# Real Elementary Approach to the Master Recurrence and Generalizations

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# Next...

# Introduction

*We introduce the standard Master Theorem and indicate two directions for generalization*

# Solving Recurrences in Computer Science

## Sources of recurrences

- Probabilistic analysis
- Combinatorial analysis
- Analysis of algorithms ( this talk )

## Divide-and-Conquer recurrences

- (Mergesort) $T(n) = 2T(n/2) + n$
- (Strassen Matrix Mult.) $T(n) = 7T(n/2) + n^2$
- (Pan Matrix Multiplication) $T(n) = 143640 \cdot T(n/70) + n^2$
- (Schönhage-Strassen Mult.) $T(n) = 2T(n/2) + n \log n \log \log n$

# The Master Recurrence

## These are instances of:

- Master Recurrence (M.R.): $\boxed{T(n) = aT(n/b) + d(n)}$
  - where $a > 0$ and $b > 1$ are real constants
  - and $d(n)$ is the driving function.

## The solution $T(n)$ is controlled by:

- the watershed function $w(n) := n^{\alpha}$
- where $\alpha := \log_b a$ ( watershed constant )

E.g., $\alpha = \log_2 7 = 2.807\ldots$ in Strassen matrix multiplication.

# The Standard Master Theorem (M.T.)

## The Master Recurrence solution satisfies a "trichotomy":

By comparing $d(n)$ with $w(n) = n^{\alpha}$,

$$T(n) = \Theta \begin{cases} n^{\alpha} & \text{if } d(n) = \mathcal{O}(w(n)n^{-\varepsilon}) \\ n^{\alpha}\log n & \text{if } d(n) = \Theta(w(n)) \\ d(n) & \text{if ``} d(n) = \Omega(w(n)n^{\varepsilon}) \text{''} \end{cases} \quad \begin{matrix} \text{Case } (-) \\ \text{Case } (0) \\ \text{Case } (+). \end{matrix}$$

## Remarks

- From [Bentley-Haken-Saxe 1980, Cormen-Leiserson-Rivest 1990]
- Regularity Condition: $d(n) = \Omega(w(n)n^{\varepsilon})$ means: $(\exists C > 1)$ s.t. $d(n) \geq C \cdot a \cdot d(n/b)$

# Two Directions for Generalization

## A. More General Driving Functions

- Trichotomy captures $d(n) = \Theta(n^{\alpha})$, or when $d(n) = \Theta(n^{\alpha \pm \varepsilon})$ ($\varepsilon > 0$)
- Does not capture: $d(n) = n^{\alpha} f(n)$ s.t. $f(n)$ is polylogarithmic
- E.g., $d(n) = n^{\alpha} \log n$ (this arises in integer GCD)

## B. Multiterm Master Recurrence (M.M.R.)

- Linear Median Algorithm: $T(n) = T(n/5) + T(7n/10) + n$
- Conjugation tree [Welzl-Edels.]: $T(n) = T(n/2) + T(n/4) + \log n$
- Generally, the M.M.R. is $T(n) = d(n) + \sum_{i=1}^{k} a_i T(n/b_i)$
  - where $a_i > 0$ and $b_i > 1$ are real constants

# Literature

## A. “Tetrachotomous” Master Theorem

- Trichotomy $\longrightarrow$ “Tetrachotomy” (4 Cases)
- [Brassard-Bratley 1996, Verma 1994, Wang-Fu 1996, Roura 1997]

## B. Multiterm Master Theorem

- Discussed in [Brown & Purdom (1985, Text, p. 243]
- 2-Term Case: [Kao 1997]
- Trichotomous Version: [Roura 1997, Akra-Bazzi 1998]

## C. Other Topics

- General Integral bounds: [Akra-Bazzi, Verma, Wang-Fu]
- Master Recurrence with $a(n), b(n)$: [Wang-Fu 1996]
- Robustness issues: [Leighton 1996, Roura 1997]

# “Tetrachotomous” Master Theorem

## The Master Recurrence solution satisfies a “tetrachotomy”:

By comparing $d(n)$ with $w(n)\log^{\delta} n$,

$$T(n) = \Theta \begin{cases} n^{\alpha} & \text{if } d(n) = \mathcal{O}(w(n)\log^{\delta} n),\ \delta < -1 & \boxed{\text{Case } (-)} \\ d(n)\log n \log\log n & \text{if } d(n) = \Theta(w(n)\log^{\delta} n),\ \delta = -1 & \boxed{\text{Case } (1)} \\ d(n)\log n & \text{if } d(n) = \Theta(w(n)\log^{\delta} n),\ \delta > -1 & \boxed{\text{Case } (0)} \\ d(n) & \text{if ``} \boxed{d(n) = \Omega(w(n)n^{\varepsilon})} \text{''} & \boxed{\text{Case } (+)} \end{cases}$$

## Remarks

- From [Brassard-Bratley 1996, Verma 1994, Wang-Fu 1996, Roura 1997]
- Still does not capture the Schönhage-Strassen recurrence,

# Next...

# Our Results

*We state our two main theorems, and illustrate their applications.*

# Overview of Results

## Two Main Theorems

- Theorem A extends the Tetrachotomous M.T. to infinitely many cases
  - A natural completion of Tetrachotomous M.T.
- Theorem B is a Multiterm generalization of Tetrachotomous M.T.
  - Proof uses a Principle of Real Induction

## Our Approach

- We propose a “real approach” to such recurrences
  - Treat all variables in recurrences as real numbers
  - This is essential for the multiterm theorem
- We introduce “elementary techniques” to derive these results
  - “Elementary” means non-calculus
  - Possible because we stress $\Theta$-order results

# Statement of Theorem B

Recall the Multiterm Master Recurrence (M.M.R.):

$$T(n) = d(n) + \sum_{i=1}^{k} a_i T(n/b_i)$$

Its watershed function $w(n) := n^{\alpha}$

where $\alpha$ satisfies $\sum_{i=1}^{k} \frac{a_i}{b_i^{\alpha}} = 1$.

## The M.M.R. solution satisfies a "tetrachotomy":

By comparing $d(n)$ with $w(n) \log^{\delta} n$,

$$T(n) = \Theta \begin{cases} n^{\alpha} & \text{if } d(n) = \mathcal{O}(w(n)\log^{\delta} n),\ \delta < -1 & \text{Case } (-) \\ d(n)\log n \log\log n & \text{if } d(n) = \Theta(w(n)\log^{\delta} n),\ \delta = -1 & \text{Case } (1) \\ d(n)\log n & \text{if } d(n) = \Theta(w(n)\log^{\delta} n),\ \delta > -1 & \text{Case } (0) \\ d(n) & \text{if ``} d(n) = \Omega(w(n) n^{\varepsilon}) \text{''} & \text{Case } (+) \end{cases}$$

# Remarks on Theorem B

- The first "tetrachotomous" Multiterm Master Theorem
- "$d(n) = \Omega(w(n)n^{\varepsilon})$" is the multiterm regularity condition:

$$(\exists C > 1) \qquad d(n) \geq C \cdot \sum_{i=1}^{k} a_i \cdot d\left(\frac{n}{b_i}\right)$$

which implies $d(n) = \Omega(w(n)n^{\varepsilon})$.

# Iterated Logarithms

To state Theorem A, we need some preparation:

## Iterated Logarithms

- $\ell\ell g_k(x) := \underbrace{\lg(\lg(\cdots(\lg(x))\cdots))}_{k \text{ times}}$
  - where $\lg := \log_2$ is "computer science logarithm"
  - E.g., $\ell\ell g_0(x) = x$ and $\ell\ell g_2(x) = \lg\lg x$
- Extend to negative indices for $k$:
  - E.g., $\ell\ell g_{-1}(x) = 2^x$ and $\ell\ell g_{-2}(x) = 2^{2^x}$

# Exponential-Logarithmic (EL) Functions

## Products of powers of iterated logs

- E.g., $f_0(x) = 2^{5x} x^4 \lg^{-3} x (\lg\lg x)^2$
- Exponent sequence of $f_0(x)$ is $\mathbf{e} = (5, 4 \,;\, -3, 2)$

## Definition

- EL function has the form $f(x) = \mathrm{EL}^{\mathbf{e}}(x) := \prod_{i \in \mathbb{Z}} \ell\ell g_i^{e_i}(x)$
  - where $e_i = \mathbf{e}(i)$ for some $\mathbf{e} : \mathbb{Z} \to \mathbb{R}$ with finite support
- Exponent sequence corresponding to $\mathbf{e} : \mathbb{Z} \to \mathbb{R}$ can be
  - written as any finite sequence $\mathbf{e} = (e_{-k}, \ldots, e_{-1}, e_0 \,;\, e_1, \ldots, e_\ell)$ s.t. $\mathbf{e}(i) \neq 0$ implies $-k \leq i \leq \ell$
  - E.g., $f_0(x) = 2^{5x} x^4 \lg^{-3} x (\lg\lg x)^2$ is denoted $\mathrm{EL}^{(5,4\,;\,-3,2)}(x)$

# Theorem A in Action

## Consider $d(n)$ near $n^\alpha$ ("at the cusp of convergence")

| Driving Function | Exponent Sequence | |
|---|---|---|
| $d_0(n) := n^\alpha \log n \log\log n$ | $\mathbf{e} = (\alpha\,;1,1)$ | (Schönhage-Strassen) |
| $d_1(n) := n^\alpha (\log\log n)^r$ | $\mathbf{e} = (\alpha\,;0,r)$ | |
| $d_2(n) := n^\alpha \frac{(\log\log\log n)^s}{\log n \log\log n}$ | $\mathbf{e} = (\alpha\,;-1,-1,s)$ | $(s \neq -1)$ |

## Conclusion of Theorem A:

| Solution | Exponent Sequence |
|---|---|
| $T_0(n) = \Theta(n^\alpha \log^2 n \log\log n$ | $\mathbf{e} = (\alpha\,;2,1)$ |
| $T_1(n) = \Theta(n^\alpha \log n (\log\log n)^r)$ | $\mathbf{e} = (\alpha\,;1,r)$ |
| $T_2(n) = \Theta \begin{cases} n^\alpha (\log\log\log n)^{s+1} \\ n^\alpha \end{cases}$ | $\left.\begin{matrix} \mathbf{e} = (\alpha\,;0,0,s+1), & s > -1 \\ \mathbf{e} = (\alpha\,;0,0,0), & s < -1 \end{matrix}\right\}$ |

# Cusp Order

- Suppose $\mathbf{e} = (\alpha ; e_1, e_2, \ldots)$
- Its cusp order is $h \geq 1$ if
  - $\mathbf{e} = (\alpha ; \underbrace{-1, -1, \ldots, -1}_{\leq h-1}, \beta, \ldots)$ for some $\beta \neq -1$
  - Also, $\beta$ is the cusp power
- Transfer these concepts to EL-functions:
- E.g., $d_2(n) = n^{\alpha} \frac{(\log\log\log n)^s}{\log n \log\log n} = \mathrm{EL}^{(\alpha ; -1, -1, s)}(n)$
- So, its cusp order is 3 and cusp power is $s$

# Statement of Theorem A

- Recall: Master Recurrence (MR) $\boxed{T(n) = aT(n/b) + d(n)}$
  - with watershed constant $\alpha = \log_b a$
- Also let $d(n) = \mathrm{EL}^{\mathbf{e}}(n)$
  - where $\mathbf{e} = (e_{-k}, e_{-k+1}, \ldots, e_0 \,;\, e_1, \ldots, e_\ell)$, and $e_{-k} \neq 0$
- If $k = 0$, let the cusp order be $h$ and cusp power be $\beta$

## The Generalized M.T.

The solution to the MR satisfies $T(n) =$

$$\Theta \begin{cases} d(n) & \text{if } (k < 0 \wedge c > 0) \text{ or } (k \geq 0 \wedge \mathbf{e}(0) > \alpha), & \boxed{\text{Case } (+)} \\ d(n) LL_h(n) & \text{if } (k = 0 \wedge \mathbf{e}(0) = \alpha \wedge \beta > -1), & \boxed{\text{Case } (h-1)} \\ n^\alpha & \text{otherwise} & \boxed{\text{Case } (-)} \end{cases}$$

where $LL_h(n) := \prod_{i=1}^{h} \ell\ell g_i(n) = \lg n \cdot \lg\lg n \cdots \ell\ell g_h(n)$.

# Remarks on Theorem A

- Infinitely many cases (for each $h = 1, 2, 3, \ldots,$)
- $h = 1$ is $\boxed{\text{Case (0)}}$ in the Standard M.T.
- $h = 2$ is $\boxed{\text{Case (1)}}$ in the "tetrachotomous" M.T.
- $h = 3$ captures the Schönhage-Strassen recurrence

# Next...

# Some Tools

*We show three slides describing our basic tools*

# Summation based on Growth Types

- Given function $f : \mathbb{R} \to \mathbb{R}$, we want to bound the summation
$$S^f(n) := \sum_{x \geq 1}^{n} f(x) = f(n) + f(n-1) + \cdots + f(n - \lfloor n \rfloor + 1)$$
where $n, x$ are real variables
- Classify functions $f : \mathbb{R} \to \mathbb{R}$ as: polynomial-type, increasing or decreasing exponential-type
- THEOREM: $S^f(n) = \Theta \begin{cases} nf(\Theta(n)) & \text{if } f \text{ is polynomial-type,} \\ f(n) & \text{if } f \text{ increases exponentially,} \\ 1 & \text{if } f \text{ decreases exponentially.} \end{cases}$

REMARK: Thus we reduce the problem of summation to classifying growth-types, which is an easier problem. Moreover, growth-types are closed under various basic operations

# Elementary Sums

- In case $f$ is an EL-function, $f(n) = \mathrm{EL}^{\mathbf{e}}(n)$,
  we write $S^{\mathbf{e}}(n)$ for the sum $S^{f}(n)$.
- Call $S^{\mathbf{e}}(n)$ an elementary sum
- THEOREM:
  Up to $\Theta$-order, an elementary sum is an EL-function.
  I.e., $S^{\mathbf{e}}(n) = \Theta(\mathrm{EL}^{\mathbf{e}'}(n))$
  where $\mathbf{e}'$ can be explicitly constructed from *bfe*

REMARK: THEOREM A can be reduced to this result on elementary sums.

# Principal of Real Induction

- Let $P(x)$ be a real predicate.
- Principle of Archimedean Induction:

  Suppose there exists real numbers $x_1$ (cutoff constant) and $\gamma > 0$ (gap constant) such that
  Real Basis (RB): For all $x < x_1$, $P(x)$ holds
  Real Induction (RI): For all $y \geq x_1$, if $(\forall x \leq y - \gamma)P(x)$, then $P(y)$

REMARK: Proof of THEOREM B makes essential use of this Principle. The principle is valid because of the Archimedean property of the reals.

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# Final Remarks

# Where are the Initial Conditions?

- We deliberately ignored initial conditions
- We may simply specify a "Default Initial Condition" (DIC):

  $T(n) = C$ for all $n \leq n_0$ and for some $n_0, C \geq 0$

- All our $\Theta$-bounds are robust under any choice of DIC

# Conclusion

- Our results provide “Cookbook” Theorems for easy application
    - Theorems A and B have the cookbook form of the standard M.T.
- Our real *and* elementary approach simplifies current literature
- The full paper will discuss robustness issues, and unified generalization of Theorems A and B.

# Thanks for Listening!

*"A rapacious monster lurks within every computer, and it dines exclusively on accurate digits."*

— B.D. McCullough (2000)