Tutorial: Exact Numerical Computation in Algebra and Geometry

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PART 2

Explicitization and Subdivision

"It can be of no practical use to know that π is irrational, but if we can know, it surely would be intolerable not to know."

— E.C. Titchmarsh

Coming Up Next

Towards Exact Numerical Computation (ENC)

Beyond the Universal Solution

Design algorithms directly incorporating the principles of EGC

- What do we need? What are its features?
  - It must be numerical in nature
  - It must be arbitrary precision
  - It must respect zero
  - It must be adaptive
    - actively control precision
    - exploit filters

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Computational Ring Approach

Computational Ring $(\mathbb{D}, 0, 1, +, -, \times, \div 2)$

- $\mathbb{D}$ is countable, dense subset of $\mathbb{R}$
- $\mathbb{D}$ is a ring extension of $\mathbb{Z}$
- Efficient representation $\rho : \{0,1\}^* \dashrightarrow \mathbb{D}$ for implementing ring operations, and exact comparison.

Examples of $\mathbb{D}$

- BigFloats or dyadic numbers:

$$\mathbb{F} := \{m2^n : m, n \in \mathbb{Z}\} = \mathbb{Z}[\frac{1}{2}]$$

- Rationals: $\mathbb{Q}$ (avoid, if possible)
- Real Algebraic Numbers: $\mathbb{A}$ (AVOID!)

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What else is needed in ENC Algorithms?

Intervals

- $\square\mathbb{D}$: set of dyadic intervals
- $\square\mathbb{D}^n$: set of n-boxes

Box Functions

Box function

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From Implicit to Explicit Representation

- Mesh generation [Problem (IV)]
- Discrete Morse-Smale complex [Problem (V)]
- Arrangement of hypersurfaces
- Voronoi diagram of a collection of objects
- Cell complex approximation of algebraic variety
- Representation of Flow fields

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- **Projection Based** (Refinements of CAD)

  E.g., [Mourrain and Tecourt (2005); Cheng, Gao, and Li (2005)]

- **Algebraic Subdivision Schemes**

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- **Properties** Exact; complete (usually); slow (in general); hard to implement

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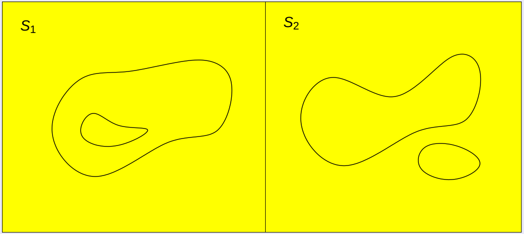
Two Criteria of Meshing

I. Topological Correctness

The approximation $\widetilde{S}$ is isotopic to the S.



- S_1 and S_2 are homeomorphic, but not isotopic
- Ambient space property!

(contd.) Two Criteria of Meshing

II. Geometrical Accuracy (ε-closeness)

For any given $\varepsilon > 0$, the Hausdorff distance $d(S, \widetilde{S})$ should not exceed ε.

- Set $\varepsilon = \infty$ to focus on isotopy.

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- Focus on (purely) Numerical Subdivision methods
- Algorithms for Meshing Curves (and Surfaces)
- What will be New?
  Numerical methods that are exact and can handle singularities

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Coming Up Next

Subdivision Algorithms

- Viewed as generalized binary search, organized as a quadtree.
- Here is a typical output:

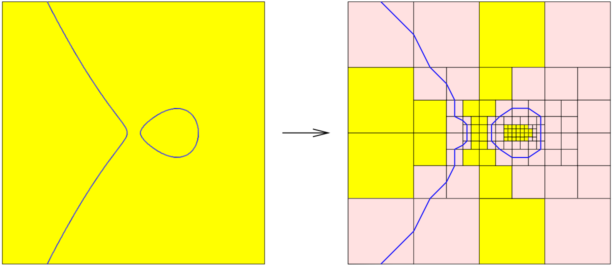

Figure: Approximation of the curve $f(X,Y) = Y^2 - X^2 + X^3 + 0.02 = 0$

The Generic Subdivision Algorithm

- INPUT: Curve $S = f^{-1}(0)$, box $B_0 \subseteq \mathbb{R}^2$, and $\varepsilon > 0$
- OUTPUT: Graph $G = (V, E)$,
  representing an isotopic ε-approximation of $S \cap B_0$.

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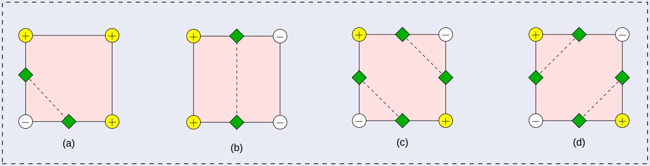
E.g., Marching Cube

Subdivision Phase

Subdivide until size of each box $\leq \varepsilon$.

Construction Phase

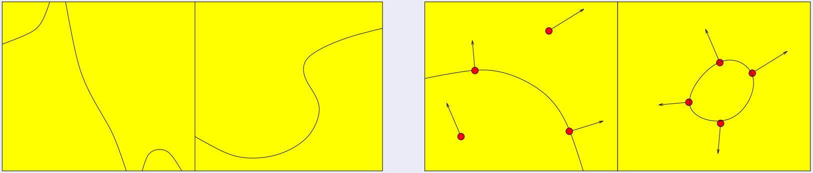
(1) Evaluate sign of f at grid points, (2) insert vertices, and (3) connect them in each box:



Cannot guarantee the topological correctness

Parametrizability and Normal Variation

Parametrizable in X-direction



(a) (b) (c) (d)

- (a) Parametrizable in X-direction
- (b) Non-parametrizable in X- or Y-direction
- (c) Small normal variation
- (d) Big normal variation

Box Predicates

Three Conditions (Predicates)

C0	$0 \notin \square f(B)$	Exclusion
Cxy	$0 \notin \square f_x(B)$ or $0 \notin \square f_y(B)$	Parametrizability
C1	$0 \notin \square f_x(B)^2 + \square f_y(B)^2$	Small Normal Variation

Implementation: e.g., $f(x,y) = x^2 - 2xy + 3y$

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Snyder's Algorithm

Subdivision Phase

For each box B:

- $C_0(B) \Rightarrow$ discard
- $\neg C_{xy}(B) \Rightarrow$ subdivide B

Construction Phase

- Determine intersections on boundary
- Connect the intersections
- (Non-trivial, unbounded complexity)

Boundary Analysis is not good (may not even terminate).

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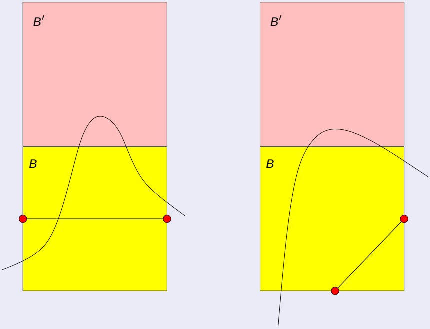
Idea of Plantinga and Vegter

Introduce a strong predicate $C1$ predicate

- Allow local NON-isotopy
  Local incursion and excursions



- Locally, graph is not isotopic

- Simple box geometry
  (simpler than Snyder, less simple than Marching Cube)

Plantinga and Vegter's Algorithm

Exploit the global isotopy

- Subdivision Phase: For each box B:
  - $C_0(B) \Rightarrow$ discard
  - $\neg C_1(B) \Rightarrow$ subdivide B
- Refinement Phase: Balance!

(contd.) Plantinga and Vegter's Algorithm

Global, not local, isotopy

- Construction Phase:

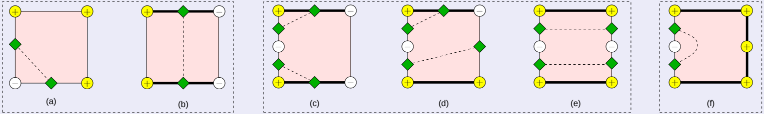


Figure: Extended Rules

- Local isotopy is NOT good !

Coming Up Next

Idea of Cxy Algorithm

Replace C1 by Cxy

- $C_1(B)$ implies $C_{xy}(B)$
- This would produce fewer boxes.

Exploit local non-isotopy

- Local isotopy is an artifact!
- This also avoid boundary analysis.

Obstructions to Cxy Idea

Replace C1 by Cxy

- Just run PV Algorithm but using C_{xy} instead:
- What can go wrong?

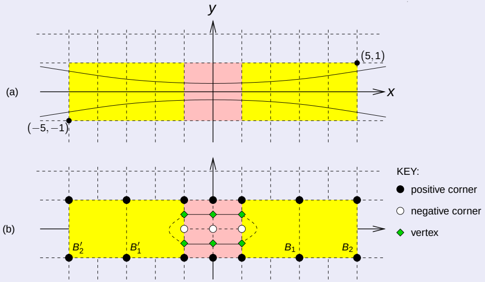


Figure: Elongated hyperbola

Cxy Algorithm

- Subdivision and Refinement Phases: As before
- Construction Phase:

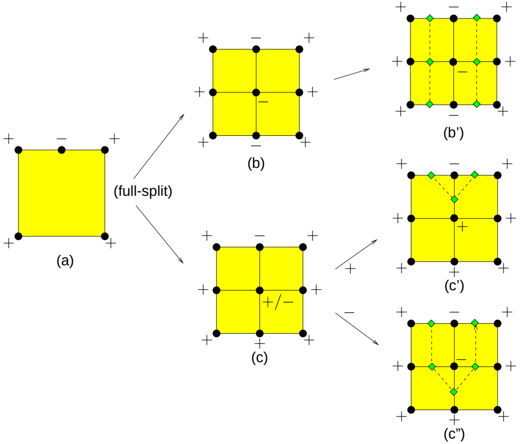


Figure: Resolution of Ambiguity

Mini Summary

- What has Cxy Algorithm done?
  - Exploit Parametrizability (like Snyder)
  - Rejected local isotopy (like PV)
- Up Next: More improvements

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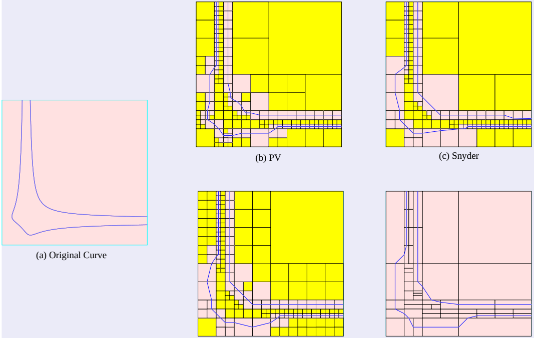
Coming Up Next

Idea of Rectangular Cxy Algorithm

Exploit Anisotropy



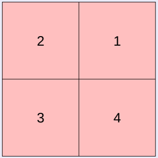
(a) Original Curve (b) PV (c) Snyder (d) Balanced Cxy (e) Rectangular Cxy

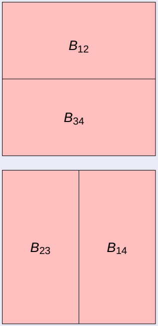
- "Heel Curve" $X^2Y^2 - X + Y - 1 = 0$ in box $B = [(-2, -10), (10, 2)]$
- Comparing PV, Snyder, Cxy, Rect Cxy

Partial Splits for Rectangles

Splits

Full-splits:
$B \longrightarrow (B_1, B_2, B_3, B_4)$





- Horizontal Half-split:
  $B \longrightarrow (B_{12}, B_{34})$
- Vertical Half-split:
  $B \longrightarrow (B_{14}, B_{23})$

Rectangular Cxy Algorithm

What is needed

- **Aspect Ratio Bound:** $r > 1$ arbitrary but fixed.
- **Splitting Procedure:** do full-split if none of these hold

L_0 :	$C_0(B), C_{xy}(B)$	Terminate
L_{out} :	$C_0(B_{12}), C_0(B_{34}), C_0(B_{14}), C_0(B_{23})$	Half-split
L_{in} :	$C_{xy}(B_{12}), C_{xy}(B_{34}), C_{xy}(B_{14}), C_{xy}(B_{23})$	Half-split

- Axis-dependent balancing: each node has a X-depth and Y-depth.

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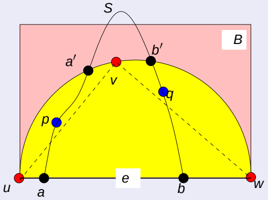
Ensuring Geometric Accuracy

Buffer Property of C1 predicate

- Aspect Ratio ≤ 2:



Half-circle argument

- Generalize $C_1(B)$ to $C_1^*(B)$. for any box B

Comparisons

- Compare Rect Cxy to PV (note: Snyder has degeneracy).
  - Curve $X(XY-1)=0$, box $B_s := [(-s,-s),(s,s)]$, Aspect ratio bound $r=5$: (JSO=Java stack overflow)

#Boxes/Time(ms)	$s=15$	$s=60$	$s=100$
PV	5686/157	JSO	JSO
Cxy	2878/125	45790/2750	JSO
Rect	258/32	3847/766	11196/7781

- Increasing r can increase the performance of Rect Cxy.
  - $r=80, s=100 \Rightarrow Boxes/Time(ms) = 751/78$

Comparisons (2)

- Compare to Snyder's Algorithm.
  - Curve $X(XY-1)=0$, box $B_n := [(-14\times 10^n, -14\times 10^n), (15\times 10^n, 15\times 10^n)]$. Maximum aspect ratio $r=257$.

#Boxes/Time(ms)	$n=-1$	$n=0$	$n=1$
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Cxy	13/0	1510/62	JSO
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Summary of Experimental Results

- Cxy combines the advantages of Snyder & PV Algorithms.
- Can be significantly faster than PV & Snyder's algorithm.
- Rectangular Cxy Algorithm can be significantly faster than Balanced Cxy algorithm.

Coming Up Next

Boundary (Summary)

- An Obvious Way and a Better Way
  - Exact Way : Recursively solve the problem on ∂B_0
  - Better Way : Exploit isotopy
- Price for Better Way: Weaker Correctness Statement

  For some $B_0 \subseteq B_0^+ \subseteq B_0 \oplus B(\varepsilon)$,

  G is isotopic to $S \cap B_0^+$.
- APPLICATIONS:

  Singularity (below)

  Input region B_0 to have "any" geometry, even holes, provided it contains no singularities.

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Coming Up Next

Singularity : Algebraic Preliminary

- Square-free part of $f(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$:
  $$\frac{f}{\mathtt{GCD}(f, \partial_1 f, \ldots, \partial_n f)} = \frac{f}{\mathtt{GCD}(f, \nabla(f))}$$
- For $n = 1$: square-free implies no singularities
- Generally:
  Singular set $sing(f) := \mathtt{Zero}(f, \nabla(f))$ has co-dimension ≥ 2.
- For Curves:
  we now assume $f(X, Y) \in \mathbb{Z}[X, Y]$ has isolated singularities.

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Some Zero Bounds

Evaluation Bound Lemma

If $f(X,Y)$ has degree d and height L then

$$\boxed{-\log EV(f) = O(d^2(L + d\log d))}$$

where $EV(f) := \min\{|f(\alpha)| : \nabla(\alpha) = 0, f(\alpha) \neq 0\}$

Singularity Separation Bound [Y. (2006)]

Any two singularities of $f = 0$ are separated by

$$\boxed{\delta_3 \geq (16^{d+2} 256^L 81^{2d} d^5)^{-d}}$$

Closest Approach Bound

The "locally closest" approach of a curve $f = 0$ to its own singularities is

$$\boxed{\delta_4 \geq (6^2 e^7)^{-30D} (4^4 \cdot 5 \cdot 2^L)^{-5D^4}}$$

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Isolating Singularities

Mountain Pass Theorem

Consider $F := f^2 + f_X^2 + f_Y^2$.

Any 2 singularities in B_0 are connected by paths $\gamma : [0,1] \to \mathbb{R}^2$ satisfying

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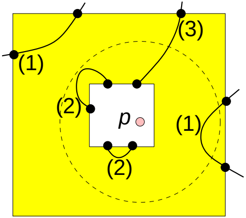
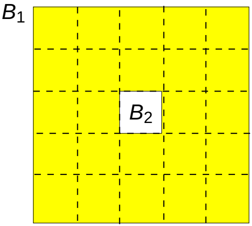
Degree of Singularites

- Degree of singularity := number of half-branches
- Use two concentric boxes $B_2 \subseteq B_1$:
  inner box has singularity, outer radius less than δ_3, δ_4



(a)



(b)

(a) Singularity p with 3 types of components

(b) Concentric boxes (B_1, B_2)

Mini Summary

- We have seen how to combine Snyder and PV, and make several practical improvements
- Future Work: Extend *3D* (and beyond?)
- Improve efficiency of refinement
- Improve efficiency of singularity
  - Nonsingular case is fast
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  Explicitization Problems
- ENC Algorithms for them are novel
- Numerical Treatment of Singularity and Degeneracy
  - Possible in theory, but severe practical challenge

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