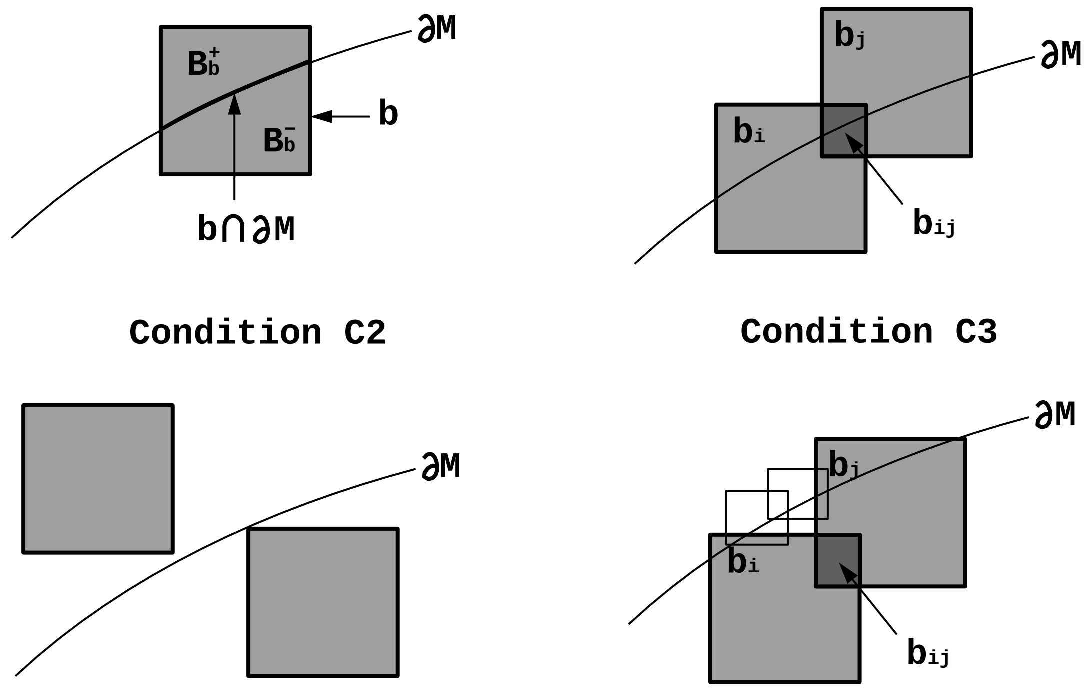
If $\mathcal{B}$ satisfies the following conditions, then, $M^{\mathcal{B}}$ is approximately equal to M:

C1. $\{Int(b_i),\ b_i \in \mathcal{B}\}$ is a cover of ∂M.

C2. Each member b of $\mathcal{B}$ intersects ∂M generically; in particular, $b \cap \partial M$ is a (closed) disk that separates b into two (closed) balls, B_b^+ and B_b^-, and

C3. Whenever $b_i \cap b_j \neq \emptyset$, then $b_{ij} = b_i \cap b_j$ is a box that satisfies **C2**, for $b_i, b_j \in \mathcal{B}$.

Conditions C2 and C3



Condition C2

Condition C3

Violation of condition C2

Violation of condition C3

Example

File format: STEP

Size: ~0.04m X ~0.06m X ~0.14m

Types of surfaces:
- 16 B-spline surfaces
- 3 cylindrical surfaces
- 5 planes

Global uncertainty: 1e-6

Number of topological entities
- V = 40 vertices
- E = 62 edges
- F = 24 faces
- L_i = 0 inner loops
- S = 1 shell

$V-E+F-L_i = 2(S-G)$

Genus $G = 0$

Model Validity Verification

Purpose

Given a B-rep model, verify its topological correctness, and its geometric consistency at certain resolutions.

Procedure

1. Verify the topological structure.
2. Grow the widths of the underlying curves of the edges.
3. Compute curve-surface intersections.
4. Construct the interval faces using the given topological structure.

Boundary Reconstruction

Purpose

Given a B-rep model, construct interval models at certain resolutions, using only the underlying surfaces.

Procedure

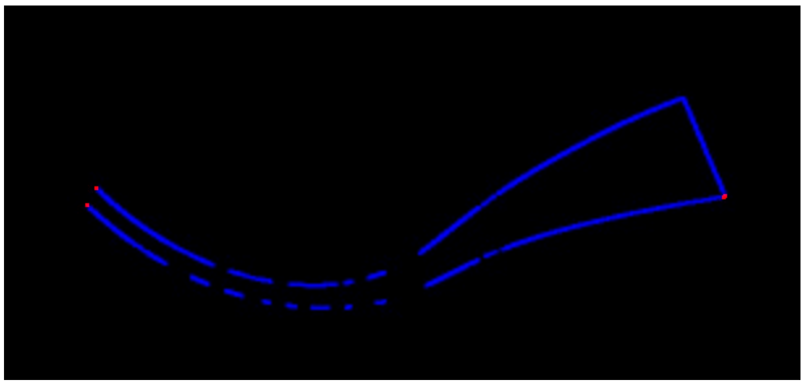
1. Grow the widths of the underlying surfaces of the faces.
2. Compute surface-surface intersections.
3. Construct the interval model.

Results

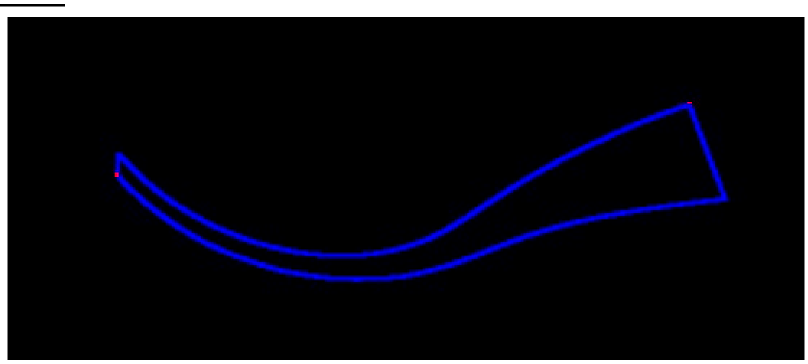
1. Can the model be reconstructed at the given resolution? No.
2. At what resolution, can the model be reconstructed?
   $\underline{5 \times 10^{-5} m.}$

Results

1. Is the model valid at the given resolution?
   No.



2. At what resolution the model is valid?
   $5 \times 10^{-4} m$.



‘N’ for Numerics

Increase accuracy and make geometric operations robust.

- good efficiency
- good data compactness
- relevance to practice

Robustness in Geometric Computations

Computation has inter-related predicates and constructors.
Interdependence usually fully evident.

- Predicate failure:
  - Computation fails catastrophically
  - Locally inconsistent structures created
  - Globally inconsistent, locally consistent
- Constructor failure:
  - Cracks, overlaps, interpenetrations
  - Geometry/topology mismatch

Approaches in the Literature

Exact arithmetic;

- e.g., Manocha and Keyser.
- Efficiency problems increase with larger algebraic degree.
- Input must be understood to be exact as written.

Issues:

- Output can be used as input to follow-on operations, hence input exactness assumption is justified.
- Problems with surface contacts of higher order.

Approaches in the Literature+

Exact, lazy predicates;

- e.g., Fortune, others.
- Distinguish predicates from constructors, permit inaccuracies in constructors, but do exact predicates where necessary.
- Input must be understood to be exact as written.

Issues: Output cannot be used as input to subsequent operations unless the problem domain is simple or general model rectification is available. Good results for polyhedra (some concept of rectification) but progress hard for higher degrees.

Industry Practice

- Pragmatically engineer the numerics such that, for the *applications envisioned*, acceptable output is generated that can be used downstream.
- Approach is inherently incomplete and does not address the underlying fundamentals of the problem.

Possible Role for Interval Arithmetic

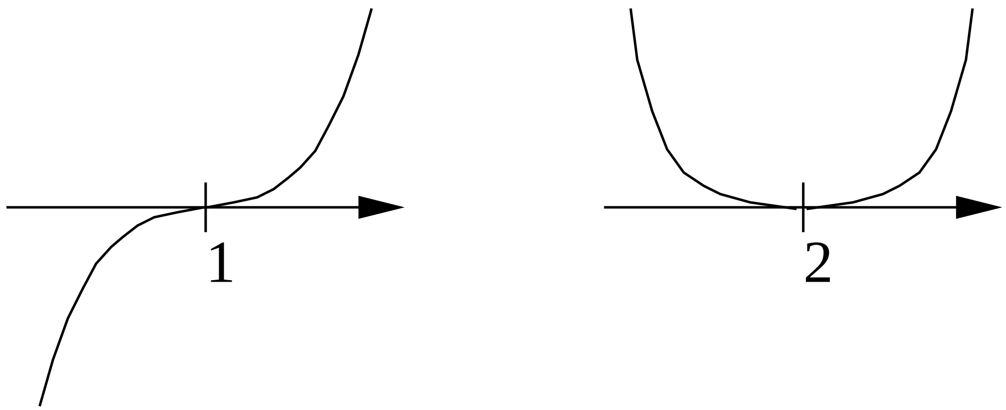
Numerical analysis usually works with *a priori* bounds, but run-time instance bounds can be sharper and can be provided by validated techniques, suitably augmented.

Perceived issues for interval arithmetic:

- pessimistic enclosures versus MIT work
- not an algebra
- but also little awareness of fix-point iterations and of the CG predicates used in exact, lazy work

Polynomial Graphs

$x^4 - 8x^3 + 24x^2 - 32x + 16$



$-x^3 + 3x^2 - 3x + 1$

Example

Polynomial evaluation near multiple roots.

Horner vs. interval, exact inner product:

Method	negative	zero	positive	sign rev.
$x^4 - 8x^3 + 24x^2 - 32x + 16$				
Horner	15	41	25	29
Interval	0	1	80	0
$-x^3 + 3x^2 - 3x + 1$				
Horner	25	41	15	9
Interval	40	1	40	1

Problems that must be addressed:

- Real intervals vs machine-representable intervals, implications for theory.
  Directed rounding, careful in implementations including I/O.
- Achievable precision. Exact inner product; directed intervals; extended intervals; ...
- Model interpretation and rectification.

Industrial Perspective

- MIT research in GK intersector
- Initial roots along boundaries
- Used daily at Boeing

Impact on Many Engineering Disciplines

- Manufacturing: NC programming
  (Beyond visual inspection.)
- Aerodynamics: Computational Fluid Dynamics (CFD)
- Observables: Computational Electro-Magnetics (CEM)
- Structures: Computational Structural Mechanics (FEA)

Root Causes

Surface intersections, except in trivial cases, *always* lead to gaps and self-intersections

- **For CAGD, exact representation not possible!**
  - Need to approximate.
  - Need for comprehensive theory.
- Intersections may be ill-conditioned
- The ideal of an algebra of regular closed sets remains unattained within software development with B-rep models.

NOT fixed in next software release!

Industrial Consortium Lessons Learned

¿From PDES, Inc. Geometry Accuracy Team

History

- 2+ years, mostly on model transfer
- broader integration beyond scope

Lessons Learned

- Current COTS tools are inadequate
- Long term effort from CARGO

Boeing Contributions

Geometric Software Libraries

- DT_NURBS
- GEML

Already Tangible Boeing Interaction

- Intern program
- Chris Mow, Summer 01
- Example shown in demo
- Model space error bounds
- His research integrated well
- Beta - testing for GEML, with Bob Ames (NSWC)

Economic Scope of Problems

NIST, March 99

U.S. Automotive Industry:

- At least \$1B/year
- from lack of model interoperability.

Intersection a hidden, critical problem.

(Cross-reference the Geometry Accuracy Team)

Industry – Academic Cooperation

In addition to employees and projects, consider

SIAM Meeting, June 23 - 25, 2003,
Mathematics in Industry, Challenges and Frontiers (Toronto)

Summary

- Analytic error bounds & topological fidelity,
- Robust & efficient solutions of nonlinear polynomial systems of equations, especially in the presence of multiple roots,
- Neighborhoods and rigorous interval spline enclosures of intersection sets.

Project Approach

Rigorous bounded enclosures of intersection sets, subject to user input, with attention to computing approximations due both to algorithmic truncation and floating-point arithmetic.