(Extended Abstract)

Complete Subdivision Algorithms, I: Intersection of Bezier Curves

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Abstract

We give the first complete subdivision algorithm for the intersection of two Bezier curves F, G, possibly with tangential intersections. Our approach to robust subdivision algorithms is based on geometric separation bounds, and using a criterion for detecting non-crossing intersection of curves. Our algorithm is adaptive, being based only on exact bigfloat computations. In particular, we avoid manipulation of algebraic numbers and resultant computations. It is designed to be competitive with current algorithms on "nice" inputs. All standard algorithms assume F, G to be relatively prime – our algorithm needs a generalization of this.

1 Introduction

The intersection and analysis of algebraic curves and surfaces is a fundamental problem in many areas of geometric modeling [13]. Most practical algorithms are based on free-form curves and surfaces [7, 6]. In this paper, we consider one class of free-form curves, Bezier curves. All current algorithms for intersecting Bezier curves are inexact, leading to well-known nonrobustness problems. Let us look at a fundamental reason for this.

A Bezier curve F is a finite curve segment, represented by a sequence $P(F) = (p_0, \ldots, p_m)$ of control points [7, 6]. Let $CH(F)$ denote the convex hull of $p_0, \ldots, p_m$. A pair (F, G) of Bezier curves is called a *candidate pair* if $CH(F) \cap CH(G)$ is non-empty. Standard algorithms for intersecting Bezier curves are based on two ideas. First, using the property that a Bezier curve F is contained in $CH(F)$, the algorithm can discard non-candidate pairs. Second, the principal algorithmic operation is *subdivision*, which divides a curve F into two subcurves F_0, F_1 using De Casteljau's algorithm. The *generic intersection algorithm* simply keeps a queue Q of candidate pairs. As long as Q is non-empty, it extracts a candidate pair (F, G) from Q. If the diameter[1] of $F \cup G$ is less than ε, it outputs this pair; otherwise it subdivides the curve with the larger diameter, say F, into subcurves F_0, F_1, and appends (F_0, G) and (F_1, G) to Q.

This algorithm depends on a constant $\varepsilon > 0$: pairs (F, G) in the output have diameter less than ε and are treated as intersecting. For display purposes, such constants are justifiable. But for topological analysis of curve arrangements, we want output pairs (F, G) that represent *unique* intersections. But the generic algorithm might output a pair (F, G) that has *no* intersection, or has *multiple* intersections.

Let p be an intersection point of F and G. Under the *standard assumption* that F, G have no common component, then p is an isolated point of $F \cap G$. The intersection at p can be *tangential* or *transversal*,

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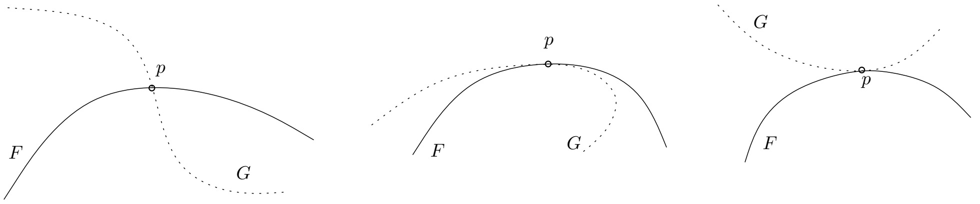
[1]The diameter $\mathrm{diam}(X)$ of a closed set $X \subseteq \mathbb{R}^2$ is the distance between a farthest pair of points in X.

depending on whether the tangents of F and G at p are coincident or not. Alternatively, we can classify p as a *crossing* or *non-crossing* intersection, depending on whether the curves cross each other at p or not. Non-crossing intersections must be tangential, and transversal intersections must be crossing. Thus we have 3 possibilities, as illustrated in Figure 1.



(a) Tansversal (crossing)   (b) Tangential, crossing   (c) Non-crossing (tangential)

Figure 1: Intersections: (a) Transversal (b) Tangential

Can the use of ε be avoided? It seems plausible that if F and G have only crossing intersections, then we can design an intersection algorithm based on subdivision that does not use any ε-cutoff. This is not obvious, but it is implied by our intersection algorithm. In any case, the key issue is how to detect non-crossing intersections. Recently, Wolpert [20, 19] addressed this issue for nonsingular algebraic curves. The class of curves she addresses and the ones addressed in this paper are not directly comparable – although Bezier curves are rather special algebraic curves, they may be singular (see Figure 4). She introduced the technique of generalized Jacobi curves to detect non-crossing intersection of non-singular curves. Like us, she uses subdivision of space to avoid the manipulation of algebraic numbers (unlike the traditional approach based on cylindrical decomposition). But her approach still uses strong algebraic tools such as resultants and root isolation. Such algebraic techniques are[2] expensive and reduce the effectiveness of any adaptivity in the rest of the algorithm. A more recent paper of Seidel and Wolpert [17] addresses computing the topological arrangement of plane algebraic curves; again, a combination of subdivision and algebraic methods are used. In contrast, the only algebraic information we use are algebraic zero bounds. Otherwise, we perform purely numerical computations using bigfloat numbers and primitive geometric operations such as computing convex hulls and intersecting curves with a line. It is not at all obvious that we can achieve our exact curve intersection goals using only geometric and numerical approximations. For instance, a purely adaptive/numerical version in the Wolpert-Seidel setting is not known.

Current subdivision algorithms deploy a variety of criteria for detecting intersection points. These are typically *partial criteria*: either a *rejection criterion* that affirms non-intersection or an *acceptance criterion* that affirms an intersection. A *complete criterion* is one that is both an acceptance and rejection criterion. The generic algorithm above uses only the *convex hull criterion*. Since this is a rejection criterion, the generic algorithm could never affirm intersections. Sederberg and Meyers [16] gave an acceptance criterion based on hodographs which affirms the presence of a transversal intersection. However there are no known acceptance criterion for non-crossing intersections; we will provide one in Section 3. In general, partial criteria is the geometric analogue of the concept of numerical *filters* [5]. Partial criteria can be very useful as a heuristic for quick reject/accept. But ultimately, a correct algorithm must use some complete criteria or some other global guarantee of completeness.

Overview of Our Approach. 1. The fundamental motivation of this work is to design an exact subdivision algorithm for Bezier intersection. Subdivisioning implies adaptivity. But in contrast to approaches that use a combination of algebraic methods with numerical methods, as in [19, 20, 17], ours is "fully adaptive".

2. The main analytical tool we introduce is geometric separation bounds. They answer such questions as: *How close can two curves be, if they do not intersect?* or *How close can a point q be to a curve if q is not on the curve?*. These bounds are denoted by various Δ's in this paper, and are derived in Sections 2.

3. The Δ-bounds provide stopping criteria for our numerical and subdivision procedures. The bounds are easily computed at the start of the algorithm. The structure of the algorithm is oblivious to the values of

[2] Nicola Wolpert and Raimund Seidel (personal communication, and talk at Dagstuhl Workshop 2005) observed that their implementation does not necessarily run faster for easy examples. This is attributable to the non-adaptive aspects of their algorithm.

these bounds. Thus, if improved Δ-bounds are available in the future, they can be automatically incorporated without changing the algorithmic logic.

4. Adaptivity means that these Δ-bounds are invoked only in the worst case scenario. For "nice inputs", an iteration may terminate long before the bound of that iteration is reached. To fully exploit this, one should introduce effective semi-criteria or filters. Although filters are not emphasized in this paper, they influence the basic design of our algorithms. In the future, filters could be added within the framework of our solution.

5. Section 3 provides the first complete criterion for detecting non-crossing intersection of *elementary Bezier Curves*. By an "elementary curve" we mean the graph of a convex or concave function. It is non-trivial to apply this criterion: hence we design in Section 6 a *coupling process* to create the necessary preconditions for applying the criterion.

6. Our algorithm uses various numerical and geometric approximations as subroutines. Two key subroutines are for intersecting an elementary Bezier curve with a line (Section 4) and for evaluating signs of the "alpha function" (Sections 5).

7. What about non-elementary Bezier curves? A general Bezier curve has *critical points*; elementary curves have no critical points. There are techniques for breaking up a general algebraic curve at such critical points (e.g., [2, 12]). But these involve algebraic, non-adaptive methods which we wish to avoid. Instead, Section 7 shows how subdivisioning with separation bounds can detect and completely isolate such critical points. The overall intersection algorithm is described in Section 8.

8. All our numerical computations are ultimately reduced to ring operations $(+, -, \times)$ on (binary) *bigfloats*, i.e., rational numbers of the form $n2^m$ where $n, m \in \mathbb{Z}$. These operations are[3] carried out *exactly*. For reasons of efficiency, we do not manipulate algebraic or even general rational numbers. We also do not manipulate polynomials or perform subresultant calculations, such as is found in the known algebraic intersection algorithms.

9. To emphasize the role of bigfloats in our representations, it is useful to introduce some terminology. Assume points, lines and Bezier curves are represented by their "standard" parameters: a point p is given by its coordinates $p = (x, y)$, a line ℓ is given by the coefficients of its equation $\ell : ax + by + c$, and a Bezier curve is given by its control points (which are in turn given by coordinates). When such parameters x, y, a, b, c, etc, are bigfloats, we called them *direct objects*; otherwise they are *indirect objects*. For instance, intersecting a "direct line" ℓ with a "direct Bezier curve" F may yield a point p whose coordinates are algebraic numbers. So p is an indirect object. We must then provide means of (a) representing, and (b) approximating, indirect objects by direct objects. For (a), we use "expressions" over direct objects. For instance, assuming that ℓ, F have a unique intersection, we may introduce the expression "$Point[\ell, F]$" to represent $p = \ell \cap F$. For (b), to approximate p, let us assume F has the standard Bezier parametrization $F = \{F(t) : 0 \leq t \leq 1\}$. Then we can compute a pair of bigfloat numbers t_0, t_1 such that $p = F(t)$ and $0 \leq t_0 \leq t \leq t_1 \leq 1$. Now the expression "$Point[\ell, F, t_0, t_1]$" is an approximation to p, in analogy to the well-known interval representation of algebraic numbers. This approximation can be arbitrarily refined by narrowing the interval $[t_0, t_1]$. *Computations with approximations of indirect objects are necessarily iterative*, with stopping criteria given by appropriate Δ-bounds. For instance, to do an (error-free) test whether $Point[\ell, F, t_0, t_1]$ lies on another Bezier curve G, we can narrow the interval $[t_0, t_1]$ until $\|F(t_0) - F(t_1)\| < \Delta/2$, and we subdivide G until $\mathrm{diam}(G) < \Delta/2$. Finally, we conclude that p lies on G iff the line segment $[F(t_0), F(t_0)]$ intersects $CH(G)$.

10. An appendix contains all the omitted proofs. Our full paper is available from `http://cs.nyu.edu/cs/faculty/yap/papers/`.

Related Work. The computational literature on algebraic curves and surfaces is very large and diverse. We may roughly divide the computational approaches into two distinct viewpoints: (A) The *Algebraic Viewpoint* treats curves and surfaces as systems of algebraic equations to be solved, usually using symbolic or algebraic techniques. Such "algebraic algorithms" are exact and are (or can be made) complete. (B) The *Geometric Viewpoint* prefers curves and surfaces in parametric form, usually solved using numerical techniques such as homotopy or subdivision. Such "geometric algorithms" are often incomplete but are widely used in practice. The Algebraic Viewpoint is more general, with impressive advances made in the last 20 years (e.g., [3]). Nevertheless, many algebraic algorithms are not considered practical. The curves

[3]There is an alternative view of bigfloats which do not support exact ring operations, but provide only approximations to any desired error bound. E.g., the floating point class `mpfr` in gnu's multiprecision package `gmp`.

and surfaces in applications are usually bounded subsets ("patches") of an algebraic set. The geometric algorithms directly manipulate such patches; the algebraic algorithms treat complete algebraic sets, often assumed to be irreducible. This fact reduces the applicability of algebraic algorithms. To specify patches of an algebraic set, one could use semi-algebraic formulas (i.e., introduce inequalities). But it is not easy, say, to specify a particular branch of a curve in the neighborhood of a self-intersection using this method. The computation and topological analysis of real plane curves is a well studied problem [1, 15], but the worst case complexity is prohibitive, reported as $O(n^{27}L^3)$ time for a curve $F(X,Y)=0$ of degree n with L-bit coefficients. In any case, such algorithms are not considered feasible [8]. When algebraic algorithms are combined with numerical techniques, more practical algorithms can be achieved, as demonstrated by Hong [11]. Recently, computational geometers have begun to address curves and surfaces [4, 10, 18]. These papers provide efficient and complete algebraic algorithms for low degree surfaces and curves. More generally, their goal is to make algorithms under the algebraic viewpoint more efficient (by careful considerations of the primitives) and complete (by explicit treatment of degeneracies); both these issues tend to be glossed over by more theoretical papers. In contrast, *our goal is to make algorithms under the geometric viewpoint completely robust while preserving their adaptive efficiency.* More generally, our ostensible goal is to do exact algebraic computation without algebraic manipulations. We only do numerical approximations, but achieve exactness by exploiting algebraic zero bounds.

The work most directly comparable to ours is Wolpert and Seidel's [20, 19, 17], mentioned above. Recently, Plantinga and Vegter [14] gave a topologically-correct algorithm for isotopic approximation of implicit nonsingular surfaces. Their subdivision-based approach is geometric and fully adaptive; however it is currently unknown whether it can be extended to handle singularities.

2 Geometric Separation Bounds for Algebraic Curves

Our key mathematical tool is the idea of separation bounds.

Suppose $A(x,y), B(x,y) \in \mathbb{Z}[x,y]$. Let $\|A\|_k$ denote the k-norm of the vector of the coefficients of A (we use $k=1,2$ and $k=\infty$). If $p \in \mathbb{R}^n$ is a point in Euclidean space, its k-norm has a similar notation, $\|p\|_k$. We omit the subscript k when referring to the 2-norm, i.e., Euclidean distance. Consider the curves F and G defined by the equations $A=0$ and $B=0$, respectively. A pair (p,q) where $p \in F$ and $q \in G$ such that the line through p,q is normal to p at F and to q at G is called an *(F,G)-antipodal pair.* In particular, if F and G intersect tangentially at p, then (p,p) is an (F,G)-antipodal pair. In the following, we assume there are only finitely many pairs of (F,G) antipodal pairs. This implies that A,B are relatively prime (i.e., they have no common component). We use a multivariate root bound of Yap [21] (Theorem 11.45).

Our first result is a lower bound on the distance between distinct points in an antipodal pair.

Theorem 1. *Suppose F and G have finitely many antipodal pairs, and satisfy the equations $A=0$ and $B=0$. Let $m=\deg(A)$, $\quad n=\deg(B)$ and $\|A\|_2=a$, $\quad \|B\|_2=b$. If (p,q) is a (F,G)-antipodal pair and $p \neq q$ then $\|p-q\| \geq \Delta_1(m,n,a,b) := (2^{\frac{3}{2}}NK)^{-D}\ 2^{-12m^2n^2}$ where*

$$K=\max\{\sqrt{13}, 4ma, 4nb\}, \quad N=\binom{3+2m+2n}{5}, \quad D=m^2n^2(3+\frac{4}{m}+\frac{4}{n}).$$

We next show give separation bound between distinct intersection points:

Theorem 2. *Let the curves $A=0$ and $B=0$ be relatively prime. If p,q are distinct points in their intersection then $\|p-q\| \geq \Delta_2(m,n,a,b) := (2^{\frac{3}{2}}NK)^{-D}\ 2^{-12m^2n^2}$ where*

$$K=\max\{\sqrt{13}, m, n\}, \quad N=\binom{3+2m+2n}{5}, \quad D=m^2n^2(3+\frac{4}{m}+\frac{4}{n}).$$

How close can a point p can get to an algebraic curve, without actually being on it? By *(L,ℓ)-bit floating point numbers* (floats, for short) we mean numbers of the form $x=m2^{k-L}$ where $m,k \in \mathbb{Z}$ with $|m| < 2^L$ and $0 \leq k \leq \ell$. Note that $-L \leq \lg|x| < \ell$ (where lg is $\log_2$). We simply say "L-bit floats" for (L,L)-bit floats.

THEOREM 3. *Let $q = (u, v)$ be a point whose coordinates are (L, ℓ)-bit floats, $\ell \geq 2$, and $A(u, v) \neq 0$. If the curve $A = 0$ does not contain a circle centered at q, and p is a point on the curve $A = 0$ then* $\|p - q\| \geq \Delta_3(m, a, L, \ell) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-8m^2}$ *where*

$$K = 2^{L+\ell+1} \max\{2^L, ma)\}, \quad N = \binom{3+2m}{3}, \quad D = m^2(3 + \frac{4}{m}).$$

The requirement that $A = 0$ does not contain a circle centered at q does not affect our application to Bezier curves, because circular arcs are non-Bezier.

The above bounds hold for general algebraic curves. To apply this to a Bezier curve F, we bound its 2-norm bound in terms of its control polygon, $(p_0, \ldots, p_m)$.

THEOREM 4. *Let F be a Bezier curve of degree m, with control points which are L-bit floating point numbers. Then F satisfies an an integer polynomial equation $A(x, y) = 0$ of degree m where*

$$\|A\|_2 \leq (16^L 9^m)^m$$

The Δ-Separation Property. For any $\Delta > 0$, we say F, G have the *Δ-separation property* if for all $p \neq q$, if (p, q) is an antipodal pair of (F, G), or if $p, q \in F \cap G$, then $\|p - q\| > \Delta$. The above theorems give us an explicit bound for Δ. It is important to note that this Δ depends only on the initial input curves F, G; subsequent subdivision of these curves does not change Δ.

3 Elementary Curves

This section introduces the notion of elementary curves. The main result is a complete criterion for non-crossing intersection of elementary curves.

Let f be a bounded, continuously differentiable real functions defined on the interval $[c, d]$ with $c < d$. Its *graph* is the parametrized curve $F = \{F(t) : t \in [c, d]\}$ where $F(t) = (t, f(t)) \in \mathbb{R}^2$. Let $\mathcal{G}[c, d]$ denote the set of all such graphs. Write "$F = F[c, d]$" to indicate that $F \in \mathcal{G}[c, d]$. Call $F(t)$ the *graph parametrization* of F, in contrast to another parametrization to be introduced later. If $F = F[c, d]$, the line segment connecting $F(c)$ and $F(d)$ is called the *base segment* of F. Call F an *elementary curve* if it is the graph of a convex or concave function f. If F lies *above* (resp., *below*) its base segment, we call it *A-elementary* (resp., *B-elementary*).

Let $n_F(t)$ denote the *normal line* through $F(t)$, and let $\theta_F(t) \in [0, \pi)$ be the angle that $n_F(t)$ makes with the positive x-axis. Call $n_F(c)$ and $n_F(d)$ the *end normals* of $F = F[c, d]$. Divide the normal line $n_F(t)$ into two *half-normals*: an upper part $a_F(t)$ and a lower part $b_F(t)$ with $F(t)$ as the common end point of the half-lines. For an A-elementary F, define its *upper swept region* $U(F)$ to be bounded by $a_F(c)$ and $a_F(d)$ and F. See Figure 2. When we extend $a_F(c)$ and $a_F(d)$ until they meet, we obtain a cone $C(F)$ that contains $U(F)$. The important property is that $U(F)$ is "stratified" by the upper half-normals $a_F(t)$ $(t \in [c, d])$

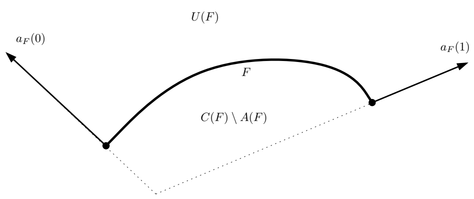


Figure 2: Upper swept region $U(F)$

Elementary Couples. Fix an A-elementary curve F; for simplicity, assume $F \in \mathcal{G}[0, 1]$. Suppose $G \in \mathcal{G}[c, d]$ is another elementary curve. We call (F, G) an *elementary couple* if (i) $G(c) \in a_F(0)$ and $G(d) \in a_F(1)$, and (ii) The entire curve G lies inside the cone $C(F)$. See Figure 3. Note that G can be an A- or B-elementary; accordingly, we call (F, G) an AA- or an AB-couple.

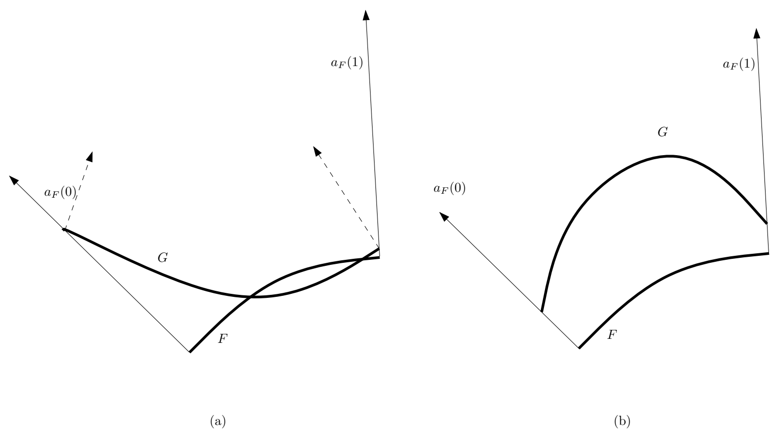


Figure 3: Elementary Couple (F, G): (a) AB-couple, (b) AA-couple

LEMMA 5. *Let $(F, G) \in \mathcal{G}[0,1] \times \mathcal{G}[c,d]$ be an elementary couple. If $G \subseteq U(F)$ then there is a unique continuous function $s : [0,1] \to [c,d]$ such that for all $t \in [c,d]$, $G(s(t))$ lines on the upper half-normal $a_F(t)$.*

COROLLARY 6. *With (F, G) as in the previous lemma, if $G \subseteq U(F)$ then the angular function*

$$\alpha : [0,1] \to (-\pi, \pi) \tag{1}$$

given by $\alpha(t) = \theta_F(t) - \theta_G(s(t))$ is well-defined and continuous.

If the dependence on F, G is to be indicated, we may write $\alpha_{F,G}$ and $s_{F,G}$ for these functions. Our main result for testing non-crossing intersection is as follows:

THEOREM 7 (Non-crossing Intersection Criterion (NIC)). *Let (F, G) be an elementary couple and α given by (1). Suppose F, G have the Δ-separation property and the diameter of $F \cup G$ is less than Δ.*
(i) If $\alpha(0)\alpha(1) \leq 0$ then F and G intersect tangentially, in a unique point.
(ii) If $\alpha(0)\alpha(1) > 0$ then F and G are disjoint.

In the following sections, we need to address two issues: (1) how create the necessary preconditions for applying the NIC above, and (2) device an adaptive method to determine the signs of the angles $\alpha(0)$ and $\alpha(1)$ in the NIC.

4 Intersection With A Line

We give a complete algorithm for intersecting an elementary Bezier curve with a straight line. This special case (and its extensions) is a basic subroutine in the general algorithm. We omit the relatively straightforward algorithm in this abstract. The main innovation is that, because of separation bounds, we can explicitly detect tangential intersection.

An application of this line-curve intersection algorithm is to implement a form of ray-shooting query: for points $q \neq v$, let $Ray(q, v)$ denote the ray originating at q and passing through v. The *Beneath-Beyond Test*, on input (F, q, v), produces one of three outputs: ON if $q \in F$; BEYOND if $Ray(q, v)$ intersects F but $q \notin F$; BENEATH otherwise. This test is used in the Coupling Process (Section 6).

REMARK: A Bezier curve F has a well-known parametrization which we may call the *Bezier parametrization* of F. Although the domain of this parametrization is normally $[0, 1]$, we can generalize it to any interval $[c, d]$ (typically, we have $[c, d] \subseteq [0, 1]$). In case F is also the graph of a function, this parametrization ought to be distinguished from the previous graph parametrization of F. In either Bezier or graph parametrizations, we write "$F = F[c, d]$" to indicate that the parametrization domain is $[c, d]$.

5 Adaptive Sign of the Alpha Angle

The NIC criterion (Theorem 7) requires sign of the angles $\alpha(0)$ and $\alpha(1)$. We now develop an adaptive procedure for this sign determination.

Let F be an elementary curve with the Bezier parametrization $F(t) = (F_1(t), F_2(t))$, and ℓ be a line with parametrization $L(u) = (cu+d, eu+f)$ for constants c, d, e, f. Let ℓ intersect F at the point $F(t^*)$ for some t^*. We want to determine the sign of the angle

$$\alpha(t^*) = \theta_F(t^*) - \theta_\ell.$$

The basic idea is that the sign of $\alpha(t^*)$ is equal to the sign of s_0 where

$$s_0 := cF_1'(t^*) + eF_2'(t^*). \tag{2}$$

From (2), we can develope a lower bound for s_0, and thus adaptively determine the sign of s_0.

6 Coupling Process

In this section, assume that F, G are elementary Bezier curves satisfying the Δ-separation property, and such that $F \cup G$ has diameter less than Δ. This implies that $|F \cap G| \leq 1$. Our goal is to determine whether they intersect by ultimately reducing them to NIC, at least implicitly. This procedure is called the *coupling process*.

OVERVIEW. By way of motivation, note that if we try to directly apply the NIC, we must carefully subdivide F, G until some pair (F', G') of subcurves that forms an elementary couple. This seems difficult to achieve in general. But since the pair (F, G) has diameter less than Δ, we know that they have at most one intersection. Then things are greatly simplified. In this case, we can give an "extended NIC" criterion that is derived from the NIC.

The first step is to detect if F and G have crossing intersection. If so, we are done. So assume otherwise. Up to symmetry, we may assume that F is A-elementary and G is *above* F. *By definition, G is above F means some open interval of G is contained in $U(F)$, the upper swept region of F.* Next, we use binary search to find a t in the domain of $F = F[a, b]$ such that $a_F(t)$ intersects G. This search will always succeed. A slight complication might arise because $a_F(t)$ might intersect G in two points. In any case, we can reduce the search to one or two *half-couples*, i.e., one end normal, but not the other end normal, of F passes through an endpoint of G. Thus half-couples satisfy only "half" the conditions for coupled pairs. Finally, we adapt our NIC to half-couples. This completes our coupling process.

7 Critical Points and Non-Elementary Curves

We now address the issue of non-elementary Bezier curves. A Bezier curve $F[0, 1]$ can be subdivided into finitely many elementary subcurves. The points $F(t)$ at which such subdivisions occur are the "critical points". Figure 4(i) shows a cubic Bezier with a critical point.

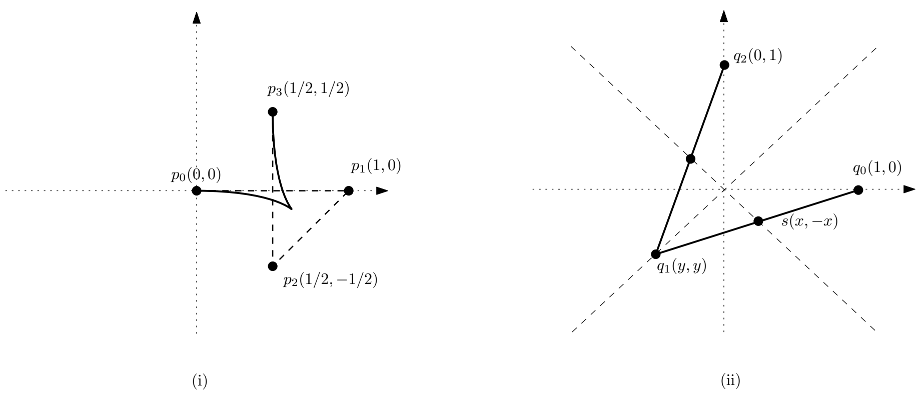


Figure 4: Singular cubic Bezier curve: (i) control polygon, (ii) hodograph

Let $F_x(t), F_y(t)$ be the coordinate functions of F, i.e., $F(t) = (F_x(t), F_y(t))$, and let and $F' = (F_x', F_y')$ denote derivatives with respect to t. A point $F(t)$, $0 < t < 1$, on $F = F[0, 1]$ is *critical* if one of the following conditions hold (cf. Kim and Lee [12]):

- (i) $F(t)$ is *stationary*, i.e., $F'_x(t) = F'_y(t) = 0$;
- (ii) $F(t)$ is *x-extreme*, i.e., $F'_x(t) = 0$, $F'_x(t^-)F'_x(t^+) < 0$ and $F'_y(t) \neq 0$;
- (iii) $F(t)$ is an *inflection point*, i.e., $H_F(t) = 0$ and $H_F(t^+)H_F(t^-) < 0$, where

$$H_F(t) := F'_x(t)F''_y(t) - F'_y(t)F''_x(t).$$

Lemma 8. *If F has no critical points in its relative interior then F is elementary. Conversely, if F is elementary, then it has no x-extreme or inflection points in its relative interior.*

In general, the parameter t at which $F(t)$ is critical is most likely an irrational number. Hence we do not propose to subdivide Bezier curves into elementary parts by subdividing at critical points. We will insist, as usual, on subdividing at parameters t which are bigfloat values. Moreover, we will subdivide the Bezier curve into subcurves that are either elementary or contain at most one critical point. Call a curve containing exactly one critical point an *elementary critical curve*. Finally, we appeal to additional separation bounds based on a new number $\Delta^* > 0$ which we next explain.

In the rest of this section, assume that the control polygons of F and G are $P(F) = (p_0, \ldots, p_m)$ and $P(G) = (q_0, \ldots, q_n)$. Moreover, the coordinates of the p_i's and q_j's are L-bit bigfloats. First, we bound the degree and norms of critical points:

Lemma 9.
(i) If $F(t) = (F_x(t), F_y(t))$ is a stationary or x-extreme point, then $F_x(t)$ and $F_y(t)$ has degree $\leq m-1$ and 2-norm at most $(4^L 9^m m)^m$.
(ii) If $F(t)$ is an inflexion point, then $F(t)$ has degree $\leq 2m-3$ and 2-norm at most $(2m^4 16^L 81^m)^m$.

This gives us:

Corollary 10. *Let p, q be two distinct critical points of F. If $2 \leq m \leq n$ then*

$$|p.x - q.x| \geq \Delta_4 := (16^{m+2} 256^L 81^{2m} m^5)^{-m}.$$

A similar inequality holds for $|p.y - q.y|$.

Next, we generalize Theorem 3 to the case where q is any algebraic point:

Theorem 11. *Let $q = (q.x, q.y)$ where $q.x, q.y$ are algebraic numbers with 2-norm $\leq c$ and degree $\leq d$. Let the polynomial $A(X,Y)$ have 2-norm a and degree m. If the curve $A(X,Y) = 0$ does not contain a circle centered at q then the distance from q to $A = 0$ is at least $\Delta_5(m, a, L, c, d) = (2^{3/2}NK)^{-D} 2^{-12m^2d^2}$ where*

$$K = \max\{\sqrt{13}, 4ma, c\}, \quad N = \binom{5 + 2m + 2d}{5}, \quad D = m^2 d^2 (3 + \frac{4}{m} + \frac{4}{d}).$$

Let $\Delta_6 = \Delta(m, n, L)$ be the minimum separation between the curve G and a critical point p of F, assuming $p \notin G$. This bound comes from plugging Lemma 9 into Theorem 11. Finally, choose Δ^* to be the minimum of the bounds Δ_1 (Theorem 1), Δ_2 (Theorem 2), Δ_4 and Δ_6. If $\mathrm{diam}(F) < \Delta^*$, we may call F a *micro curve*.

It is well-known that the derivative with respect to t is given by

$$F'(t) = m \sum_{i=1}^{m} (p_i - p_{i-1}) B_{m-1}^{i-1}(t) = m \sum_{i=1}^{m} \nabla p_i B_{m-1}^{i-1}(t)$$

where we define $\nabla p_i := p_{i+1} - p_i$. Also, let $\nabla P(F) = (\nabla p_1, \ldots, \nabla p_m)$. Thus, we see that $m \nabla P(F)$ is the control polygon for the curve $F'(t)$, known as the *hodograph* of F. Hence F contains a stationary point iff $F'(t)$ passes through the origin. E.g., Figure 4(ii) shows $\nabla P(F)$ for the cubic Bezier of Figure 4(i); we may check that the hodograph passes through the origin in this case.

Theorem 12 (Stationary Points). *Let $\mathrm{diam}(F) < \Delta_3(m-1, ma, 2)/m$. Then F contains a stationary point iff the convex hull of $\nabla P(F)$ contains the origin $(0,0)$.*

THEOREM 13 (x-Extreme Points). *Let* $\text{diam}(F) < \Delta^*$. *Then* F *contains an* x*-extreme point iff* $(\nabla p_1).x(\nabla p_m).x \leq 0$ *and* $(\nabla p_1).y(\nabla p_m).y > 0$.

If p, q, r are planar points, let $\det(p, q, r)$ be the determinant of the 3×3 matrix whose rows are p^h, q^h, r^h, where $p^h = (p.x, p.x, 1)$. Also let $\det(p, q)$ denote $p.xq.y - p.yq.x$.

THEOREM 14 (Inflexion Points). *Let* $\text{diam}(F) < \Delta^*$. *Then* F *contains an inflexion point iff* $\det(p_0, p_1, p_2)\det(p_{m-2}, p_{m-1}, p_m) < 0$.

The preceding three theorems give us complete criteria for checking if a micro curve is an elementary critical curve.

8 Intersection Algorithm

We now present the global algorithm for intersecting two arbitrary Bezier curves. We design the algorithm so that the generic intersection algorithm (see Introduction) is naturally embedded as the first phase of our algorithm. Simplicity rather than practical efficiency is the aim of the following description.

We have two work queues Q_0 and Q_1. Each queues is just a list of candidate pairs. A candidate pair (F, G) is called a *micro pair* if the diameter of $CH(F) \cup CH(G)$ is less than Δ^*; it is a *macro pair* otherwise. Call Q_0 and Q_1 the *macro queue* and *micro queue*, as they contain macro pairs and micro pairs, respectively. The algorithm operates in two phases.

Macro Phase: This is essentially the generic intersection algorithm on Section 1, operating on Q_0. The key difference is that after we split a pair (F, G) into (F_i, G) $(i = 0, 1)$, we place (F_i, G) into Q_0 or Q_1, depending on whether it is a macro or micro pair. If we did nothing else in the macro phase, we would still be correct. But in practice, we would perform a variety of efficient partial criteria (e.g., a test for transversal intersection).

Micro Phase: Now we operate on Q_1. For each pair (F, G) extracted from Q_1, we first check if F contains a critical point (see Section 8). If so, we can output the pair. Otherwise, we apply the coupling process (see Section 7).

REMARKS. The macro phase is "standard" and easy to implement, while the micro phase treats degenerate and unlikely situations. Thus for nice inputs, it is possible that the micro queue is empty.

9 Open Problems

Beyond solving the problem of complete intersection of Bezier curves, this paper opens up the possibility of achieving adaptive and geometric algorithms with exactness guarantees in related computational problems. This new direction is intrinsically bound to introduction of geometric separation bounds. Clearly, many open problems remain. We list some main ones here:

- Our algorithm for intersecting F, G has this *Antipodal Assumption*: F, G *have finitely many antipodal pairs.* As noted, it generalizes the standard assumption that F, G are relatively prime Clearly, if F is an offset of G, or vice-versa, then the Antipodal Assumption fails. We conjectured that the converse holds. Sungwoo Choi[4] recently proved this conjecture. Choi's theorem implies that a Δ-separation property holds even when the Antipodal Assumption fails. Thus, our Antipodal Assumption is not essential. Unfortunately, we do not know how to bound Δ without the Antipodal Assumption.

- Give a complexity analysis of our algorithm, or some variant thereof. In general, the complexity analysis of most subdivision algorithms is not well understood.

- Implement an adaptive algorithm such as ours, and compare to non-robust subdivision algorithms, or algebraic algorithms.

- Improve our separation bounds. We have somewhat improved bounds which exploit the fact that the curves are Bezier; these have been omitted for brevity. Developing novel geometric techniques for geometric bounds would be of great interest.

[4] Indeed, his proof allows F and G to be real analytic.

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APPENDIX: Proofs

We provide all the missing proofs of the paper. The results are re-stated for the readers' convenience.

THEOREM 1. *Suppose F and G have finitely many antipodal pairs, and are defined by the equations $A = 0$ and $B = 0$ where*

$$m = \deg(A), \quad n = \deg(B), \qquad \|A\|_2 = a, \quad \|B\|_2 = b.$$

If (p, q) is a (F, G)-antipodal pair and $p \neq q$ then $\|p - q\| \geq \Delta_1(m, n, a, b) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-12m^2n^2}$ where

$$K = \max\{\sqrt{13}, 4ma, 4nb\}, \quad N = \binom{3 + 2m + 2n}{5}, \quad D = m^2n^2(3 + \frac{4}{m} + \frac{4}{n}).$$

Proof. By definition of antipodal pairs, $p = (p.x, p.y)$ and $q = (q.x, q.y)$ and the distance $h = \|p - q\|$ satisfy the following system of five-variate polynomial equations:

$$\begin{aligned}
\mathcal{A}_1 &: \quad A(p) = 0,\\
\mathcal{A}_2 &: \quad B(q) = 0,\\
\mathcal{A}_3 &: \quad h^2 - \|p - q\|^2 = 0\\
\mathcal{A}_4 &: \quad \left\langle (\frac{dA}{dy}(p),\ -\frac{dA}{dx}(p)), (p - q) \right\rangle = 0,\\
\mathcal{A}_5 &: \quad \left\langle (\frac{dB}{dy}(q),\ -\frac{dB}{dx}(q)), (p - q) \right\rangle = 0,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. Since A, B has finitely many antipodal pairs, this system is zero dimensional. To apply [21, Theorem 11.45], note that $\|\mathcal{A}_1\|_2 = a$, $\|\mathcal{A}_2\|_2 = b$ and $\|\mathcal{A}_3\|_2 = \sqrt{13}$. Next, $\|A_y\|_2 \leq ma$ where $A_y = dA/dy$. Since $\|A + B\|_2 \leq \|A\|_2 + \|B\|_2$, we get $\|A_y(p) \cdot (p.x - q.x)\|_2 \leq 2ma$. Hence

$$\|\mathcal{A}_4\|_2 = \|A_y(p) \cdot (p.x - q.x) - A_x(q) \cdot (p.y - q.y)\|_2 \leq 4ma. \tag{3}$$

Similarly, $\|\mathcal{A}_5\|_2 \leq 4nb$. Also, $d_1 = d_3 = m, d_2 = d_4 = n, d_5 = 2$. The result follows by plugging into [21, Theorem 11.45]. **Q.E.D.**

THEOREM 2. *Let the curves $A = 0$ and $B = 0$ be relatively prime. If p, q are distinct points in the intersection of these two curves then $\|p - q\| \geq \Delta_2(m, n, a, b) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-12m^2n^2}$ where*

$$K = \max\{\sqrt{13}, m, n\}, \quad N = \binom{3 + 2m + 2n}{5}, \quad D = m^2n^2(3 + \frac{4}{m} + \frac{4}{n}).$$

Proof. Now p and q satisfy the following system of equations:

$$\begin{aligned}
\mathcal{A}_1 &: \quad A(p) = 0,\\
\mathcal{A}_2 &: \quad B(q) = 0,\\
\mathcal{A}_3 &: \quad h^2 - \|p - q\|^2 = 0\\
\mathcal{A}_6 &: \quad A(q) = 0,\\
\mathcal{A}_7 &: \quad B(p) = 0.
\end{aligned}$$

This system is zero-dimensional because A, B are relatively prime. The application of [21, Theorem 11.45] is direct in this case. **Q.E.D.**

THEOREM 3. *Let $q = (u, v)$ be a point whose coordinates are (L, ℓ)-bit floats, $\ell \geq 2$, and $A(u, v) \neq 0$. If the curve $A = 0$ does not contain a circle centered at q, and p is a point on the curve $A = 0$ then*

$$\|p - q\| \geq \Delta_3(m, a, L, \ell) := (2^{\frac{3}{2}} NK)^{-D}\ 2^{-8m^2}$$

where

$$K = 2^{L+\ell+1} \max\{2^L, ma)\}, \quad N = \binom{3 + 2m}{3}, \quad D = m^2(3 + \frac{4}{m}).$$

Proof. We may assume p is a closest point on A to q and $h = \|p - q\|$. Thus three variables $p.x, p.y, h$ satisfy the system,

$$\begin{array}{lcl} \mathcal{A}_1 & : & A(p) = 0, \\ \mathcal{A}_3 & : & h^2 - \|p-q\|^2 = 0, \\ \mathcal{A}_4 & : & \left\langle (\frac{dA}{dy}(p),\ -\frac{dA}{dx}(p)), (p-q) \right\rangle = 0. \end{array}$$

This system is zero-dimensional since the curve $A = 0$ does contain a circle about q. We have $\|\mathcal{A}_1\| = a$, $\|\mathcal{A}_3\| \le \sqrt{1 + 2(2^{2\ell} + 2^{\ell+1} + 1)} < 2^{1+\ell}$ (assuming $\ell \ge 2$). For $\|\mathcal{A}_4\|$, we have $\|A_x(p) \cdot q.x\| < 2^{\ell} ma$ but $\|A_x(p)\| = \|A_x(p) \cdot p.x\| \le ma$ (because $p.x$ is a variable). Hence $\|A_x(p) \cdot (p.x - q.x)\| \le ma(1 + 2^{\ell})$. But instead of $\|\mathcal{A}_3\|$ and $\|\mathcal{A}_4\|$, we use the integer polynomials $2^{2L}\mathcal{A}_3$ and $2^L\mathcal{A}_4$, with $\|2^{2L}\mathcal{A}_3\| < 2^{2L+\ell+1}$ and $\|2^L\mathcal{A}_4\| \le ma(1+2^{\ell})2^L < ma2^{1+\ell+L}$. The result follows by plugging into [21, Theorem 11.45]. **Q.E.D.**

THEOREM 4. *Let F be a Bezier curve of degree m, with control points which are L-bit floating point numbers. Then F satisfies an an integer polynomial equation $A(x, y) = 0$ of degree m where*

$$\|A\|_2 \le (16^L 9^m)^m$$

Proof. The curve $F = (F_1, F_2)$ satisfies the equation $B(x, y) = 0$ where $B(x, y) = \det(M)$ where M is a suitable matrix (see the full paper). The numbers in M are are (L, ℓ)-bit floats, so the matrix $2^L M$ has integer values. Let $A(x, y) = \det(2^L M) = 2^{2mL} B(x, y)$ is an integer matrix. The multivariate version of the Graham-Goldstein lemma [21, Sect. 11.11] says that $\|A\|_2$ is bounded by the product of the 2-norms of each row of the matrix $2^L M_1$ where M_1 is another matrix (see full paper). We can then bound the 2-norm of each row of $2^L M_1$ by $2^{2L}3^m$, Since there are $2m$ rows, the product is $(2^{2L}3^m)^{2m}$, as stated in the theorem. **Q.E.D.**

LEMMA 5. *Let $(F, G) \in \mathcal{G}[0, 1] \times \mathcal{G}[c, d]$ be an elementary couple. If $G \subseteq U(F)$ then there is a unique continuous function $s : [0, 1] \to [c, d]$ such that each point $G(s(t))$ is a G-projection of $F(t)$.*

Proof. Since G lies in the upper swept region $U(F)$, for all $t \in [0, 1]$, $a_F(t)$ intersects G. Our lemma amounts to saying that $a_F(t)$ intersects G exactly once. Clearly $a_F(t)$ intersects G an odd number of times. However, since the region bounded by G and its base segment is convex, no line can intersect G more than twice. Hence G is intersected exactly once by $a_F(t)$, say at the point $G(s(t))$. The continuity of the function $s(t)$ follows from the continuity of the parameterization of $a_F(t)$ and the continuity of the curve G. **Q.E.D.**

THEOREM 7[NON-CROSSING INTERSECTION CRITERION (NIC)].
Let (F, G) be an elementary couple. Suppose F, G have the Δ-separation property and the diameter of $F \cup G$ is less than Δ.
(i) If $\alpha(0)\alpha(1) \le 0$ then F and G intersect tangentially, in a unique point.
(ii) If $\alpha(0)\alpha(1) > 0$ then F and G are disjoint.

Proof. If $F \cap G$ contains two distinct points p and q, then $\|p - q\| \le \operatorname{diam}(F \cup G) \le \Delta$. But this contradicts the Δ-separation property. We conclude that $F \cap G$ is either empty or contains a unique point. Thus, if $F \cap G$ intersects, they intersect tangentially. This also shows that $G \subseteq U(F)$. Therefore, the functions $s : [0, 1] \to [c, d]$ and α in (1) are well-defined and continuous.
(i) Assume $\alpha(0)\alpha(1) \le 0$. So there exists $0 < t < 1$ such that $\alpha(t) = 0$. The pair $(p, q) = (F(t), G(s(t)))$ is therefore antipodal. If $p \ne q$, the Δ-separation property implies $\|p - q\| > \Delta$. But this contradicts our assumption that $F \cup G$ has diameter $\le \Delta$. Thus $p = q$, i.e., F and G intersect tangentially.
(ii) Suppose, by way of contradiction, that F and G intersect at some (necessarily tangential) point p. Write $p = F(t_0)$ for some t_0. Then $\alpha(t_0) = 0$. Since $\alpha(0)\alpha(1) > 0$, we may assume $\alpha(0) > 0$ and $\alpha(1) > 0$ (the other case, $\alpha(0) < 0$ and $\alpha(1) < 0$, is similar). Since p is a tangential intersection and F is assumed to be below G, we see that $\alpha(t_0^-) > 0$ and $\alpha(t_0^+) < 0$. Then, by continuity of the function $\alpha(t)$, there exists $t_1 \in (t_0, 1)$ such that $\alpha(t_1) = 0$. Thus $(F(t_1), G(s(t_1)))$ is an antipodal pair. Again, we argue that $F(t_1) = G(s(t_1))$ must be a tangential intersection of F and G. This contradicts the Δ-separation property. **Q.E.D.**

Lemma 9. *Let $P(F) = (p_0, \ldots, p_m)$ where coordinates of the p_i's are (L, ℓ)-bit floats.*
(i) If $F(t) = (F_x(t), F_y(t))$ is a stationary or x-extreme point, then $F_x(t)$ and $F_y(t)$ has degree $\leq m-1$ and 2-norm at most $(4^L 9^m m)^m$.
(ii) If $F(t)$ is an inflexion point, then $F(t)$ has degree $\leq 2m-3$ and 2-norm at most $(2m^4 16^L 81^m)^m$.

Proof. (i) If $F(t) = (X, Y)$ is stationary or x-extreme, then X satisfies the equation $R(X) = \mathtt{res}_t(F_x(t) - X, F_x'(t))$. If $F_x(t) = \sum_{i=0}^m a_i t^i$, then $R(X)$ is the determinant of the following matrix M',

$$M' = \begin{bmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 - X & \cdots & \\ & a_m & \cdots & a_2 & a_1 & \cdots & \\ & & \ddots & & & \cdots & \\ & & & a_m & a_{m-1} & \cdots & a_0 - X \\ ma_m & (m-1)a_{m-1} & \cdots & 2a_2 & a_1 & & \\ & ma_m & \cdots & 3a_3 & 2a_2 & \cdots & \\ & & \ddots & & & \cdots & \\ & & & ma_m & (m-1)a_{m-1} & \cdots & a_1 \end{bmatrix} \tag{4}$$

Similar to the proof of Theorem 4, we consider the matrix

$$M_2 = \begin{bmatrix} 2^{L+m}\binom{m}{0} & 2^{L+m-1}\binom{m}{1} & \cdots & 2^{L+1}m & 2^L+1 & & \\ & 2^{L+m}\binom{m}{0} & \cdots & 2^{L+2}\binom{m}{2} & 2^{L+1}m & & \\ & & \ddots & & & \ddots & \\ & & & 2^{L+m}\binom{m}{0} & 2^{L+m-1}\binom{m}{1} & \cdots & 2^L+1 \\ m2^{L+m}\binom{m}{0} & (m-1)2^{L+m-1}\binom{m}{1} & \cdots & 2.2^{L+2}\binom{m}{2} & 2^{L+1}m & & \\ & m2^{L+m}\binom{m}{0} & \cdots & 3.2^{L+3}\binom{m}{3} & 2.2^{L+2}\binom{m}{2} & & \\ & & \ddots & & & \ddots & \\ & & & m2^{L+m}\binom{m}{0} & (m-1)2^{L+m-1}\binom{m}{1} & \cdots & 2^{L+1}m \end{bmatrix}. \tag{5}$$

The first $m-1$ rows of M_2 have 2-norms $2^L 3^m$ (see full paper). The last m rows of M_2 have 2-norms $\leq m2^L3^m$. Applying the generalized Hadamard bound, $\|R(X)\| \leq (2^L3^m)^{m-1}(m2^L3^m)^m$, which is $2^{(2m-1)L}3^{2m^2-m}m^m < (4^L9^m m)^m$, as we claim.
(ii) We proceed as in (i), but the matrix M' now has $2m-3$ rows of the coefficients of $F_x(t) - X$, and has m rows of the coefficients of $F_x'(t)F_y''(t) - F_x''(t)F_y'(t)$. Thus the degree of $R(X) = \det(M')$ is $2m-3$. We have $\|F_x'(t)\| \leq m2^L3^m$, $\|F_y''(t)\| \leq m^2 2^L 3^m$. Hence $\|F_x'(t)F_y''(t)\| \leq m(m2^L3^m)(m^22^L3^m) = m^4 4^L 9^m$ using the fact that $\|AB\| \leq m\|A\| \cdot \|B\|$ where A, B have degrees $\leq m$. Hence $\|F_x'(t)F_y''(t) - F_x''(t)F_y'(t) \leq 2m^4 4^L 9^m$. Thus $\|R(X)\| \leq (2^L3^m)^{2m-3}(2m^44^L9^m)^m < (2m^4 16^L 81^m)^m$. **Q.E.D.**

Corollary 10. *Suppose p, q are two distinct critical points of F. If $2 \leq m \leq n$ then*

$$|p.x - q.x| \geq \Delta_5 := (16^{m+2} 256^L 81^{2m} m^5)^{-m}.$$

A similar inequality holds for $|p.y - q.y|$.

Proof. By application of the previous lemma. See full paper. **Q.E.D.**

Theorem 11. *Let $q = (q.x, q.y)$ where $q.x, q.y$ are algebraic numbers with 2-norm $\leq c$ and degree $\leq d$. If $A = 0$ does not contain a circle centered at q then the distance from q to A is at least $\Delta_4(m, a, L, c, d) = (2^{3/2}NK)^{-D}2^{-12m^2d^2}$ where*

$$K = \max\{\sqrt{13}, 4ma, c\}, \quad N = \binom{5+2m+2d}{5}, \quad D = m^2d^2(3 + \frac{4}{m} + \frac{4}{d}).$$

Proof. We may assume p is a closest point on A to q and $h = \|p - q\|$. Thus q, p, h satisfy the system,

$$\begin{aligned}
\mathcal{A}_1 &: \quad A(p) = 0, \\
\mathcal{A}_3 &: \quad h^2 - \|p - q\|^2 = 0, \\
\mathcal{A}_4 &: \quad \left\langle (\frac{dA}{dy}(p),\ -\frac{dA}{dx}(p)), (p - q) \right\rangle = 0, \\
\mathcal{A}_9 &: \quad P(q.x) = 0, \\
\mathcal{A}_{10} &: \quad Q(q.y) = 0,
\end{aligned}$$

where $P(X), Q(X)$ are polynomials of degree $\leq d$ and 2-norm $\leq c$ satisfied by $q.x$ and $q.y$, respectively. This system is zero-dimensional since $A = 0$ does contain a circle centered at q. We have $\|\mathcal{A}_1\|_2 = a$, $\|\mathcal{A}_3\|_2 \leq \sqrt{13}$, $\|\mathcal{A}_4\|_2 \leq 4ma$, $\|\mathcal{A}_9\|_2 \leq c$, and $\|\mathcal{A}_{10}\|_2 \leq c$. The result follows by plugging into [21, Theorem 11.45]. **Q.E.D.**

THEOREM 12. *Let* $\operatorname{diam}(F) < \Delta_3(m-1, ma, 2)/m$. *Then F contains a stationary point iff the convex hull of $\nabla P(F)$ contains the origin* $(0, 0)$.

Proof. If F contains a stationary point, then clearly its hodograph passes through $(0, 0)$. Hence the convex hull of $m\nabla P(F)$ contains $(0, 0)$. Equivalently, the convex hull of $\nabla P(F)$ contains $(0, 0)$. Conversely, suppose F does not contain a stationary point. Note that F' has degree $m - 1$ and $\|F'\|_2 \leq m\|F\|_2 = ma$. By Theorem 3, the distance from $(0, 0)$ to the curve F' is at least $\Delta_3(m-1, ma, 2)$ since $(0, 0)$ is a 2-bit float. Thus the distance from $(0, 0)$ to F'/m is at least $\Delta_3(m-1, ma, 2)/m$. On the other hand, the diameter of $\nabla P(F)$ is at most the diameter of F, i.e., less than $\Delta_3(m-1, ma, 2)/m$. This means $(0, 0)$ could not be inside the convex hull of $\nabla P(F)$. **Q.E.D.**

THEOREM 13. *Suppose* $\operatorname{diam}(F) < \Delta^*$. *Then F contains an x-extreme point iff $(\nabla p_1).x(\nabla p_m).x \leq 0$ and $(\nabla p_1).y(\nabla p_m).y > 0$.*

Proof. If $(\nabla p_1).x(\nabla p_m).x \leq 0$ and $(\nabla p_1).y(\nabla p_m).y > 0$ then clearly F contains an x-extreme point. Conversely, suppose $F(t)$ is an x-extreme point, then we know that both $F[0, t]$ and $F[t, 1]$ lies strictly to one side of the vertical line through $F_x(t)$. Since $\operatorname{diam}(F)$ is small, there are no other critical points, not even y-extreme points. Thus, the two subcurves are elementary. Now it is clear that the first and last edges ∇p_1 and ∇p_m on the control polygon $\nabla P(F)$ has the claimed properties. **Q.E.D.**

THEOREM 14. *Suppose* $\operatorname{diam}(F) < \Delta^*$. *Then F contains an inflexion point iff* $\det(p_0, p_1, p_2) \det(p_{m-2}, p_{m-1}, p_m) < 0$.

Proof. We make the connection between $\det(p_0, p_1, p_2)$ and inflection points. Let $H(t) = F'_x(t)F''_y(t) - F''_x(t)F'_y(t)$. Note that $F'(0) = p_1 - p_0$ and $F''(0) = p_2 - 2p_1 + p_0$. Hence $H(0) = \det(p_1 - p_0, p_2 - 2p_1 + p_0) = \det(p_1 - p_0, p_2 - p_1)$. $H(0) = -\det(p_0 - p_1, p_2 - p_1) = \det(p_1, p_0, p_2) = -\det(p_0, p_1, p_2)$. Similarly, $H(1) = -\det(p_{m-2}, p_{m-1}, p_m)$. Therefore, $\det(p_0, p_1, p_2) \det(p_{m-2}, p_{m-1}, p_m) < 0$, there must be some t such that $H(t) = 0$, and this would be an inflection point. Conversely, if $F(t)$ is an inflection point, then $H(t^-)H(t^+) < 0$. Since $\operatorname{diam}(F)$ is small, there are no other inflection points and therefore we conclude that $H(0)H(1) < 0$. **Q.E.D.**