(Extended Abstract) Integral Analysis of Evaluation-Based Root Isolation

Michael Burr, Felix Krahmer and Chee Yap
Courant Institute
New York University
{burr,krahmer,yap}@cims.nyu.edu

April 11, 2008

Abstract

Let f be a univariate polynomial with real coefficients, $f \in \mathbb{R}[X]$. Subdivision methods are widely used for isolating the roots of f in a given interval. In this paper we consider **evaluation-based subdivision** algorithms that use using simpler primitives than well-known subdivision methods such as Sturm or Descartes methods. In particular, we study the EVAL algorithm based on an algorithm by Mitchell, which can be seen as a 1-dimensional analogue of Plantinga-Vegter's meshing algorithm.

(1) First we give a general framework for performing an adaptive analysis of the EVAL algorithm. This leads to an upper bound, defined via an integral, for the complexity of the EVAL algorithm. This novel technique for complexity analysis can be viewed as a continuous amortization technique. In addition, this framework is quite general and promises to be applicable for analyzing the complexity of other evaluation-based algorithms.

(2) Next we consider the benchmark case of a square-free integer polynomial f of degree d and logarithmic height L. We give a priori worst-case upper bounds of the form $O(d^3(\log d + L))$ for the size of the subdivision tree. This bound exploits the Mahler-Davenport root bound as well as new evaluation analogues of such bounds.

1 Introduction

A basic problem in the computational geometry of surfaces is the meshing of implicit surfaces. This problem asks for an isotopic ε-approximation $\widetilde{S}$ of a surface S in $\mathbb{R}^n$ given by the equation $f = 0$ where $f : \mathbb{R}^n \to \mathbb{R}$. A survey of recent literature on meshing can be found in [4]. When f is a polynomial, there are many algebraic methods for solving this problem (e.g., [5]). However, numeric and geometric methods based on **subdivision** are widely used by practitioners because these methods are easier to implement than algebraic methods. Furthermore, such methods have an adaptive complexity that can be quite efficient on most inputs. A standard example of subdivision methods is the Marching Cube [22]. This algorithm, as is the case with most non-algebraic algorithms, is incomplete [38]; hence, hybrid methods that combine algebraic primitives with subdivision [32] are usually necessary to ensure complete algorithms. The first purely numerical subdivision method that is provably complete for non-singular surfaces is from Plantinga-Vegter [27, 28]. They provided algorithms in 2 and 3-D, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ for $n = 2$ or 3.

In this paper, we provide a complexity analysis for the 1-D version of the Plantinga-Vegter algorithm, which we call the EVAL algorithm. The algorithm amounts to real root isolation (and refinement); there are many well-known subdivision algorithms in this case, e.g., Sturm method [11, 30, 21] or the Descartes method [8, 15, 2]. The computational model [28] of the EVAL algorithm is purely numerical: like the Marching Cubes algorithm, it is based on evaluation of functions, but it also uses interval versions of the function and its derivatives. We call such algorithms **evaluation-based**. In contrast, Sturm or Descartes methods more sophisticated primitives which restrict f to polynomials. So evaluation-based methods are more widely applicable (e.g. f could be analytic, although the more general setting is outside our present scope). The 1-D version of Plantinga-Vegter's algorithm was first formulated by Mitchell [23], based on an algorithm of Moore [24]. In [6], we extended the EVAL algorithm to the case where f may have multiple roots.

The **adaptive complexity** of subdivision algorithms is a topic of growing interest. But what is the proper way to quantify the adaptivity of an algorithm? Most approaches in the literature are based on the condition number of the problem. For instance, Mourrain and Pavone [25] use the condition number to bound the complexity of Bernstein-type subdivision for isolating multivariate zeros. Condition number approaches to complexity are extensively used in the Smale school [3]. Another quantity is based on the precision sensitivity [33], the bit-version of output sensitivity which is well-known in computational geometry. In this paper we introduce a different quantity, defined via an integral, which can be viewed as a **continuous amortization** technique.

Amortization is a standard analysis technique for discrete algorithms [9]. In the continuous domain, Davenport [10] was the first to give an amortization argument yielding the optimal subdivision tree complexity for Sturm method. Recently, amortization arguments have been used in [12] for the Sturm method and [15] for the Descartes method. These complexity bounds are dependent on the Mahler-Davenport root separation bounds [10, 39]. In this paper, we also require bounds analogous to Mahler-Davenport type bounds, but in the form of evaluation bounds.

Subdivision methods for root isolation may be classified by their stopping predicates. The Sturm predicate is based on Sturm sequences, and Descartes predicate is based on the Descartes rule of sign. In the EVAL algorithm, we use an extremely simple principle: *in an interval* $[a, b]$ *where* $f(a)f(b) < 0$ *and* f *is continuous, there exists* $c \in (a, b)$ *such that* $f(c) = 0$. This is known as the Bolzano Theorem, a special case of the Intermediate Value Theorem. For this reason, the EVAL algorithm could be called the "Bolzano method". These predicates represent a progression of decreasing strength:

$$STURM > DESCARTES > BOLZANO \tag{1}$$

Sturm is the strongest predicate and is algebraic in nature, working only for polynomials. Bolzano is the weakest but is more general, being purely numerical in nature. The computational complexity of the predicates also decreases in this sequence. This may work to the advantage of simpler predicates due to the trade off between the number of subdivisions and the complexity of each step. Thus, the Descartes method is empirically faster than Sturm for a wide range of inputs [16, 31], even discounting the overhead of computing the Sturm sequence. This difference is attributable to the cheaper Descartes predicate because the number of subdivisions in Sturm method is minimal among all subdivision methods. In [6] we offer evidence that evaluation-based methods might similarly be competitive with the Descartes method.

For the purposes of complexity analysis, however, we find a reverse ordering in (1): the simpler predicates are harder to analyze. It is standard to judge these algorithms using the **benchmark problem** of isolating all the real roots of an integer polynomial of degree d and logarithmic height L. *What is the size of the subdivision tree in terms of d and L?* Davenport [10] proves that the tree size is $O(d(\log d + L))$ for the Sturm predicate. The Descartes method is also $O(d(\log d + L))$ but more subtle to show [15]. In this paper, we prove that the EVAL has a complexity of $O(d^3(\log d + L))$.

¶1. Overview of Paper. In Section 2, we describe the Plantinga-Vegter computational model and the algorithm EVAL. In Section 3, we describe the general framework of "stopping functions" for analyzing the complexity of EVAL. In Section 4, we give our main result, an a priori tree size bound of $O(d^3(\log d + L))$ on EVAL. In Section 5, we bound the gamma integral which is a component of the main bound. We conclude in Section 6. An appendix gives missing proofs and some additional material.

¶2. Additional Background. Root isolation has a large literature; we touch on a few results.

It appears that evaluation-based methods, in order to be complete, are necessarily tied to interval arithmetic. Other examples of evaluation-based root isolation are based on interval forms of the Newton operator. Moore, Krawcyk and others have provided such algorithms [24]. Mitchell [23] presented a form of EVAL, noting that his algorithm is simpler than Moore's [24]. His version is incomplete because he implicitly adopted the numerical analyst's view of fixed precision arithmetic. Snyder [5] introduced interval-based methods for meshing which recursively solves the problem on the boundary of subdivision boxes. Kearfott [18, 17] has provided empirical evaluation of Newton-type subdivision algorithms, and also provided a complexity analysis. Evaluation bounds were used in [7] for numerical solution of zero-dimensional triangular systems.

A comprehensive treatment of the Descartes method including its historical roots is available in an upcoming thesis of Eigenwillig [13]. The Descartes method can be developed into algorithms such as the Bisection Algorithm of Collins-Akritas [8] or the Continued Fraction Algorithm [1, 35, 34]. The Bernstein polynomial approach [20, 26] may also be viewed as a variant of the Descartes method [15]. Rouillier and Zimmermann [31] describe various improvements on the basic algorithm of [8]. The almost optimality of its tree size was established in [15]. The Descartes method can be generalized to a setting where coefficients are viewed as on-demand bit-streams [14].

2 An Evaluation-based Algorithm.

Fix f to be a square-free polynomial in $\mathbb{R}[X]$ of degree d. In the Plantinga-Vegter computational model, we need the box (i.e., interval) version of f and its derivatives.

¶3. Box Functions. For any set $S \subseteq \mathbb{R}$, let $\square S$ denote the set of closed intervals in S. If $I = [a, b]$, let $m(I)$ denote the **midpoint** $(a+b)/2$ of I and let $w(I)$ denote the **width** $b-a$ of I. A **partition** of I is a finite subset $P \subseteq \square I$ such that the union of the intervals in P is equal to I, and any two intervals in P have disjoint interiors. The **size** $\#(P)$ of P is the number of intervals in P. Our partitions of I mostly come from repeated bisections: for any interval $X = [a, b]$, the term **children**

of X refers to the two intervals $[a, m(X)], [m(X), b]$. Note that $\{X\}$ and $\{[a, m(X)], [m(X), b]\}$ are both partitions of X. In general, if P is a partition of I, and $X \in P$, then to **bisect** X **in** P means to replace X by its two children in P. As a result $\#(P)$ increases by 1. A partition of I that arises from repeated bisections of the initial $\{I\}$ is called a **subdivision** of I.

A **box function** for f **over** I is a function of the form $\square f : \square I \to \square\mathbb{R}$ such that for all $X \in \square I$, we have $f(X) \subseteq \square f(X)$. $f(X)$ is the set extension of f where $f(S) = \{f(a) : a \in S\}$ for any set $S \subseteq \mathbb{R}$. To ensure the termination of the EVAL algorithm, we need $\square f$ to be continuous, i.e., if for $X_1, X_2, \cdots$ a decreasing sequence sequence of intervals, i.e. $X_i \supseteq X_{i+1}$ of finite width in $\square I$ such that $\cap X_i = \{p\}$, then $\square f(X_i) \to \{f(p)\}$.

¶4. The Evaluation Algorithm. We now present the Evaluation Algorithm, EVAL, also discussed in [6]. Given an interval $I = [a, b]$, EVAL will isolate all the real roots of $f(x)$ in the open interval (a, b). Specifically, it will output a sequence of isolating intervals, one for each real root of f in the interval (a, b). The isolating intervals are either $[c, c]$, implying that $f(c) = 0$ or $[c, d]$, implying that there is a root in (c, d). The idea is to maintain a subdivision P of I. Initially, $P = \{I\}$. The algorithm operates in two phases.

Phase 1: Repeatedly bisect each $X \in P$ until each interval X in P is **terminal**. By this, we mean that one of the following two conditions hold for X:

$$C_0(X) \quad : \quad 0 \notin \square f(X) \tag{2}$$

$$C_1(X) \quad : \quad 0 \notin \square f'(X) \tag{3}$$

If $f(m(X)) = 0$, then we also output $[m(X), m(X)]$.

Phase 2: Let P_I denote the subdivision of I at the end of Phase 1. For each $X \in P_I$, we take one of two actions: If $C_0(X)$ holds, we discard X. If $C_1(X)$ holds, we evaluate the signs of f at the two end points of $X = [c, d]$. If $f(c)f(d) < 0$ we output (c, d), else we discard X. Let P'_I be the set of output intervals. Proving correctness of this algorithm amounts to showing that P'_I is a set of isolating intervals for the roots of f in I.

The first phase of the algorithm terminates by the continuity of $\square f$ and $\square f'$ and the fact that all zeros of f in $[a, b]$ are simple. We can easily determine if the endpoints of I are roots by evaluation to extend the algorithm to find all roots of f in the closed interval $[a, b]$. When f is an integer polynomial, EVAL can be implemented exactly using bigfloats.

An important property of the EVAL algorithm is that it only depends on the existence of box functions for f and f', and ability to evaluate sign of f, f' at dyadic points. For many C^1 functions, these conditions are satisfied. So the EVAL algorithm can be used on such functions to isolate its roots in an interval I, provided I contains a finite number of roots, all simple.

¶5. The Centered Interval Functions. For our complexity results, we need additional convergence properties of the box functions. So we assume $\square f$ is the **centered form** box function [29], defined as follows:

$$\square f(X) := \sum_{i=0}^{d} \frac{|f^{(i)}(m(X))|}{i!} \left(\frac{w(X)}{2}[-1, 1]\right)^i . \tag{4}$$

where $f = \deg f$ and $\left(\frac{w(X)}{2}[-1, 1]\right)^i$ is an interval arithmetic expression. To quantify the range of our box functions, for any interval X, define

$$K_X = K_X(f) := \max_{a \in X} \sum_{i=1}^{d} \frac{|f^{(i)}(a)|}{i!} (w(X))^{i-1} . \tag{5}$$

Also, we write K'_X for $K_X(f')$ where $f' = f^{(1)}$ is the first derivative. Note that $X \subseteq Y$ implies $K_X \le K_Y$. Call K_X a **Lipschitz constant** as it can be seen that $|f(a) - f(b)| \le K_X|a - b|$ for $a, b \in X$ follows from the following proposition:

PROPOSITION 1. *Let $Y \subseteq X$ be an interval, then:*
(i) $\square f(Y) \subseteq \square f(X)$.
(ii) $w(\square f(Y)) \le K_X \cdot w(Y)$.
(iii) $w(\square f(Y)) - w(f(Y)) \le K_X \cdot w(Y)^2$.

Property (i) shows that $\square f$ is **inclusion isotone** and Property (iii) is called quadratic convergence for $\square f$. However, we do not use this property.

Our goal is to find an upper bound for the size $\#(P_I)$ of P_I from the EVAL algorithm. $\#(P_I)$ is one more than the number of bisection steps. We begin our analysis with a simple observation:

LEMMA 2. *If $a \in Y \subseteq X$ and $0 \in \square f(Y)$ then $w(Y) \ge |f(a)|/K_X$.*

Proof. Since $\{0, f(a)\} \subseteq \square f(Y)$, we have $w(\square f(Y)) \ge |f(a)|$. By Proposition 1(ii), $w(Y) \ge w(\square f(Y))/K_X$ and hence $w(Y) \ge |f(a)|/K_X$. **Q.E.D.**

3 General Framework of Stopping Functions.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. If X is any interval, we will call X **big** (relative to g) if

$$w(X) \ge \frac{1}{2} \max_{a \in X}\{g(a)\}. \tag{6}$$

For convenience, we say X is **large** (relative to g) if $w(X) \ge \max_{a \in X}\{g(a)\}$. Clearly, if X is large, then X is big, and both of the children of X are also big. A partition P of I is **big** if each $X \in P$ is big. Our key definition is this: call g a **stopping function** (over an interval I) if any interval $X \subseteq I$ that is *not* large relative to g must be terminal. Recall that the notion of "terminal interval" is based on the EVAL algorithm. The following is immediate:

LEMMA 3. *If g_1, g_2 are stopping functions over I, then so is* $\max\{g_1, g_2\}$.

We remark that this simple device of using $\max\{g_1, g_2\}$ is useful for achieving complexity bounds; it acts as a regularizing device when we integrate.

LEMMA 4. *If P is a big partition of I relative to stopping function g, then its size is at most*

$$S := 2 \int_I \frac{1}{g(a)} da \tag{7}$$

In addition, if g is never zero in I, then the integral S is finite.

Proof. If g is never zero, $1/g$ is continuous and never infinite. As I is compact and $1/g$ is continuous, $1/g$ is bounded in I, thus the integral is finite. S can be rewritten as $S = \sum_{X \in P} \int_X \frac{2}{g(a)} da$. It remains to show that this sum is at least $\#P$. It suffices to show that each summand is at least 1. For any $X \in P$, if we choose $c = \arg\max_{a \in X}\{g(a)\}$. Then we have

$$\int_X \frac{2}{g(a)} da \ge \int_X \frac{2}{g(c)} da = w(X) \cdot \frac{2}{g(c)} \ge 1, \tag{8}$$

where the last step uses the fact that X is big. **Q.E.D.**

To apply these lemmas, consider the following conceptual Procedure G ('G' for generic). We say "conceptual" because it is not meant to be implemented, but to be used as an analysis tool.

Procedure G:
  Input: interval I, and a stopping function g
  Output: partition P of I
    If P is not large relative to g
      Return $P = \{I\}$.
    Initialize a queue $Q = \{I\}$, and an empty partition $P = \emptyset$.
(A)   While Q is non-empty
      Remove any J from Q
(B)   If J is not large, insert J into P
(C)   Else bisect J into J_0, J_1 and insert them into Q.
    Return P.

¶6. Subdivision Tree. Let $T_g = T_g(I)$ denote the **subdivision tree** rooted at I that is constructed by Procedure G on input I, g. Thus T_g is a binary tree whose nodes are subintervals of I, and each non-leaf J has two children obtained by bisection. Moreover, the set of leaves of T_g forms a partition $P_g = P_g(I)$ of I. Procedure G is essentially EVAL in which we have used g in place of the normal termination criteria based on C_0 and C_1 ((2) and (3)).

THEOREM 5. *Let $T_g = T_g(I)$ be the subdivision tree constructed by Procedure G using interval I and stopping function g.*
(i) Then $T_g(I)$ is a refinement of the corresponding subdivision tree produced by EVAL.
(ii) If the set of leaves of T_g is $P_g = P_g(I)$, then

$$\#(P_g) \leq \max\left\{1, \int_I \frac{2}{g(a)} da\right\}. \tag{9}$$

Proof. (i) By inclusion isotonicity, if an interval J is terminal, then any subinterval of J must also be terminal. Since g is a stopping function, if a node $J \subseteq I$ is a leaf of T_g, then J is terminal. Clearly, EVAL must terminate at some node J' along the path from I to J; possibly $J' = J$. Thus the tree from EVAL is a subtree of T_g. (ii) If $\#(P_g) = 1$, the bound (9) is immediate. Assuming $\#(P_g) > 1$, then I is non-terminal and so I is must be large. Consider the while loop (Line A in Procedure G): we have the invariant that $P \cup Q$ is a partition of I, P contains only big intervals and Q contains only children of large intervals. Line B transfers big intervals of Q into P, and Line C converts a large interval into two large or big intervals for Q. Thus the loop invariant is thus preserved by Lines B and C. At termination, Q is empty and P becomes our output $P_g(I)$. Thus $P_g(I)$ is a big partition of I. Now Lemma 4 implies that $\#(P_g) \leq \int_I \frac{2}{g(a)} da$. **Q.E.D.**

Thus we see the utility of a stopping function g: it is an analysis tool for bounding the complexity of EVAL. We next investigate possible g's and discuss the information that each provides.

So far, we have not seen any explicit stopping functions. We now give a first example by defining for $X \in \square I$,

$$f_X(a) := \max\left\{\frac{|f(a)|}{K_X}, \frac{|f'(a)|}{K'_X}\right\}. \tag{10}$$

It is easy to show that $g = f_I$ (i.e. $X = I$ in (10)) is a stopping function, as given in the appendix. The use of the "global constant" K_I in this stopping function limits its usefulness; its main merit lies in its simplicity. The Appendix introduces more useful stopping functions, based on local Lipschitz constants K_a ($a \in I$).

4 An Integral Bound based on Refined Stopping Function

In the remainder of this paper, we will use stopping functions to analyze the complexity of the EVAL algorithm for the benchmark problem: $f \in \mathbb{Z}[X]$ square free of height $\|f\| \leq 2^L$ and the endpoints of $I = [a, b]$ are integers. We may assume that $a, b \le 2^L$ since all real zeros of f lie in this range [39]. We also assume that f' is square free; this removal assumption will be treated in the full paper.

The height $\|f\|$ is the maximum absolute value of the coefficients of f and the logarithmic height is $\log \|f\|$. In particular, we want *a priori* complexity bounds in terms of d and L

The bound given in this section is an *a priori* worst-case bound. Thus, it is *non-adaptive*, and does not replace the utility of adaptive integral bounds such as Theorem 5, which are adapted to the individual f and I. Our main result is the following:

THEOREM 6 (Main Result). *The number of bisections performed by EVAL on input f and an interval I is $O(d^3(\log d + L))$.*

Making the mild assumption of $L \ge \log d$, the bound in this theorem is becomes $O(d^3 L)$. This should be contrasted with the optimal bound of $O(dL)$ known for Descartes method [15].

Our proof exploits the "gamma function" that is central in Smale's theory of point estimates [3, 36]. This is defined as

$$\gamma(x) = \gamma_f(x) := \max_{i \ge 2} \left(\frac{|f^{(i)}(x)|}{i!|f'(x)|} \right)^{1/(i-1)}. \tag{11}$$

Intuitively, $\gamma(x)^{-1}$ is the radius of Newton convergence of f at x. Write $\gamma'(x)$ for $\gamma_{f'}(x)$; in the literature, $\gamma'(x)$ is also written as $\gamma_2(x)$. Thus, $\gamma'(x)$ should not be confused with the derivative of $\gamma(x)$ which is not used in this paper.

LEMMA 7. *Let $b \in J$ such that $w(J) \le \frac{1}{2\gamma(b)}$. Then $K_J \le 2d|f'(b)|$.*

This is proved by replacing each $f^{(i)}(a)$ in the definition of K_J (5) by its Taylor expansion at b.

We now introduce the stopping function that we use in our analysis:

$$G(a) := \min \left\{ \frac{1}{2\gamma(a)}, \frac{|f(a)|}{2d|f'(a)|} \right\}. \tag{12}$$

Also let $G'(a)$ denote the function analogous to $G(a)$ based on f' instead of f. Again, $G'(a)$ is not the derivative of G.

LEMMA 8. *G is a stopping function.*

Proof. Suppose J is not large relative to G. This means there exists $b \in J$ such that $w(J) < G(b)$. We must show that J is terminal. It suffices to show that $C_0(J)$ holds. Since $w(J) < G(b)$, we have $w(J) < \frac{|f(b)|}{2d|f'(b)|} \le \frac{|f(b)|}{K_J}$ where the second inequality follows from Lemma 7. The conclusion that $C_0(J)$ holds now follows from Lemma 2. **Q.E.D.**

By a similar argument, $G'(a)$, and hence $\max\{G(a), G'(a)\}$, is a stopping function. We can now use the function $g(a) := \max\{G(a), G'(a)\}$ in our generic Procedure G, as in Theorem 5. However, this function g need not lead to a finite integral bound. Specifically, the integral becomes unbounded when $G(a) = G'(a) = 0$. This is addressed in the next paragraph.

¶7. Avoiding Zeros of ff'. By definition, $G(a) \ge 0$ and $G'(a) \ge 0$ for all a. Now, $G(a) = 0$ iff $f(a) = 0$ or $f'(a) = 0$. Similarly, $G'(a) = 0$ iff $f'(a) = 0$ or $f''(a) = 0$. Thus, if the integral $\int \frac{da}{\max\{G(a), G'(a)\}}$ of Theorem 5 is taken over a set of intervals $I' \subseteq I$ that avoid the zeros of f and f', the integral will be bounded. We next show how to construct such a set I'.

For each zero $\alpha \in \textsc{Zero}(f)$, let $\rho(\alpha)$ denote the distance from α to the nearest zero of $\textsc{Zero}(f)$ different from α. Since f is square-free, $\rho(\alpha) > 0$. Similarly, if $\beta \in \textsc{Zero}(f')$, let $\rho'(\beta)$ be the corresponding function for f'. The ρ' function is also positive as f' and are square free (by assumption). Since f and f' have no roots in common, we can merge these two ρ functions into one, $\overline{\rho} : \textsc{Zero}(ff') \to \mathbb{R}_{>0}$ where $\overline{\rho}(\alpha) = \rho(\alpha)$ if $f(\alpha) = 0$ and $\overline{\rho}(\alpha) = \rho'(\alpha)$ if $f'(\alpha) = 0$.

We now provide a conceptual Procedure H, viewed as a two-staged refinement of Procedure G:

Procedure H:
  Input: interval I
  Output: partition P_2 of I
    Start with the partition $P = \{I\}$.
    Stage 1:
      While there is a $J \in P$ satisfying (a) or (b) below, split J in P:
        (a) $\#(J \cap \textsc{Zero}(ff')) > 1$.
        (b) $\#(J \cap \textsc{Zero}(ff')) = 1$, and $w(J) \geq \min\left\{B(\alpha), \frac{\overline{\rho}(\alpha)}{4d(d-1)}\right\}$
          where $\alpha \in \textsc{Zero}(ff') \cap J$ and $B(\alpha)$ is a technical bound discussed below
    Stage 2:
      For each $J \in P$, partition J using Procedure G.

¶8. Partitions P_1 and P_2 of I. We consider two partitions of I: Let P_1 be the partition at the end of Stage 1, and P_2 be the partition at the end of Stage 2. An interval $J \in P_1$ is said to be **special** if $\#(J \cap \textsc{Zero}(ff')) = 1$ and **non-special** otherwise. Let the number of special intervals be s_I. Clearly, $s_I \leq 2d-1$. Let $P_1' \subseteq P_1$ denote the set of non-special intervals of P_1 and $I' = \bigcup P_1'$ is the union of all non-special intervals. The following lemma will be shown in the next paragraph:

Lemma 9. *If $J \in P_1$ is special then it is terminal,*

In view of Lemma 9, we have the following bound on P_2 the final partition of Procedure H:

$$\begin{aligned} \#(P_2) &\leq s_I + \sum_{J \in P_1'} \max\left\{1, \int_J \frac{2da}{\max\{G(a), G'(a)\}}\right\} \leq s_I + \sum_{J \in P_1'} 1 + 2 \sum_{J \in P_1'} \int_J \frac{da}{\max\{G(a), G'(a)\}} \\ &\leq \#(P_1) + 2 \int_{I'} \frac{da}{\max\{G(a), G'(a)\}}. \end{aligned} \tag{13}$$

Below we will show that $\#(P_1) = O(d(\log d + L))$, and in the next section, we show $\int_{I'} \frac{da}{\max\{G(a), G'(a)\}} = O(d^3(\log d + L))$ (Theorem 14). These calculations will complete the proof of our main theorem.

¶9. Proof that Special Intervals are Terminal. We now prove Lemma 9. Let J be a special interval. Then there is a unique $\alpha \in J \cap \textsc{Zero}(ff')$. There are two cases: when $\alpha \in \textsc{Zero}(f)$, we show that $C_1(J)$ holds, and when $\alpha \in \textsc{Zero}(f')$, we show that $C_0(J)$ holds. We now define the technical bound $B(\alpha)$ in Procedure H. Define

$$B(\alpha) = \begin{cases} \infty & \text{if } \alpha \text{ is zero of } f \\ \sqrt{\frac{|f(\alpha)|}{3|f''(\alpha)|}} & \text{if } \alpha \text{ is zero of } f' \end{cases}. \tag{14}$$

The technical bound $\frac{\overline{\rho}(\alpha)}{4d(d-1)}$ is designed to bound $w(J)$ by $\frac{1}{8\gamma(\alpha)}$ or $\frac{1}{8\gamma'(\alpha)}$. This follows from an application of the following bound from [37]:

Proposition 10. $\frac{1}{\gamma(\alpha)} > \frac{2\rho(\alpha)}{d(d-1)}$.

Now we can use this bound on $w(J)$ to ensure that C_0 or C_1 hold; the full calculations can be found in the appendix.
(i) $\alpha \in \textsc{Zero}(f)$: Since $B(\alpha) = \infty$, it plays no role in the stopping condition (b) of Procedure H. Since there is a zero of f in J, we require $C_1(J)$ to hold. By Lemma 30, $C_1(J)$ would hold provided $w(J) < \frac{|f'(\alpha)|}{K'_J}$. Using the bound on $w(J)$, a Taylor expansion about α, and the triangle inequality one can show $K'_J \le \frac{7}{9}\frac{|f'(\alpha)|}{w(J)}$, giving the desired bound. See Lemma 28.
(ii) $\alpha \in \textsc{Zero}(f')$: By Lemma 30, $C_0(J)$ would hold provided $w(J) < \frac{|f(\alpha)|}{K_J}$. Again, using the bound on $w(J)$, a Taylor expansion about α, and the triangle inequality we see that $K_J \le 3|f''(\alpha)|w(J)$, giving the bound when combined with the additional condition supplied by $B(\alpha)$. See Lemma 29.

¶10. Bounding the Size of P_1. We will bound the size of the partition P_1 as follows:

Lemma 11. *$\#(P_1) = O(d(\log d + L))$.*

For this purpose, we focus on the (at most) $2d-1$ special intervals. Consider the subdivision tree T_1 whose leaves are labeled by P_1. Clearly, $\#(T_1) = 2\#(P_1) - 1$. A leaf is said to be special iff it is labeled by a special interval. Let T_2 be the result of pruning all non-special leaves from T_1. Every non-special leaf has a sibling which is either special or an interior node, and the root has no sibling. Hence $\#(T_2) \ge \frac{\#(T_1)-1}{2} = \#(P_1) - 1$. Thus, we now want to upper bound $\#(T_2)$.

The **external path length**, denoted $EPL(T)$, of a tree T is the sum of lengths of paths from the root to each leaf of T (Knuth [19, p. 399]). Then, $\#(T_2) \le EPL(T_2) + 1$. Lemma 11 follows from:

Lemma 12. *$EPL(T_2) = O(d(\log d + L))$.*

In some subsequent application, we will need the stronger result of Lemma 12 instead of Lemma 11. To prove Lemma 12, let $S = \textsc{Zero}(f) \cap I$ and $S' = \textsc{Zero}(f') \cap I$ where I is the input interval. Each leaf of T_2 is associated with a unique $\alpha \in S \cup S'$; the corresponding interval will be denoted I_α. Let T_3 (resp., T'_3) denote the subtree of T_2 comprising all the paths from a leaf I_α where $\alpha \in S$ (resp., $\alpha \in S'$) to the root of T_2. Clearly $EPL(T_2) = EPL(T_3) + EPL(T'_3)$. Moreover,

$$EPL(T_3) \le \sum_{\alpha \in S} \lg(w(I)/w(I_\alpha)), \qquad EPL(T'_3) \le \sum_{\alpha \in S'} \lg(w(I)/w(I_\alpha)),$$

where $\lg = \log_2$. Recall our assumption that $w(I) \le 2^{L+1}$. Hence

$$EPL(T_3) \le d(L+1) - \sum_{\alpha \in S} \lg w(I_\alpha) \tag{15}$$

But $w(I_\alpha) \ge \frac{1}{2} \min\left\{B(\alpha), \frac{\rho(\alpha)}{4d(d-1)}\right\}$, by our stopping condition in Procedure H. If $\alpha \in S$, this reduces to $w(I_\alpha) \ge \frac{1}{2}\frac{\rho(\alpha)}{4d(d-1)}$, and hence we obtain:

$$-\lg \prod_{\alpha \in S} w(I_\alpha) \le -\lg \prod_{\alpha \in S} \frac{\rho(\alpha)}{8d(d-1)} = O(d \log d + dL). \tag{16}$$

The last relation is from [12, p. 126]. Combining (15,16), we get $EPL(T_3) = O(d \log d + dL)$.

Next, we consider the case $\alpha \in S'$. In this case, $B(\alpha) = \sqrt{\frac{|f(\alpha)|}{3|f''(\alpha)|}}$. Therefore $w(I_\alpha) \geq \min\{B(\alpha)/2, \rho'(\alpha)/8d(d-1)\}$. We split S' into $S'_0 \cup S'_1$ where $\alpha \in S'_0$ iff $\min\{B(\alpha)/2, \rho'(\alpha)/8d(d-1)\} = B(\alpha)/2$. Thus $\prod_{\alpha \in S'} w(I_\alpha) \geq \prod_{\alpha \in S'_0} B(\alpha) \prod_{\alpha \in S'_1} \frac{\rho'(\alpha)}{8d(d-1)}$. We have

$$-\lg \prod_{\alpha \in S'_1} \frac{\rho'(\alpha)}{8d(d-1)} = O(d \log d + dL) \tag{17}$$

as in (16). The final bound we need is

$$-\lg \prod_{\alpha \in S'_0} \sqrt{\frac{|f(\alpha)|}{3|f''(\alpha)|}} = O(d \log d + dL). \tag{18}$$

From (17) and (18), we see that $EPL(T'_3) = O(d \log d + dL)$, completing the proof of Lemma 12.

¶11. The Evaluation Bound. The final bound (18) above comes from an application of the following evaluation bound.

Theorem 13. *Let $\phi(x), \eta(x) \in \mathbb{C}[x]$ be complex polynomials of degrees m and n. Let $\beta_1, \ldots, \beta_n$ be all the zeros of $\eta(x)$.*
(a)

$$\prod_{i=1}^{n} |\phi(\beta_i)| \leq ((m+1)\|\phi\|)^n \left(\frac{M(\eta)}{\mathrm{lc}(\eta)}\right)^m. \tag{19}$$

(b) Suppose there exists relatively prime $F, H \in \mathbb{Z}[x]$ such that $F = \phi\overline{\phi}$, $H = \eta\overline{\eta}$ for some $\overline{\phi}, \overline{\eta} \in \mathbb{C}[x]$. If the degrees of $\overline{\phi}$ and $\overline{\eta}$ are $\overline{m}$ and $\overline{n}$, then

$$\prod_{i=1}^{n} |\phi(\beta_i)| \geq \frac{1}{\mathrm{lc}(\overline{\eta})^m \cdot ((m+1)\|\phi\|)^{\overline{n}} M(\overline{\eta})^m \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n+\overline{n}} M(H)^{\overline{m}}}. \tag{20}$$

This theorem has independent interest. Its proof and applications (in particular, to the bound (18)) are in the appendix.

5 Bounding the Integral $\int_{I'} \frac{dx}{\max\{G(x), G'(x)\}}$.

This section bounds the following expression from (13), used in the main result (Theorem 6):

Theorem 14. $\int_{I'} \frac{dx}{\max\{G(x), G'(x)\}} = \int_{I'} \frac{dx}{G(x)} = O(d^3(\log d + L))$.

The trick in Theorem 14 is to replace the first integral by the second integral, which involves only $G(x)$ but not $G'(x)$. The second integral is finite because I' excludes both zeros of f and f'. Next, we bound the second integral by a sum of two integrals:

$$\frac{1}{2}\int_{I'} \frac{dx}{G(x)} = \int_{I'} \max\left\{\gamma(x), \frac{d|f'(x)|}{|f(x)|}\right\} dx \leq \int_{I'} \gamma(x) dx + d\int_{I'} \left|\frac{f'(x)}{f(x)}\right| dx = \Gamma + dR$$

where $\Gamma := \int_{I'} \gamma(x) dx$ ("gamma integral") and $R := \int_{I'} \left|\frac{f'(x)}{f(x)}\right| dx$ ("logarithmic-derivative integral"). The next two lemmas provide bounds on these integrals:

Lemma 15. $R = \int_{I'} \left|\frac{f'(x)}{f(x)}\right| dx = O(d^2(\log d + L))$.

Lemma 16. $\Gamma = \int_{I'} \gamma(x)dx = O(d^2(\log d + L))$.

Thus Theorem 14 follows from Lemmas 15 and 16.

The proof of Lemma 15 is in the appendix. The basic idea is to write I' as a union of intervals, $I' = \bigcup_{i=0}^{k}[a_i, b_i]$ and note that $f'(x)/f(x)$ has constant sign on each interval $[a_i, b_i]$. Thus so we are integrating the logarithmic derivative of f, with the result $\int_{a_i}^{b_i} |f'(x)/f(x)|dx = \log|f(b_i)/f(a_i)|$. We then apply the evaluation bound on $\log\prod_{i=0}^{k}|f(b_i)|$ and $\log\prod_{i=0}^{k}|f(a_i)|$, exploiting the fact that the a_i's and b_i's from Procedure H have nice bounds.

In the rest of this section, we outline the proof of Lemma 16 which bounds the gamma integral. Let $\beta_1, \cdots, \beta_d$ be the foots of f'. The gamma function satisfies a key inequality:

Lemma 17.

$$\Gamma = \int_{I'} \gamma(x)dx \leq \sum_{i=2}^{d} \int_{I'} \frac{dx}{2|x-\beta_i|}$$

The proof exploits the relation $f^{(i)}(x)/f'(x) = \sum'_{(j_2,\ldots,j_i)} \prod_{\ell=2}^{i} \frac{1}{x-\beta_{j_\ell}}$, where j_ℓ's are taken from the set $\{1, \ldots, d-1\}$, and the prime in the summation indicates that the j_ℓ's are pairwise distinct.

Next, we write $\beta_i = r_i + \mathbf{i}s_i$ where $r_i = \mathtt{Re}(\beta_i)$ and $s_i = \mathtt{Im}(\beta_i)$ are the real and imaginary parts, and $\mathbf{i} = \sqrt{-1}$. Furthermore, assume $s_i = 0$ iff $i \leq k$, so all the real roots of f' are given by $r_2, \ldots, r_k$. In the appendix, we construct two integer polynomials $R(X)$ (Lemma 22) and $S(X)$ (Lemma 23) of degrees $\leq d^2$ whose zero set contains r_i and s_i (resp.). We split the summation from Lemma 17 into the real and complex parts:

Lemma 18 (Real Part).

$$\sum_{i=2}^{k} \int_{I'} \frac{dx}{2|x-r_i|} = O(d^2(\log d + L)).$$

We outline the proof used in the appendix: note that I' is equal to I minus the special intervals. The special intervals contain the real zeros of f and of f'. Let I'' be I minus the special intervals that contain real zeros of f. Since $I' \subseteq I''$, it is enough to bound the integral of Lemma 18 over I''. If $[a_j, b_j]$ $(j = 0, \ldots, \ell)$ is a connected component of I'', the integral $\int_{a_j}^{b_j} dx/|x-r_i|$ is $\log|a_j - r_i|/|b_j - r_i|$. Summing over all i and j, we obtain the bound of the form $\sum_j \sum_i \log|a_j - r_i|/|b_j - r_i| = \sum_j \log|\Phi_R(a_j)/\Phi_R(b_j)|$ where $\phi_R(X) = \prod_{i=2}^{k}(X - r_i)$. We can now apply the evaluation bound of Theorem 13.

For the complex part, we obtain the better bound:

Lemma 19 (Complex Part).

$$\sum_{i=k+1}^{d} \int_{I'} \frac{dx}{2|x-\beta_i|} = O(d(d+L)).$$

The proof in the appendix uses a very similar argument as for the real part.

This completes the proof of Lemma 16.

6 Conclusion

In this paper, we introduced novel techniques for analyzing the complexity of evaluation-based algorithms. Our bounds are based on an integral formula (9) and an amortized evaluation bound (Appendix). This can be viewed as a continuous amortization. We pose several open problems:
(a) Is the inherent complexity of EVAL $\Theta(d^2L)$ or $\Theta(dL)$ in the benchmark case (with $L \geq \lg d$)?
(b) Extend integral analysis to the Plantinga-Vegter algorithms in 2- and 3-D.

References

[1] A. G. Akritas and A. Strzeboński. A comparative study of two real root isolation methods. *Nonlinear Analysis:Modelling and Control*, 10(4):297–304, 2005.

[2] A. Alesina and M. Galuzzi. A new proof of Vincent's theorem. *L'Enseignement Mathémathique*, 44:219–256, 1998.

[3] L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation*. Springer-Verlag, New York, 1998.

[4] J.-D. Boissonnat, D. Cohen-Steiner, B. Mourrain, G. Rote, and G. Vegter. Meshing of surfaces. In Boissonnat and Teillaud [5]. Chapter 5.

[5] J.-D. Boissonnat and M. Teillaud, editors. *Effective Computational Geometry for Curves and Surfaces*. Number 59 in Mathematics and Visualization. Springer, 2006.

[6] M. Burr, V. Sharma, and C. Yap. Evaluation-based root isolation, Nov. 2007. In preparation.

[7] J.-S. Cheng, X.-S. Gao, and C. K. Yap. Complete numerical isolation of real zeros in general triangular systems. In *Proc. Int'l Symp. Symbolic and Algebraic Computation (ISSAC'07)*, pages 92–99, 2007. Waterloo, Canada, Jul 29-Aug 1, 2007. DOI: http://doi.acm.org/10.1145/1277548.1277562.

[8] G. E. Collins and A. G. Akritas. Polynomial real root isolation using Descartes' rule of signs. In R. D. Jenks, editor, *Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computation*, pages 272–275. ACM Press, 1976.

[9] T. H. Corman, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to Algorithms*. The MIT Press and McGraw-Hill Book Company, Cambridge, Massachusetts and New York, second edition, 2001.

[10] J. H. Davenport. Computer algebra for cylindrical algebraic decomposition. Tech. Rep., Royal Inst. of Technology, Dept. of Numer. Analysis and Computing Science, Stockholm, Sweden, 1985. Reprinted as Tech. Rep. 88-10, U. of Bath, School of Math. Sciences, Bath, England. URL http://www.bath.ac.uk/ masjhd/TRITA.pdf.

[11] Z. Du, V. Sharma, and C. Yap. Amortized bounds for root isolation via Sturm sequences. In D. Wang and L. Zhi, editors, *Proc. Internat. Workshop on Symbolic-Numeric Computation*, pages 81–93, School of Science, Beihang University, Beijing, China, 2005. Int'l Workshop on Symbolic-Numeric Computation, Xi'an, China, Jul 19–21, 2005.

[12] Z. Du, V. Sharma, and C. Yap. Amortized bounds for root isolation via Sturm sequences. In D. Wang and L. Zhi, editors, *Symbolic-Numeric Computation*, Trends in Mathematics, pages 113–130. Birkhäuser Verlag AG, Basel, 2007. Proc. Int'l Workshop on Symbolic-Numeric Computation, Xi'an, China, Jul 19–21, 2005.

[13] A. Eigenwillig. *Real Root Isolation for Exact and Approximate Polynomials using Descartes' Rule of Signs*. PhD thesis, University of Saarlandes, (to appear) May 2008.

[14] A. Eigenwillig, L. Kettner, W. Krandick, K. Mehlhorn, S. Schmitt, and N. Wolpert. A Descartes algorithm for polynomials with bit stream coefficients. In *8th Int'l Workshop on Comp.Algebra in Sci.Computing (CASC 2005)*, pages 138–149. Springer, 2005. LNCS 3718.

[15] A. Eigenwillig, V. Sharma, and C. Yap. Almost tight complexity bounds for the Descartes method. In *Proc. Int'l Symp. Symbolic and Algebraic Computation (ISSAC'06)*, 2006. Genova, Italy. Jul 9-12, 2006.

[16] J. Johnson. Algorithms for polynomial real root isolation. In B. Caviness and J. Johnson, editors, *Quantifier Elimination and Cylindrical Algebraic Decomposition*, Texts and monographs in Symbolic Computation, pages 269–299. Springer, 1998.

[17] R. B. Kearfott. Abstract generalized bisection with a cost bound. *Math.Comp.*, 49:187–202, 1987.

[18] R. B. Kearfott. Empirical evaluation of innovations in interval branch and bound algorithms for nonlinear systems. *SIAM J. Sci.Comput.*, 18(2):574–594, 1997.

[19] D. E. Knuth. *The Art of Computer Programming: Fundamental Algorithms*, volume 1. Addison-Wesley, Boston, 2nd edition edition, 1975.

[20] J. M. Lane and R. F. Riesenfeld. Bounds on a polynomial. *BIT*, 21:112–117, 1981.

[21] T. Lickteig and M.-F. Roy. Sylvester-Habicht sequences and fast Cauchy index computation. *J. of Symbolic Computation*, 31:315–341, 2001.

[22] W. E. Lorensen and H. E. Cline. Marching cubes: A high resolution 3D surface construction algorithm. In M. C. Stone, editor, *Computer Graphics (SIGGRAPH '87 Proceedings)*, volume 21, pages 163–169, July 1987.

[23] D. Mitchell. Robust ray intersection with interval arithmetic. In *Graphics Interface'90*, pages 68–74, 1990.

[24] R. E. Moore. *Interval Analysis*. Prentice Hall, Englewood Cliffs, NJ, 1966.

[25] B. Mourrain and J.-P. Pavone. Subdivision methods for solving polynomial equations. Technical Report 5658, INRIA, 2005.

[26] B. Mourrain, F. Rouillier, and M.-F. Roy. The Bernstein basis and real root isolation. In J. E. Goodman, J. Pach, and E. Welzl, editors, *Combinatorial and Computational Geometry*, number 52 in MSRI Publications, pages 459–478. Cambridge University Press, 2005.

[27] S. Plantinga. *Certified Algorithms for Implicit Surfaces*. Ph.D. thesis, Groningen University, Institute for Mathematics and Computing Science, Groningen, Netherlands, Dec. 2006.

[28] S. Plantinga and G. Vegter. Isotopic approximation of implicit curves and surfaces. In *Proc. Eurographics Symposium on Geometry Processing*, pages 245–254, New York, 2004. ACM Press.

[29] H. Ratschek and J. Rokne. *Computer Methods for the Range of Functions*. Horwood Publishing Limited, Chichester, West Sussex, UK, 1984.

[30] D. Reischert. Asymptotically fast computation of subresultants. In *ISSAC 97*, pages 233–240, 1997. Maui, Hawaii.

[31] F. Rouillier and P. Zimmerman. Efficient isolation of [a] polynomial's real roots. *J. Computational and Applied Mathematics*, 162:33–50, 2004.

[32] R. Seidel and N. Wolpert. On the exact computation of the topology of real algebraic curves. In *Proc. 21st ACM Symp. on Comp. Geometry*, pages 107–116, 2005. Pisa, Italy.

[33] J. Sellen, J. Choi, and C. Yap. Precision-sensitive Euclidean shortest path in 3-Space. *SIAM J. Computing*, 29(5):1577–1595, 2000. Also: 11th ACM Symp. on Comp. Geom., (1995)350–359.

[34] V. Sharma. *Complexity Analysis of Algorithms in Algebraic Computation*. Ph.D. thesis, New York University, Department of Computer Science, Courant Institute, Dec. 2006. From `http://cs.nyu.edu/exact/doc/`.

[35] V. Sharma. Complexity analysis of real root isolation using continued fractions. In *Proc. Int'l Symp. Symbolic and Algebraic Computation (ISSAC'07)*, 2007. Waterloo, Canada, Jul 29-Aug 1, 2007.

[36] V. Sharma, Z. Du, and C. Yap. Robust approximate zeros. In G. S. Brodal and S. Leonardi, editors, *Proc. 13th European Symp. on Algorithms (ESA)*, volume 3669 of *Lecture Notes in Computer Science*, pages 874–887. Springer-Verlag, Apr. 2005. Palma de Mallorca, Spain, Oct 3-6, 2005.

[37] V. Sharma and C. Yap. Complexity of strong root isolation, 2007. In preparation.

[38] B. T. Stander and J. C. Hart. Guaranteeing the topology of an implicit surface polygonalization for interactive meshing. In *Proc. 24th Computer Graphics and Interactive Techniques*, pages 279–286, 1997.

[39] C. K. Yap. *Fundamental Problems of Algorithmic Algebra*. Oxford University Press, 2000.

APPENDIX

7 An Amortized Evaluation Bound

Our main complexity result is based on two distinct kinds of bounds. The first is the usual Mahler-Davenport bounds (e.g., [15]) that involves root separation bounds. But like [7], we need another class of bounds that we call **evaluation bounds**. Evaluation bounds refer to upper and lower bounds on $|f(\alpha)|$ where $f \in \mathbb{C}[X]$ and $\alpha \in \mathbb{C}$. Of course, lower bounds are only possibly with the additional assumption that f, α are algebraic. Our bounds are described as "amortized bounds" because they bound a product of $|f(\alpha)|$'s. Remark that the evaluation bound here is distinct from the multivariate version used in [7].

Let $f = \sum_{i=t}^{d} c_i X^i \in \mathbb{C}[X]$ $(t \le d)$ where $c_0 c_t \ne 0$. Recall that the height of f is $\|f\| = \max_{i=t}^{d} |c_i|$. Let $\mathrm{lc}(f) = |c_d|$ and $\mathrm{tc}(f) = |c_t|$ (resp.) denote the absolute values of the **leading coefficient** and **tail coefficient** (i.e., smallest non-zero coefficient) of f. We write $\mathtt{res}(f,g)$ for the resultant of two polynomials f, g. In addition to heights, we use the Mahler measure of polynomials, defined as $M(f) = \mathrm{lc}(f) M_1(f)$ where

$$M_1(f) := \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$$

where $\alpha_1, \ldots, \alpha_d$ are all the complex roots of f.

We restate Theorem 13 here; we add an additional clause (b') which is just a variation of (b).

Theorem 13. *Let $\phi(X), \eta(X) \in \mathbb{C}[X]$ be complex polynomials of degrees m and n. Let $\beta_1, \ldots, \beta_n$ be all the zeros of $\eta(X)$.*
(a)

$$\prod_{i=1}^{n} |\phi(\beta_i)| \le ((m+1)\|\phi\|)^n \left(\frac{M(\eta)}{\mathrm{lc}(\eta)}\right)^m . \tag{21}$$

(b) Suppose there exists relatively prime $F, H \in \mathbb{Z}[X]$ such that $F = \phi\overline{\phi}, H = \eta\overline{\eta}$ for some $\overline{\phi}, \overline{\eta} \in \mathbb{C}[X]$. If the degrees of $\overline{\phi}$ and $\overline{\eta}$ are $\overline{m}$ and $\overline{n}$, then

$$\prod_{i=1}^{n} |\phi(\beta_i)| \ge \frac{1}{\mathrm{lc}(\eta)^m \cdot ((m+1)\|\phi\|)^{\overline{n}} M(\overline{\eta})^m \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n+\overline{n}} M(H)^{\overline{m}}}. \tag{22}$$

(b') As alternative to (b), we also have:

$$\prod_{i=1}^{n} |\phi(\beta_i)| \ge \frac{1}{\mathrm{lc}(\eta)^m \cdot ((m+\overline{m}+1)\|F\|)^{\overline{n}} M(\overline{\eta})^{m+\overline{m}} \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n} M(\eta)^{\overline{m}}}. \tag{23}$$

Proof. (a) We may index the β_i's such that, for some $n' \in \{0, 1, \ldots, n\}$, we have $|\beta_i| \ge 1$ iff $i > n'$. Now for $i = 1, \ldots, n'$, we have $|\phi(\beta_i)| < \|\phi\|(m+1)$ and hence

$$\prod_{i=1}^{n'} |\phi(\beta_i)| \le (\|\phi\|(m+1))^{n'} . \tag{24}$$

This inequality is strict iff $n' > 0$. For $i = n'+1, \ldots, n$, we have $|\phi(\beta_i)| \le \|\phi\|(m+1)|\beta_i|^m$. So

$$\prod_{i=n'+1}^{n} |\phi(\beta_i)| \le (\|\phi\|(m+1))^{n-n'} \left(\prod_{i=n'+1}^{n} |\beta_i|\right)^m = (\|\phi\|(m+1))^{n-n'} \left(\frac{M(\eta)}{\mathrm{lc}(\eta)}\right)^m \tag{25}$$

Part (a) follows from (24) and (25).

(b) We have $\texttt{res}(F,H) = \mathrm{lc}(H)^{m+\overline{m}} \prod_{i=1}^{n+\overline{n}} F(\beta_i)$ where $\beta_1,\ldots,\beta_n,\beta_{n+1},\ldots,\beta_{n+\overline{n}}$ are all the zeros of H ([39, p. 167]). Thus,

$$\begin{aligned}
1 \le |\texttt{res}(F,H)| &= \mathrm{lc}(H)^{m+\overline{m}} \cdot \prod_{i=1}^{n} |\phi(\beta_i)| \left(\prod_{i=n+1}^{n+\overline{n}} |\phi(\beta_i)| \cdot \prod_{i=1}^{n+\overline{n}} |\overline{\phi}(\beta_i)| \right) \\
\prod_{i=1}^{n} |\phi(\beta_i)| &\ge \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot \prod_{i=n+1}^{n+\overline{n}} |\phi(\beta_i)| \cdot \prod_{i=1}^{n+\overline{n}} |\overline{\phi}(\beta_i)|} \\
&\ge \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot ((m+1)\|\phi\|)^{\overline{n}} \left(M(\overline{\eta})/\,\mathrm{lc}(\overline{\eta})\right)^{m} \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n+\overline{n}} \left(M(H)/\,\mathrm{lc}(H)\right)^{\overline{m}}},
\end{aligned}$$

where the last inequality is an application of the bound in part (a). Since $\mathrm{lc}(H) = \mathrm{lc}(\eta)\,\mathrm{lc}(\overline{\eta})$, the last expression simplifies to the bound in the (20). Alternatively, we could proceed thus:

$$\begin{aligned}
\prod_{i=1}^{n} |\phi(\beta_i)| &\ge \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot \prod_{i=n+1}^{n+\overline{n}} |F(\beta_i)| \cdot \prod_{i=1}^{n} |\overline{\phi}(\beta_i)|} \\
&\ge \frac{1}{\mathrm{lc}(H)^{m+\overline{m}} \cdot ((m+\overline{m}+1)\|F\|)^{\overline{n}} \left(M(\overline{\eta})/\,\mathrm{lc}(\overline{\eta})\right)^{m+\overline{m}} \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n} \left(M(\eta)/\,\mathrm{lc}(\eta)\right)^{\overline{m}}}.
\end{aligned}$$

which simplifies to the bound in (23). **Q.E.D.**

¶12. Proof of (18). We want to show

$$-\lg \prod_{\alpha\in S_0'} \sqrt{\frac{|f(\alpha)|}{3|f''(\alpha)|}} = O(d(\log d + L)).$$

Since $|S_0'| \le d$, it suffices to show that $-\lg \prod_{\alpha\in S_0'} \frac{|f(\alpha)|}{|f''(\alpha)|} = O(d(\log d + L))$. This in turn reduces to

$$-\lg \prod_{\alpha\in S_0'} |f(\alpha)| = O(d(\log d + L)) \tag{26}$$

and

$$\lg \prod_{\alpha\in S_0'} |f''(\alpha)| = O(d(\log d + L)). \tag{27}$$

To prove (26), we substitute into Theorem 13(b) the following: let $\phi(X) = F(X)$ be equal to f, and let $H(X)$ be equal to $f'(X)$ and also $\eta(X) = \prod_{\alpha\in S_0'}(X-\alpha)$. We use the fact that

$$M(\overline{\eta}) \le M(f') \le \|f'\|_2 \le \sqrt{d}\|f'\| \le d^{3/2}2^L$$

Since $\overline{n} = \overline{m} = 0$, $\overline{\phi} = 1$ and $\mathrm{lc}(\eta) = 1$, the bound (22) gives

$$\begin{aligned}
-\lg \prod_{\alpha\in S_0'} |f(\alpha)| &= -\lg \prod_{i=1}^{n} |\phi(\beta_i)| \\
&\le \lg\left(\mathrm{lc}(\eta)^m \cdot ((m+\overline{m}+1)\|F\|)^{\overline{n}}\, M(\overline{\eta})^{m+\overline{m}} \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n} M(\eta)^{\overline{m}} \right) \\
&\le \lg\left(M(\overline{\eta})^d \right)
\end{aligned}$$

$$= \quad O(d(\log d + L)).$$

Similarly, we obtain (27) from the upper bound in (21) by choosing f' for $\phi(X) = F(X)$, but choosing $H(X), \eta(X)$ as before. The bound (21) gives

$$\begin{aligned}
\lg \prod_{\alpha \in S_0'} |f''(\alpha)| &= \lg \prod_{i=1}^{n} |\phi(\beta_i)| \\
&\leq ((m+1)\|\phi\|)^n \left(\frac{M(\eta)}{\mathrm{lc}(\eta)}\right)^m \\
&= O(d(\log d + L)).
\end{aligned}$$

¶13. We will also need the following bound:

LEMMA 20. *If $S \subseteq \{\alpha_1, \ldots, \alpha_d\}$ is a set of non-zero roots of f then*

$$\prod_{\alpha \in S} |\alpha| \geq \frac{\mathrm{tc}(f)}{M(f)}.$$

Proof.

$$\begin{aligned}
\prod_{\alpha \in S} |\alpha| &\geq \prod_{i=t+1}^{d} \min\{1, |\alpha_i|\} \\
&= \frac{\prod_{i=t+1}^{d} |\alpha_i|}{\prod_{i=t+1}^{d} \max\{1, |\alpha_i|\}} \\
&= \frac{\mathrm{lc}(f) \prod_{i=t+1}^{d} |\alpha_i|}{M(f)} \\
&= \frac{\mathrm{tc}(f)}{M(f)}.
\end{aligned}$$

Q.E.D.

So if f is an integer polynomial, $\prod_{\alpha \in S} |\alpha| \geq \frac{1}{M(f)}$.

8 Bound on Integral of Logarithmic Derivatives

Our goal is to show Lemma 15, which claims that $R = \int_{I'} |f'(x)/f(x)| dx = O(d^2(\log d + L))$. Let us write $I' = \bigcup_{i=0}^{k} [a_i, b_i]$ where the $[a_i, b_i]$'s are pairwise disjoint intervals, for some $k < 2d$. Note that $f'(x)/f(x)$ has constant non-zero sign over each $[a_i, b_i]$, and so we can evaluate the integral $\int_{a_i}^{b_i} |f'(x)/f(x)| dx = [\log |f(x)|]_{a_i}^{b_i} = \log |f(b_i)|/|f(a_i)|$. Therefore, using the natural logarithm ln, we have

$$R = \sum_{i=0}^{k} \ln |f(b_i)|/|f(a_i)|. \tag{28}$$

Thus, Lemma 15 follows from the upper bound

$$\lg \prod_{i=0}^{k} |f(b_i)| = O(d^2 L) \tag{29}$$

and the lower bound

$$-\lg \prod_{i=0}^{k} |f(a_i)| = O(d^2(\log d + L)). \tag{30}$$

To show (29) and (30), we first give an amortized bound on the complexity of the a_i's and b_i's:

LEMMA 21. *Let $\eta(X) := \prod_{i=0}^{k}(X-a_i)(X-b_i)$. There is an integer N such that $H(X) := 4^N\eta(X)$ is an integer polynomial and*

$$N = \lg(\mathrm{lc}(H)) = O(d(\log d + L)), \qquad \lg M(\eta) = O(dL)$$

These bounds hold even when η is replaced by any of its factors; in particular when $\eta = \prod_{i=0}^{k}(X-a_i)$ or when $\eta = \prod_{i=0}^{k}(X-b_i)$.

Proof. Note that $I = [a_0, b_k]$ and I' is I minus the special intervals which are enumerated by $[b_0, a_1], [b_1, a_2], \ldots, [b_{k-1}, a_k]$. Let $N = \sum_{i=1}^{k} n_i$ where n_i is the depth of the special interval $[b_{i-1}, a_i]$ in the tree T_2 in ¶10. But N is the external path length bound of the tree T_2, and hence $N = O(d \log d + dL)$ from Lemma 12. By assumption, the polynomial $\eta_0(X) := (X-a_0)(X-b_k)$ is an integer polynomial. Let $\eta_i(X) := (X-b_{i-1})(X-a_i)$ for $i = 1, \ldots, k$. Thus each $4^{n_i}\eta_i(X)$ ($i = 1, \ldots, k$) is an integer polynomial; this is a consequence of the way the subdivision tree is obtained. This implies $4^N\eta(X) = 4^N \prod_{i=0}^{k}\eta_i(X) \in \mathbb{Z}[X]$, as claimed. Further, from

$$M(\eta_i(X)) = \max\{1, |b_{i-1}|\}\max\{1, |a_i|\} \leq 4^L$$

we infer $M(\eta) = 4^{kL}$ or $\lg M(\eta) = O(dL)$.

Q.E.D.

To prove (29), we use Theorem 13(a) where the polynomials $\phi(x)$ and $\eta(x)$ in the theorem are replaced by $f(x)$ and $\prod_{i=0}^{k}(x-b_i)$. Also, $F(x)$ is replaced by $f(x)$, and $H(x)$ is given by Lemma 21. Using the notations of Theorem 13(a), we also have

$$m = d, \qquad n \leq 2d, \qquad \lg\|\phi\| = \lg\|f\| \leq L, \qquad \lg M(\eta) \leq dL, \qquad \lg M(\overline{\eta}) = O(d(\log d + L)). \tag{31}$$

Thus (21) gives

$$\begin{aligned} \lg \prod_{i=0}^{k} |f(b_i)| = \lg \prod_{i=1}^{n} |\phi(\beta_i)| &\leq \lg\left(((m+1)\|\phi\|)^n \cdot \left(\frac{M(\eta)}{\mathrm{lc}(\eta)}\right)^m\right) \\ &= O(d^2L). \end{aligned}$$

To prove (30), we use Theorem 13(b) where the polynomials $\phi(x), \eta(x)$ in Theorem 13 are replaced by $f(x)$, and $\eta(x) = \prod_{i=0}^{k}(x-a_i)$. As before, we have (31). Thus (22) yields

$$\begin{aligned} -\lg \prod_{i=0}^{k} |f(b_i)| = -\lg \prod_{i=1}^{n} |\phi(\beta_i)| &\leq \lg\left(\mathrm{lc}(\eta)^m \cdot ((m+1)\|\phi\|)^{\overline{n}} M(\overline{\eta})^m \cdot \left((\overline{m}+1)\|\overline{\phi}\|\right)^{n+\overline{n}} M(H)^{\overline{m}}\right) \\ &= \lg M(\overline{\eta})^m = O(d^2(\log d + L)). \end{aligned}$$

This concludes the proof of Lemma 15.

9 On the Real and Imaginary Part of Zeros.

Let $f \in \mathbb{R}[X]$ be a real polynomial of degree $d \geq 1$. Suppose its complex zeros are $\alpha_1, \ldots, \alpha_d$ and let $r_i = \mathtt{Re}(\alpha_i)$ and $s_i = \mathtt{Im}(\alpha_i)$ for each i. Our goal is to construct two integer polynomials $R(X), S(X)$ whose roots contains the r_i's and s_i's respectively. We also want to bound the heights of $R(X)$ and $S(X)$. CAVEAT: In this section, r_i, s_i here refer to real/complex parts of roots of f; elsewhere, they refer to real/complex parts of roots of f'.

¶14. Real Part. We first construct a polynomial $R(X)$ whose roots include all the r_i's (cf. [39, p. 202]).

Use the Taylor expansion of $f(X + \mathbf{i}Y)$ at the point X:

$$\begin{aligned} f(X+\mathbf{i}Y) &= f(X) + f'(X)(\mathbf{i}Y) + \frac{f''(X)}{2}(\mathbf{i}Y)^2 + \cdots + \frac{f^{(d)}(X)}{d!}(\mathbf{i}Y)^d \\ &= P(X,Y) + (\mathbf{i}Y)Q(X,Y) \end{aligned}$$

where

$$\begin{aligned} P = P(X,Y) &:= \sum_{j=0}^{\lfloor d/2 \rfloor} f_{2j}(X)(-Y^2)^j \\ Q = Q(X,Y) &:= \sum_{j=0}^{\lceil d/2 \rceil - 1} f_{2j+1}(X)(-Y^2)^j \end{aligned}$$

and $f_i(X) := (-1)^{\lfloor i/2 \rfloor} \frac{f^{(i)}(X)}{i!}$ is the "normalized" ith derivative (with sign). Note that $f_0(X) = f(X)$ and $\deg_Y(P) \geq \deg_Y(Q)$. It follows that r_i are real zeros of the resultant $R(X) := \mathtt{res}_Y(P(X,Y), Y \cdot Q(X,Y))$. It is easy to verify that

$$\mathtt{res}_Y(P, Y \cdot Q) = f_0(X)\mathtt{res}_Y(P,Q).$$

To further factor $R(X)$, let us assume $d \geq 3$, so that $\deg_Y(P) \geq \deg_Y(Q) \geq 2$. Then we can write

$$P(X,Y) = \overline{P}(X, Y^2), \qquad Q(X,Y) = \overline{Q}(X,Y^2)$$

where $\deg_Y \overline{P} = \lfloor d/2 \rfloor \geq \lceil d/2 \rceil - 1 = \deg_Y \overline{Q}$. Then we may verify

$$R(X) = f(X) \cdot \overline{R}(X)^2$$

where $\overline{R}(X) = \mathtt{res}_Y(\overline{P}, \overline{Q})$.

For the next bound, we use the 1-norm $\|f\|_1$ and 2-norm $\|f\|_2$ of f.

LEMMA 22. *The degree of* $\overline{R} = \mathtt{res}_Y(\overline{P}, \overline{Q})$ *is*

$$\binom{d}{2} = \frac{d(d-1)}{2}.$$

Also, $\|\overline{R}\|_2 \leq (2^d \|f\|_1)^{d-1}$.

Proof. The degree of $\overline{R}$ comes from looking at the main diagonal of the Sylvester matrix defining the resultant. There are two cases: Case d is odd: here $\deg_Y \overline{P} = \deg_Y \overline{Q} = (d-1)/2$. The product of the diagonal elements is $(f_0)^{(d-1)/2}(f_d)^{(d-1)/2}$. Since $\deg f_0 = d$ and $\deg f_d = 0$, the degree of this product is $d(d-1)/2$. Case d is even: here $\deg_Y \overline{P} = d/2$ and $\deg_Y \overline{Q} = (d-2)/2$. The product of the diagonal elements is $(f_0)^{(d-2)/2}(f_{d-1})^{d/2}$. Since $\deg f_0 = d$ and $\deg f_{d-1} = 1$, the degree of this product is again $\frac{d(d-2)}{2} + d/2 = d(d-1)/2$.

For the height of $\overline{R}$, we use the Goldstein-Graham bound ([39, p. 173]). Let $\mathtt{res}_Y(\overline{P}, \overline{Q}) = \det(T)$ where $T = [t_{ij}]_{i,j}$ is the $(d-1) \times (d-1)$ Sylvester matrix constructed from $\overline{P}, \overline{Q}$. For instance the first and last rows of T are (respectively) given by

$$(f_0, f_2, f_4, \ldots, f_{\lfloor d/2 \rfloor}, 0, \ldots, 0),$$
$$(0, \ldots, 0, f_1, f_3, , \ldots, f_{\lceil d/2 \rceil - 1}).$$

Let $W = [w_{ij}]_{i,j}$ be the $(d-1) \times (d-1)$ matrix whose (i,j)th entry is given by $w_{ij} = \|t_{ij}\|_1$. Each of the t_{ij} is of the form f_k for some $k = k(i,j)$. We use the simple estimate $\|f_k\|_1 \le \binom{d}{k}\|f\|_1$ and hence the 2-norm of the first row of W is

$$\left(\|f_0\|_1^2 + \|f_2\|_1^2 + \|f_4\|_1^2 + \cdots + \|f_{\lfloor d/2 \rfloor}\|_1^2\right)^{1/2} < \left(\sum_{i \ge 0} \binom{d}{i}^2 \|f\|_1^2)\right)^{1/2} \le 2^d \|f\|_1.$$

In fact, the 2-norm of every row of W is bounded by $2^d\|f\|_1$. The Graham-Goldstein bound says $\|\overline{R}\|_2$ is upper bounded by the product of these 2-norms, i.e., $\|\overline{R}\|_2 \le (2^d\|f\|_1)^{d-1}$. **Q.E.D.**

Since $\lg \|f\|_1 \le \lg d + L$, we obtain

$$\lg \|\overline{R}\|_2 = O(d(d+L)). \tag{32}$$

¶15. Complex Part. A similar procedure can be used to construct a polynomial $S(Y)$ whose roots include all the $s_i = \mathtt{Im}(\alpha_i)$. The details are somewhat different, which we proceed to derive. First, we write $f(X)$ as a sum of its even and odd parts:

$$\begin{aligned} f(X) &= f_e(X) + f_o(X) & (33)\\ &= \overline{f}_e(X^2) + X \cdot \overline{f}_o(X^2) & (34) \end{aligned}$$

where $\overline{f}_e, \overline{f}_o \in \mathbb{R}[X]$ have degrees $\lceil (d-1)/2 \rceil$ and $\lfloor (d-1)/2 \rfloor$, respectively. For $i \ge 0$, we further write the i-th derivatives of f_e and f_e in the form:

$$f_e^{(i)}(X) = \begin{cases} \overline{f}_{e,i}(X^2) & \text{if } i = even \\ X \cdot \overline{f}_{e,i}(X^2) & \text{if } i = odd, \end{cases}$$

$$f_o^{(i)}(X) = \begin{cases} X \cdot \overline{f}_{o,i}(X^2) & \text{if } i = even \\ \overline{f}_{o,i}(X^2) & \text{if } i = odd. \end{cases}$$

The polynomials $\overline{f}_{e,i}$ and $\overline{f}_{o,i}$ are implicitly defined by these equations.

Use the Taylor expansion of $f(X + \mathbf{i}Y)$ at the point $\mathbf{i}Y$:

$$\begin{aligned} f(X + \mathbf{i}Y) &= \sum_{i \ge 0} f^{(i)}(\mathbf{i}Y)\frac{X^i}{i!} \\ &= \sum_{i \ge 0} \left[f_e^{(i)}(\mathbf{i}Y) + f_o^{(i)}(\mathbf{i}Y)\right]\frac{X^i}{i!} \\ &= \sum_{i \ge 0} \left[f_{e,2i}(-Y^2) + \mathbf{i}Y f_{o,2i}(-Y^2)\right]\frac{X^{2i}}{(2i)!} + \sum_{i \ge 0} \left[\mathbf{i}Y f_{e,2i+1}(-Y^2) + f_{o,2i+1}(-Y^2)\right]\frac{X^{2i+1}}{(2i+1)!} \end{aligned}$$

$$= \sum_{i\ge 0}\left[\frac{f_{e,2i}(-Y^2)}{(2i)!} + X\frac{f_{o,2i+1}(-Y^2)}{(2i+1)!}\right]X^{2i} + \mathbf{i}Y\sum_{i\ge 0}\left[X\frac{f_{e,2i+1}(-Y^2)}{(2i+1)!} + \frac{f_{o,2i}(-Y^2)}{(2i)!}\right]X^{2i}$$
$$= P(X,Y) + \mathbf{i}YQ(X,Y)$$

where

$$\begin{array}{rcl} P(X,Y) &=& \sum_{i=0}^{2\lfloor (d-1)/2\rfloor} p_i(Y)X^i, \\ && \text{with } \ p_{2i}(Y) = \frac{f_{e,2i}(-Y^2)}{(2i)!} \ \text{ and } \ p_{2i+1}(Y) = \frac{f_{o,2i+1}(-Y^2)}{(2i+1)!}, \\ Q(X,Y) &=& \sum_{i=0}^{2\lceil (d-1)/2\rceil} q_i(Y)X^i, \\ && \text{with } \ q_{2i}(Y) = \frac{f_{o,2i}(-Y^2)}{(2i)!} \ \text{ and } \ q_{2i+1}(Y) = \frac{f_{e,2i+1}(-Y^2)}{(2i+1)!}. \end{array}$$

NOTE: we are reusing the symbols P, Q, and they should not be confused with the polynomials P, Q used in the definition of $R(X)$ above.

Now the imaginary part of the zeros of $f(X)$ are zeros of the resultant $S(Y) := \mathtt{res}_X(P, Q)$ since

$$\mathtt{res}_X(P(X,Y), Y\cdot Q(X,Y)) = Y^{2\lfloor (d-1)/2\rfloor}\mathtt{res}_X(P,Q). \tag{35}$$

Note that $S(Y)$ is the determinant of a Sylvester matrix T whose first and last rows are

$$(p_0, p_1, p_2, \ldots, p_{2\lfloor (d-1)/2\rfloor}, 0, \ldots, 0),$$
$$(0, \ldots, 0, q_0, q_1, \ldots, q_{2\lceil (d-1)/2\rceil}).$$

The dimension of T is $(d-1)\times(d-1)$, and $\deg(S(Y)) \le d(d-1)$.

To bound the height of $S(Y)$, we proceed as before: $\|p_i\|_1 \le \binom{d}{i}\|f\|_1$ and $\|q_i\|_1 \le \binom{d}{i}\|f\|_1$. Then the Goldstein-Graham bound implies $\|S\|_2 \le (2^d\|f\|_1)^{d-1}$, or $\lg\|S\|_2 = O(dL)$.

Lemma 23. *The degree of* $S = \mathtt{res}_X(P, Q)$ *is*

$$\binom{d}{2} = \frac{d(d-1)}{2}.$$

Also, $\|S\|_2 \le (2^d\|f\|_1)^{d-1}$.

10 Bounding the Gamma Integral

We first prove the key inequality of Lemma 17, restated here:

Lemma 24. *Let* $\beta_2, \ldots, \beta_d$ *be all the critical points of* $f(x)$ *(i.e., zeros of* f'*). Then*

$$\gamma(x) \le \sum_{j=2}^{d}\frac{1}{2|x-\beta_j|}$$

Proof. We have

$$\frac{f^{(i)}(x)}{f'(x)} = \sum_{(j_2,\ldots,j_i)}{}' \prod_{\ell=2}^{i}\frac{1}{x-\beta_{j_\ell}}$$

where the summation ranges over all ordered $(i-1)$-tuples $(j_2, j_3, \ldots, j_i)$ taken from $\{1, \ldots, d-1\}$, $1 \le j_2 < j_3 < \cdots < j_i \le d-1$. The prime in the summation symbol, $\sum'$, indicates the strict inequality, $j_2 < \cdots < j_i$. When we omit the prime in the summation, it means that the tuples could have duplicated components, $1 \le j_2 \le j_3 \le \cdots \le j_i \le d-1$. Thus

$$\begin{aligned}
\left|\frac{f^{(i)}(x)}{f'(x)}\right|^{1/(i-1)} &= \left|\sum_{(j_2,\ldots,j_i)}{}' \prod_{\ell=2}^{i} \frac{1}{x-\beta_{j_\ell}}\right|^{1/(i-1)} \\
&\le \left(\sum_{(j_2,\ldots,j_i)}{}' \prod_{\ell=2}^{i} \frac{1}{|x-\beta_{j_\ell}|}\right)^{1/(i-1)} \\
&\le \left(\sum_{(j_2,\ldots,j_i)} \prod_{\ell=2}^{i} \frac{1}{|x-\beta_{j_\ell}|}\right)^{1/(i-1)} \qquad \text{unprimed summation} \\
&= \left(\left(\sum_{j=2}^{d} \frac{1}{|x-\beta_j|}\right)^{i-1}\right)^{1/(i-1)} \\
&\le \sum_{j=2}^{d} \frac{1}{|x-\beta_j|}.
\end{aligned}$$

For $i \ge 2$, we have $i! \ge 2^{i-1}$, and hence

$$\left|\frac{f^{(i)}(x)}{i!f'(x)}\right|^{1/(i-1)} \le \frac{1}{2}\left|\frac{f^{(i)}(x)}{f'(x)}\right|^{1/(i-1)} \le \frac{1}{2}\sum_{j=2}^{d} \frac{1}{|x-\beta_j|}.$$

Q.E.D.

Recall that $\beta_i = r_i + \mathbf{i}s_i$ where $r_i = \mathtt{Re}(\beta_i)$, $s_i = \mathtt{Im}(\beta_i)$. Wlog, let $s_i = 0$ iff $2 \le i \le k$. We next split the analysis into the real and nonreal parts.

¶16. Real Part. Recall that Procedure H produces $s_I < 2d$ disjoint special intervals which contain real zeros of f and f'. Assume the real roots of f' in I are $r_2 < r_3 < \cdots < r_k$ for some $k \le d$, and each $r_i \in [b_i, a_{i+1}]$. Let $I'' := I \setminus \bigcup_{i=1}^{k} [b_i, a_{i+1}]$ where For consistency, let $I = [a_1, b_{k+1}]$. Writing

$$\phi_R(X) = \prod_{i=2}^{k} (X - r_i), \tag{36}$$

we have:

$$\begin{aligned}
\int_{I'} \sum_{i=2}^{k} \frac{dx}{|x-r_i|} &\le \int_{I''} \sum_{i=2}^{k} \frac{dx}{|x-r_i|} \\
&= \sum_{j=1}^{k} \int_{b_j}^{a_{j+1}} \sum_{i=2}^{k} \frac{dx}{|x-r_i|} \\
&= \sum_{j=1}^{k} \sum_{i=2}^{k} \int_{b_j}^{a_{j+1}} \frac{dx}{|x-r_i|} \\
&= \sum_{j=1}^{k} \sum_{i=2}^{k} [\log|x-r_i|]_{b_j}^{a_{j+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k}\sum_{i=2}^{k} \log |(b_j - r_i)/(a_{j+1} - r_i)| \\
&= \sum_{j=1}^{k} \log \left| \frac{\phi_R(a_{j+1})}{\phi_R(b_j)} \right| .
\end{aligned}
$$

Thus we have shown:

LEMMA 25.

$$
\int_{I'} \sum_{i=1}^{k} \frac{dx}{|x - r_i|} \le \log \prod_{j=1}^{k} \left| \frac{\phi_R(a_{j+1})}{\phi_R(b_j)} \right| .
$$

¶17. Complex Part. Consider the case where $\beta_i = r_i + \mathbf{i}s_i$ is nonreal, i.e., $i > k$. Initially, assume $a + |s_i| \le r_i \le b - |s_i|$ where $I = [a, b]$. Then

$$
\begin{aligned}
\int_{I'} \frac{dx}{|x - \beta_i|} &\le \int_a^b \frac{dx}{|x - \beta_i|} \\
&\le \int_a^b \frac{dx}{\max\{|x - r_i|, |s_i|\}} \\
&\overset{(*)}{=} \int_a^{r_i - |s_i|} \frac{dx}{r_i - x} + \int_{r_i - |s_i|}^{r_i + |s_i|} \frac{dx}{|s_i|} + \int_{r_i + |s_i|}^{b} \frac{dx}{x - r_i} \\
&= \ln\left(\frac{r_i - a}{|s_i|}\right) + 2 + \ln\left(\frac{b - r_i}{|s_i|}\right).
\end{aligned}
$$

where (*) is valid since $\max\{|x - r_i|, |s_i|\} = |s_i|$ iff $x \in [r_i - |s_i|, r_i + |s_i|]$. Next, suppose $r_i - |s_i| \le a$. Then the above bound holds, provided the term $\ln\left(\frac{r_i - a}{|s_i|}\right)$ be dropped. Similarly, if $r_i + |s_i| \ge b$ then the term $\ln\left(\frac{b - r_i}{|s_i|}\right)$ should be dropped. Combining all these cases, we obtain:

LEMMA 26.

$$
\int_{I'} \frac{dx}{|x - \beta_i|} \le \ln \max\left\{1, \left(\frac{r_i - a}{|s_i|}\right)\right\} + 2 + \ln \max\left\{1, \left(\frac{b - r_i}{|s_i|}\right)\right\} .
$$

We may assume that the roots β_i are indexed so that

$$
r_{k+1} - |s_{k+1}| \le r_{k+2} - |s_{k+2}| \le \cdots \le r_d - |s_d|. \tag{37}
$$

Then there exists $\ell \in \{k+1, \ldots, d+1\}$ such that $a < r_i - |s_i|$ iff $\ell \le i$. Note that $\ell = d + 1$ means there is no such i. Note that (37) is equivalent to

$$
r_{k+1} + |s_{k+1}| \le r_{k+2} + |s_{k+2}| \le \cdots \le r_d + |s_d|.
$$

Thus, there exists $\lambda \in \{k, k+1, \ldots, d\}$ such that $r_j + |s_j| < b$ iff $\lambda \le j$. Again, $\lambda = k$ means there is no such j. Thus Lemma 26 implies:

$$
\int_{I'} \sum_{i=k+1}^{d} \frac{dx}{|x - \beta_i|} \le \ln \prod_{i=k+1}^{d} \max\left\{1, \left(\frac{r_i - a}{|s_i|}\right)\right\} + 2(d - k) + \ln \prod_{i=k+1}^{d} \max\left\{1, \left(\frac{b - r_i}{|s_i|}\right)\right\}
$$

$$= \ln \prod_{i=\ell}^{d} \left(\frac{r_i - a}{|s_i|} \right) + 2(d-k) + \ln \prod_{i=k+1}^{\lambda} \left(\frac{b - r_i}{|s_i|} \right).$$

In order to bound the integral in Lemma 27 in terms of d and L, we introduce the polynomials

$$\phi_A(X) := \prod_{i=\ell}^{d} (r_i - X) \tag{38}$$

$$\phi_B(X) := \prod_{i=k+1}^{\lambda} (X - r_i) \tag{39}$$

$$\phi_C(X) := \prod_{i=k+1}^{d} (X - s_i). \tag{40}$$

It follows from Lemma 20 that

$$\prod_{i=\ell}^{d} |s_i| \geq \frac{1}{M(\phi_C)} \geq \frac{1}{M(S)}, \qquad \prod_{i=k+1}^{\lambda} |s_i| \geq \frac{1}{M(\phi_C)} \geq \frac{1}{M(S)} \tag{41}$$

where $S(Y)$ is the polynomial of Lemma 23. This allows us to rephrase the preceding integral bound in a compact form:

Lemma 27.

$$\int_{I'} \sum_{i=k+1}^{d} \frac{dx}{|x - \beta_i|} \leq \ln \frac{\phi_A(a)\phi_B(b)}{M(\phi_C)^2} + 2(d-k)$$

11 Applying the Evaluation Bounds.

In the previous section, we bounded the integrals for the real part (Lemma 25) and non-real parts (Lemma 27). These bounds were given in terms of the polynomials $\phi_R, \phi_A, \phi_B, \phi_C$ ((36) and (38)) evaluated at suitable points. To convert these into explicit bounds in terms of d and L, we now use the Evaluation Bound in Theorem 13.

¶18. Bound on Real Part. We want to bound the evaluation expression in Lemma 25. Define

$$\eta_A(X) = \prod_{j=0}^{\ell} (X - a_{j+1}), \qquad \eta_B(X) = \prod_{j=0}^{\ell} (X - b_j). \tag{42}$$

We split the proof into two steps. The first step is to upper bound

$$\lg \prod_{j=0}^{\ell} |\phi_R(a_{j+1})|.$$

We exploit the fact that all the zeros of ϕ_R ((36)) are also zeros of f'. Hence we have

$$\|\phi_R\| \leq 2^d M(\phi_R) \leq 2^d M(f').$$

We apply Theorem 13(a), with ϕ replaced $\phi_R(X)$, η replaced by $\eta_A(X)$, H by H_A. By Lemma 21, $\lg M(\eta_A) = O(dL)$. Hence $m \le d$ and $n \le d$, and

$$\begin{aligned}
\prod_{j=0}^{\ell} |\phi_R(a_{j+1})| &\le ((d+1)\|\phi_R\|)^d \, M(\eta_A)^d \\
&\le \Big((d+1)2^d M(f')\Big)^d \, M(\eta_A)^d. \\
\lg \prod_{j=0}^{\ell} |\phi_R(a_{j+1})| &= O(d(d+L)).
\end{aligned}$$

The second step is to lower bound

$$\lg \prod_{j=0}^{\ell} |\phi_R(b_j)|.$$

We apply Theorem 13(b), with ϕ replaced by $\phi_R(X)$ as before, but η replaced by $\eta_B(X)$, F by f', and H given by Lemma 21. We have $m \le d-1$ and $n \le d$ as before. Now $\overline{\phi}$ is given by f'/ϕ_R of degree $\overline{m} \le d-1$, and $\overline{\eta} = H_B/\eta_B = K_B$ of degree $\overline{n} = 0$. Thus:

$$\begin{aligned}
-\lg \prod_{j=0}^{\ell} |\phi_R(b_j)| &\le \lg \Big(\mathrm{lc}(H_B)^{d-1} \cdot ((d\|\phi_R\|)^0 \, M(\overline{\eta})^{d-1} \cdot ((\overline{m}+1)\|\overline{\phi}\|)^d M(H_B)^{d-1}\Big) \\
&= O(d^2(\log d + L)).
\end{aligned}$$

This concludes the proof of Lemma 18.

¶19. Bound on Complex Part. We bound the evaluation expression in Lemma 27. Consider the polynomial $\phi_A(X)$ in (38). We have ([39, p. 118])

$$\|\phi_A\| \le 2^d M(\phi_A) \le 2^d M(\overline{R}') \le 2^d \|\overline{R}'\|_2 \tag{43}$$

where $\overline{R}'$ is defined for f', analogous to the definition of $\overline{R}$ defined for f in ¶14. From (32), we conclude that $\lg \|\phi_A\| = O(d(d+L))$. Recall that w.l.o.g. a, b are integers satisfying $|a|, |b| \le 2^L$. We now apply Theorem 13(a) where we take the polynomial $\phi(X)$ to be ϕ_A, and F to be $\overline{R}'$. The polynomial $\eta(X)$ is just $X - a$, and $H = \eta$. Hence $m = d$ and $n = 1$. As $M(\eta) \le 2^L$ and we have

$$|\phi_A(a)| \le ((d+1)\|\phi_A\|) \cdot M(\eta)^d \le (d+1)M(\overline{R}') \cdot 2^{Ld}$$

and taking logs,

$$\lg \phi_A(a) = O(d(d+L)). \tag{44}$$

Similarly, $\lg \phi_B(b) = O(d(d+L))$.

From (41), we see that $-\lg \prod_i |s_i| \le \lg M(S)$. By Lemma 23, we get $\lg M(S) = O(d(d+L))$. Plugging this and (44) into Lemma 27, we obtain a bound for the integral over non-real roots β_i's:

$$\int_{I'} \sum_{i=k+1}^{d} \frac{dx}{|x - \beta_i|} = O(d(d+L)). \tag{45}$$

This concludes the proof of Lemma 19.

12 Shifting γ and γ'

We now provide the two lemmas are needed in ¶9 to guarantee that special intervals are terminal. But before this, we prove Lemma 7, restated here:

LEMMA 7. *Let $b \in J$ such that $w(J) \le \frac{1}{2\gamma(b)}$. Then $K_J \le 2d|f'(b)|$.*

Proof.

$$\begin{aligned}
K_J &= \max_{a\in J} \sum_{i=1}^{d} \frac{|f^{(i)}(a)|}{i!} w(J)^{i-1} \\
&= \max_{a\in J} \sum_{i=1}^{d} \left| \sum_{j=i}^{d} \frac{f^{(j)}(b)(b-a)^{j-i}}{i!(j-i)!} \right| w(J)^{i-1} \\
&\le \sum_{i=1}^{d} \sum_{j=i}^{d} \frac{|f^{(j)}(b)|}{i!(j-i)!} w(J)^{j-1} && |a-b| \le w(J) \\
&= \sum_{j=1}^{d} \frac{|f^{(j)}(b)|}{j!} w(J)^{j-1} \sum_{i=1}^{j} \binom{j}{i} \\
&\le \sum_{j=1}^{d} \frac{|f^{(j)}(b)|}{j!} w(J)^{j-1} 2^j \\
&\le \sum_{j=1}^{d} \frac{|f^{(j)}(b)|}{j!} \frac{2^j}{2^{j-1}\gamma(b)^{j-1}} && w(J) \le \frac{1}{2\gamma(b)} \\
&\le 2 \sum_{j=1}^{d} \frac{|f^{(j)}(b)|}{j!} \frac{j!|f'(b)|}{|f^{(j)}(b)|} && \gamma(b) \ge \left(\frac{|f^{(j)}(b)|}{j!|f'(b)|} \right)^{\frac{1}{j-1}} \\
&= 2d|f'(b)|
\end{aligned}$$

Q.E.D.

LEMMA 28. *Let J be a special interval containing α with $\alpha \in$ ZERO(f) and $w(J) < \frac{\rho(\alpha)}{4d(d-1)}$. Then $w(J) < \frac{|f'(\alpha)|}{K'_J}$.*

Proof. From Proposition 10 we know that the condition on $w(J)$ implies that $w(J) < \frac{1}{8\gamma(\alpha)}$. Now, by computing an upper bound on K'_J, we show the desired result.

$$\begin{aligned}
K'_J &= \max_{a\in J} \sum_{i=1}^{d-1} \frac{|(f')^{(i)}(a)|}{i!} w(J)^{i-1} \\
&= \max_{a\in J} \sum_{i=2}^{d} \frac{|f^{(i)}(a)|}{(i-1)!} w(J)^{i-2} \\
&= \max_{a\in J} \sum_{i=2}^{d} \left| \sum_{j=i}^{d} \frac{f^{(j)}(\alpha)(a-\alpha)^{j-i}}{(j-i)!(i-1)!} \right| w(J)^{i-2}
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=2}^{d}\sum_{j=i}^{d} \frac{|f^{(j)}(\alpha)|}{(j-i)!(i-1)!} w(J)^{j-2} && |a-\alpha| \le w(J) \\
&= \sum_{j=2}^{d} \frac{|f^{(j)}(\alpha)|}{(j-1)!} w(J)^{j-2} \sum_{i=2}^{j} \binom{j-1}{i-1} \\
&\le \sum_{j=2}^{d} \frac{|f^{(j)}(\alpha)|}{(j-1)!} w(J)^{j-2} 2^{j-1} \\
&\le \frac{1}{w(J)} \sum_{j=2}^{d} \frac{|f^{(j)}(\alpha)|}{(j-1)!} \frac{2^{j-1}}{8^{j-1}\gamma(\alpha)^{j-1}} && w(J) \le \frac{1}{8\gamma(\alpha)} \\
&\le \frac{1}{w(J)} \sum_{j=2}^{d} \frac{|f^{(j)}(\alpha)|}{(j-1)!} \frac{j!|f'(\alpha)|}{|f^{(j)}(\alpha)|} 2^{-2j+2} && \gamma(\alpha) \ge \left(\frac{|f^{(j)}(\alpha)|}{j!|f'(\alpha)|}\right)^{\frac{1}{j-1}} \\
&= \frac{|f'(\alpha)|}{w(J)} \sum_{j=2}^{d} j 2^{-2j+2} \\
&< \frac{7}{9} \frac{|f'(\alpha)|}{w(J)}
\end{aligned}
$$

Then, by rearranging $w(J)$ and K'_J, we find that $w(J) < \frac{|f'(\alpha)|}{K'_J} < \frac{7}{9}\frac{|f'(\alpha)|}{K'_J}$ as desired. **Q.E.D.**

LEMMA 29. *Let J be a special interval containing α with $\alpha \in \textsc{Zero}(f)$ and $w(J) < \min\left\{\frac{\rho'(\alpha)}{4d(d-1)}, \sqrt{\frac{|f(\alpha)|}{3|f''(\alpha)|}}\right\}$. Then $w(J) < \frac{|f(\alpha)|}{K_J}$.*

Proof. From Proposition 10 we know that the condition on $w(J)$ implies that $w(J) < \frac{1}{8\gamma'(\alpha)}$. Now by computing an upper bound on K_J, we can show the desired result.

$$
\begin{aligned}
K_J &= \max_{a\in J} \sum_{i=1}^{d} \frac{|f^{(i)}(a)|}{i!} w(J)^{i-1} \\
&= \max_{a\in J} \sum_{i=1}^{d} \left| \sum_{j=i}^{d} \frac{f^{(j)}(\alpha)(a-\alpha)^{j-i}}{(j-i)!i!} \right| w(J)^{i-1} \\
&\le \sum_{i=1}^{d}\sum_{j-i}^{d} \frac{|f^{(j)}(\alpha)|}{(j-i)!i!} w(J)^{i-1} && |a-\alpha| \le w(J) \\
&= \sum_{j=1}^{d} \frac{|f^{(j)}(\alpha)|}{j!} w(J)^{j-1} \sum_{i=1}^{j} \binom{j}{i} \\
&\le \sum_{j=1}^{d} \frac{|f^{(j)}(\alpha)|}{j!} w(J)^{j-1} 2^{j} \\
&= \sum_{j=2}^{d} \frac{|f^{(j)}(\alpha)|}{j!} w(J)^{j-1} 2^{j} && \alpha \in \textsc{Zero}(f')
\end{aligned}
$$

$$= w(J)\sum_{j=2}^{d}\frac{|f^{(j)}(\alpha)|}{j!}w(J)^{j-2}2^j$$

$$\le w(J)\sum_{j=2}^{d}\frac{|f^{(j)}(\alpha)|}{j!}\frac{2^j}{8^{j-2}\gamma'(\alpha)^{j-2}}$$

$$\le w(J)\sum_{j=2}^{d}\frac{|f^{(j)}(\alpha)|}{j!}\frac{(j-1)!|f''(\alpha)|}{|f^{(j)}(\alpha)|}2^{-2j+6} \qquad \gamma(\alpha) \ge \left(\frac{|(f')^{(j-1)}(\alpha)|}{(j-1)!|(f')'(\alpha)|}\right)^{\frac{1}{j-2}}$$

$$= |f''(\alpha)|w(J)\sum_{j=2}^{d}\frac{2^{-2j+6}}{j}$$

$$< 64|f''(\alpha)|w(J)|\left(\ln\left(\frac{4}{3}\right)-\frac{1}{4}\right)$$

$$< 3|f''(\alpha)|w(J)$$

Therefore, it follows that $\frac{|f(\alpha)|}{K_J} > \frac{|f(\alpha)|}{3|f''(\alpha)|w(J)} \ge \frac{(w(J))^2}{w(J)} = w(J)$, completing the result. **Q.E.D.**

13 Basic Stopping Functions

We now address the stopping functions mentioned at the end of ¶6.

¶20. Global Lipschitz Constants. The function $g = f_I$ defined in ¶6 is based on the "global" Lipschitz constant K_I. To show that g is a stopping function, we show that if X is not large then X is terminal:

LEMMA 30. *The functions* $\frac{|f(a)|}{K_I}$, $\frac{|f'(a)|}{K'_I}$ *and* f_I *are stopping functions over* I.

The result easily follows from Lemma 2. Using this stopping function, we can now apply Theorem 5 to get an integral bound on the complexity of EVAL. Naturally, such a bound based on a global constant K_I is not very satisfactory. The next section introduces a local local Lipschitz constants K_a ($a \in I$) and a corresponding stopping function in the next section.

¶21. Local Lipschitz Constants. One suggestion to use more local Lipschitz constants is to use K_X and f_X instead of K_I and f_I in our algorithms. This seems to lead to complicated conditions on our partitions, and it is hard to state an integral independent of the partition. Instead, we proceed as follows.

In this section, we fix the interval I, and throughout, X range over $\square I$. For any $a \in I$ and $\ell > 0$, define

$$K_{a,\ell} := \max_{\substack{X\subseteq I\\ a\in X\\ w(X)\le\ell}} K_X. \tag{46}$$

.

If we replace K_X by K'_X in (46), the resulting constant will be denoted by $K'_{a,\ell}$.

LEMMA 31. *Let* $a \in I$ *and* $\ell > 0$.
(i.a) $K_{a,\ell}$ *is monotonically non-decreasing with* ℓ.
(i.b) As $\ell \to \infty$, *we have* $K_{a,\ell} \to \infty$ *and* $\frac{|f(a)|}{K_{a,\ell}} \to \frac{|f(a)|}{K_I}$.

(i.c) As $\ell \to 0$*, we have* $K_{a,\ell} \to |f'(a)|$ *and* $\frac{|f(a)|}{K_{a,\ell}} \to \frac{|f(a)|}{|f'(a)|}$*. (Hence, define* $K_{a,0} = |f'(a)|$*.)*
(ii.a) The product $\ell \cdot K_{a,\ell}$ *in strictly increasing with* ℓ*.*
(ii.b) As $\ell \to \infty$*, we have* $\ell \cdot K_{a,\ell} \to \infty$*.*
(ii.c) As $\ell \to 0$*, we have* $\ell \cdot K_{a,\ell} \to 0$*.*

The proof is omitted. From (ii.a-c), we conclude that there is a unique $\ell = \ell_a$ such that $\ell_a \cdot K_{a,\ell} = |f(a)|$. Define $w(a) := \ell_a$ as the **local width** at a, and define $K_a := K_{a,w(a)}$ as the **local Lipschitz constant** at a. Note that $f(a) = 0$ implies $w(a) = 0$ and hence $K_a = |f'(a)|$.

We can also define the local width $w'(a)$ (resp., local Lipschitz constant K'_a) if we use $f', K'_{a,\ell}$ instead of $f, K_{a,\ell}$ in the above definitions of $w(a)$ (resp., K_a).

For all $a \in I$, we have

$$w(a) = \frac{|f(a)|}{K_a}. \tag{47}$$

From Lemma 31(ii), we immediately obtain:

Lemma 32. *Let* $a \in I$ *and* $\ell > 0$*. Then*

$$\ell \geq \frac{|f(a)|}{K_{a,\ell}} \quad \Leftrightarrow \quad \ell \geq w(a) \geq \frac{|f(a)|}{K_{a,\ell}}.$$

$$\ell \leq \frac{|f(a)|}{K_{a,\ell}} \quad \Leftrightarrow \quad \ell \leq w(a) \leq \frac{|f(a)|}{K_{a,\ell}}.$$

Moreover, equality is simultaneously achieved on both sides.

We define $f_\ell(a) = \max\left\{\frac{|f(a)|}{K_{a,l}}, \frac{|f'(a)|}{K'_{a,l}}\right\}$.

Using these facts, we define our candidate for a stopping functions:

$$\begin{aligned} f_*(a) \quad &:= \quad \max\left\{\frac{|f(a)|}{K_a}, \frac{|f'(a)|}{K'_a}\right\} && (48)\\ &= \quad \max\left\{w(a), w'(a)\right\}. && (49)\end{aligned}$$

Using these definitions, Lemma 32 can be rephrased as follows:

$$\ell \geq f_\ell(a) \quad \Leftrightarrow \quad \ell \geq f_*(a) \geq f_\ell(a). \tag{50}$$

$$\ell \leq f_\ell(a) \quad \Leftrightarrow \quad \ell \leq f_*(a) \leq f_\ell(a). \tag{51}$$

Moreover, equality occurs simultaneously on both sides.

Lemma 33. f_* *is a stopping function.*

Proof. Let $a \in X$. If $C_0(X)$ and $C_1(X)$ fail, as before, it means $w(X) \geq \max\{|f(a)|/K_X, |f'(a)|/K'_X\} = f_X(a)$. Thus $w(X) \geq f_{w(X)}(a)$. By (50), this is equivalent to $w(X) \geq f_*(a)$. Hence X is large. **Q.E.D.**

Theorem 34. *Let* P_I *be the partition of* I *at the end of Phase 1 of the Evaluation Algorithm. Then*

$$\#(P) \leq \max\left\{1, \int_I \frac{2da}{f_*(a)}\right\} = \max\left\{1, \int_I \min\left\{\frac{K_a}{|f(a)|}, \frac{K_a}{|f(a)|}\right\} 2da\right\}. \tag{52}$$

and this integral is finite.

We already know that f_* is a stopping function. This result follows from Theorem 5 if f_* is never 0. f_* is never 0 since f is square free and so f and f' do not share any roots.