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VISUAL GEOMETRY, COMPUTER GRAPHICS AND THEOREMS OF PERCEIVED TYPE

BY

PHILIP J. DAVIS

> The inborn capacity to understand through the eyes has been put to sleep and must be reawakened.
>
> Rudolph Arnheim, *Art and Visual Perception*

> This is the Visual Generation, *New York Magazine*, May 28, 1973

ABSTRACT. The author presents arguments in favor of the following two positions.

(1) Visual geometry ought to be restored to an honored position in mathematics. Computer graphics comprising animation and color offers the possibility of going far beyond conventional drawings.

(2) The classical notions of what constitutes a mathematical theorem or a mathematical truth need broadening. These notions should be recast so as to include a variety of phenomena which are systematically generated, perceived by the senses and interpreted by the brain.

1. Introduction. It is incumbent upon each mathematician and each generation of mathematicians to formulate a definition of mathematics. Granted that this is a hopeless task and also granted that no consensus can ever be reached and that all formulations are evanescent, the exercise is useful in that it compels the mathematician to think through where he believes his discipline places him in the world of experience and thought. It is also useful in that it provides future historians of science with a picture of how the past regarded itself.

A popular contemporary mathematical dictionary (James & James) defines mathematics as "the logical study of shape, arrangement, and quantity." This definition, unsophisticated though it may be, coincides with the popular view of what the subject is all about. A definition which goes back a hundred years to the writings of C. S. Peirce and which emphasizes the logical aspect of the subject

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tells us that mathematics is the science of drawing necessary conclusions. An up-date of the C. S. Peirce definition might be that mathematics is the workings-out of a universal Turing machine. Other contemporaries might talk of mathematics in terms of logical transformations, grammars, invariants, or in terms of structuralism. At the turn of the century, Bertrand Russell, focusing attention on the varieties of external interpretation that one and the same mathematical structure might carry, wisecracked that "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

2. **Theorems.** Let us open the average book on mathematics. What kind of thing do we find in it? Well, first of all, we find definitions, theorems and proofs. These constitute the Trinity of contemporary mathematizing and form the hard core of the book. But there may be other things in the book. There may be discussions of the definitions, theorems and proofs. The discussions may be historical or bibliographical or methodological or aesthetic. There may be judgments or indications of where the core material fits in with other mathematics or with other aspects of the universe. Of course, if the book is on applied mathematics then the percentage of this last type of material might very well (but not necessarily) go up.

A book on mathematics might also contain graphical or visual material. This differs from what is found in normal mathematical sentences written in the normal mathematical font of symbols. These are put in by way of elucidation or clarification but are never (by purists anyway) thought to constitute an adequate mathematical proof of anything. There is a widespread feeling that proper proofs can only be carried out in the format canonized by Euclid. This is the mathematical parallel to the feeling of mediaeval theologians that the spirit is pure while the flesh is corrupt; mathematicians are notorious puritans in their own peapatch.

Despite the theorem-olatry of the past several hundred years of mathematics, there is surprisingly little theorem-ology. What is a theorem? How does it operate? What is it for? James & James says that a theorem is a general conclusion proved or proposed to be proved on the basis of certain given assumptions. A somewhat more sophisticated definition of a theorem, adapted from a current book on mathematical logic, goes along the following inductive lines. The axioms of a formal system F are theorems. If all the hypotheses of a rule of F are theorems then the conclusion of the rule is a theorem. The axioms, i. e., the primitive statements or assumptions, are representable as certain strings of atomic symbols. The theorems are representable as certain other strings of atomic symbols. Proving is the process of passing from an axiom string to a theorem string by a finite sequence of allowable elementary transformations. To verify that the next man's putative theorem is, in fact, the theorem he claims it to be is merely to verify that the sequence of string transformations is in order. The whole thing is in principle perfectly mechanizable.

Now that we know what a theorem is, what can we say about theorems in a general way *apart from comments on specific theorems*? Books on mathematics or metamathematics say very little. One authority I consulted told me categorically that the only assertion one would want to make is that a theorem is either true or false (if it is proposed to be proved) or true (if it has been proved), in which case there is no reason at all for mentioning the fact. This is an extreme point of view.

In the mathematical sense one can, e.g., talk about the range of a theorem (whether or not it applies to anything at all) or the generality of a theorem. There is even a recent mathematical theory of the depth of a theorem.

In the extramathematical sense, one can talk about the utility of a theorem, the beauty of a theorem, the popularity of a theorem, the revolutionary quality of a theorem, etc. (A recent mathematical article contains the following sentence: "Theorem 7.4. Hilbert's Tenth Problem is unsolvable!" The exclamation point here is not mathematical notation. Presumably the author is trying to convey to the reader his own sense of elation or surprise at the result.) One can even talk about the possible evolution of the notion of a theorem and not treat the thing as if it were a fixed concept frozen for all future time. There is obviously much that can be said about the theorems in general, although I have the distinct impression that there is a dearth of such talk in the mathematical literature.

3. The visual image. In the early 19th century the greatest accolade that could have been accorded one mathematician by another was to have called him a "geometer." The irony is that at the very time this honorific was in use, the reasons which called it into being were themselves almost dead. The title was a splendiferous archaism.

What are some of the reasons for the decline of the visual image in mathematics?

(1) The tremendous impact of Descartes' *Discours de la Méthode* (1637) by which geometry was reduced to algebra; also the subsequent turnabout wherein the medium (algebra) became the message (algebraic geometry).

(2) The collapse, in the early 19th century, of the view, derived largely

from limited sense experience, that Euclidean geometry has a priori truth for the universe; that it is *the* model for physical space.

(3) The incompleteness of the logical structure of Euclidean geometry as discovered in the 19th century and as corrected by Hilbert an others (Euclid debugged).

(4) The limitations of two or three physical dimensions hich form the natural backdrop for visual geometry.

(5) The limitations of the visual ground field over which visual geometry is built as opposed to the great generality that is possible abstractly (finite geometries, complex geometries, etc.) when geometry has been algebraicized.

(6) The limitations of the eye in its perception of mathematical "truths" (e.g., the existence of continuous everywhere nondifferentiable functions, optical illusions, suggestive but misleading special cases, etc.).

These perceptions and historical developments have been of overwhelming importance. The visual image went into a tailspin from which it has not yet recovered. The little boy played with matches and got his fingers burned, so civilization abolished all the matches instead of training the boy. It is time to restore the image. The image has much that is new to offer. It can be done through the medium of computer graphics.

4. What computer graphics offers. By an interactive computer graphics installation I shall mean—leaving the jargon of computer hardware aside—a television tube hooked up to a computer. This combination is to be addressable by typewriter, lightpen, joystick, control dials or other analogue devices and the whole is to be backed up by sufficient graphics hardware and software that the programming of visual images of the ordinary mathematical variety can be carried out as easily as, say, computation in some well-known interactive languages such as BASIC or APL. Admittedly, at the time of writing (August 1973), this combination is available at very few university computer centers. The availability of really advanced graphical facilities such as color tubes, sketchpads, opportunities for computer animation are correspondingly much more limited.

What are some of the mathematical potentialities of computer graphics?

(1) Insight into situations of a mathematically conventional but possibly difficult nature.

(2) Computer-generated art.

(3) Creation of mathematical theorems of "perceived type."

I shall discuss these points separately.

5. Generation of conventional theorems via graphics. A computer graphics installation can, of course, be used to illustrate a wide variety of principles of elementary mathematics for purposes of instruction. This can be of enormous importance for didactics. Much effort has been spent in the past decade to illustrate various principles of calculus, probability and statistics, mechanics, higher-dimensional geometry, etc. by means of the scope. There have also been illustrations of more advanced things such as mappings induced by analytic functions of a complex variable, certain geometrical principles occurring in the theory of functions of two complex variables such as Bergman's distinguished boundaries, solutions of partial differential equations animated according to the time parameter, the solution of the many-body problems assuming general force laws, studies of singularities of algebraic curves, iterations of nonlinear transformations, projections of higher-dimensional objects and transformations of these objects, etc.

Graphical displays can suggest theorems or truths which the mathematician might then attempt to prove in a conventional way. Conventional proofs of what has in fact been observed may be extremely difficult to obtain. For example, in celestial mechanics one renowned authority (Carl Ludwig Siegel) writes off the possibility of analytic progress in certain areas of the subject. Does this mean that there can be no knowledge in such an area? Nonsense, as any practical man would tell you. Students fooling around with, e.g., orbits in the many-body problem that have been graphically displayed have found periodic solutions whose existence defies our keenest analytical analysis. A systematic graphical exploration of certain topics might lead to a consistent, extensive, interconnected, interesting and important corpus of material which might not have been available through research that is pursued according to the conventional mathematical methodology. To an experimental scientist this point of view is, of course, old stuff. To a mathematical conservative, this might be magnificent *mais ce n'est pas la guerre.*

In the investigation by computer graphics of conventional mathematical problems one also moves to knowledge or experience which I shall call here, for lack of a better term, "theorems or structures of perceived type."

This type of knowledge might be perceived by the individual as a gut feeling. To quote one team of investigators (Banchoff and Strauss):

> Using control dials, joysticks, and other analog input devices, a mathematician can get immediate portrayal of the geometric effect of continuously varying parameters. He also has finger-tip control over the current values and rate of changes of these parameters, encouraging the development of a visceral feeling for the effect of these parameter variations.

This visceral feeling might be of importance in experiences ranging from the highly practical training of aircraft pilots via simulated cockpits to space intuition that might be achieved by moving around objects computer-wise in a higher-dimensional space. The pilot-in-training is learning a body of theorems of "perceived type." There is obviously a close relation here to kinematics and kinaesthetics.

6. **Computer-generated art and animated films.** The variety of output devices in a computer center offers the possibility of computer-generated art and films. The line printer, the plotter, and the scope have all been used; masters of the craft have produced pieces and effects which are nothing short of amazing. At the very lowest level, the computer-driven output device can be regarded as a new medium with characteristic effects, similar to technological processes such as acrylics of silk-screening. Each process has a certain scope and certain strengths and weaknesses. At the very lowest level, computer art might attempt to imitate certain effects obtained by conventional art media. At a higher level, the unique nature of the medium comes into play and one obtains effects which might be difficult or pointless, if not impossible, to create conventionally. At a still higher level, the relationship between the visual effects and the language used to create the effects comes into great prominence. One might even posit an advanced Descartean stage (I have not seen it yet) in which the language turns about and supersedes the visual effect.

Computer art can be carried out for sensual or craft pleasure, for amusement, for aesthetic values, for shock, for practice in programming, or as an adjunct to mathematical investigations of conventional type. It can be carried out for *l'art pour l'art,* or simply because the output devices are there.

I recall seeing Abraham Lincoln's face produced by computer-driven typewriters in the late 40's, done as a demonstration piece for a laboratory "open house." But serious computer art is only about ten years old. It is too early for an iconography to have developed which might lend value independently of the image *qua* image.

On the purely utilitarian level, computer art moves imperceptibly towards commercial art (I have seen some very beautiful stamps with a computer art figure on them) and towards the design of commercial and industrial shapes and thence into the automatic fabrication of such shapes. As such, computer art becomes a genuine topic of applied mathematics.

One paradigm for the production of computer art goes along the following lines. Starting from some mathematical scheme (spirographic geometry, number theory or, for that matter, any illustrable mathematical theory, or digitalized conventional pictures) and employing certain mathematical transformations with considerable parametric freedom and possible even built-in "randomness," one produces output. This output is then monitored and accepted or rejected on the basis of some internalized criterion. This leads to parameter adjustments, program modifications, etc., and a new generation of outputs.

The resulting piece of computer art may very well be accidental or serendipitous in the sense that the artist-programmer may not be able to foresee in advance precisely what will be created, but at the same time it represents a tight control by the artist-programmer over his work in that the output results from a fixed program and is reproducible, given the parameters and the initializing values in the case of a randomizer.

The field of computer art appears to me to be wide open; at the same time, as with all seedlings, its future is moot. I personally feel that the potentialities are much greater with animated images than with static images. I should have liked to have included some instances of animation with my illustrative material, but obviously cannot.

7. **Creation of mathematical theorems of perceived type.** I come to the nub of my argument. The Cartesian program–i.e., the algebraicization of geometry and of vast portions of mathematics with geometric content–represents a major revolution in the history of mathematics. Nonetheless, as with all revolutions, a certain loss was incurred when the culture of the *ancien régime* was undermined. The algebraicization of geometry must be regarded as a prosthetic device of great power which maps certain aspects of space into analytical symbols. The blind might be unabled to manipulate space through the instrumentality of these symbols, but since one channel of sense experience is denied to the blind, one feels that a corresponding fraction of the mathematical world must be lost to them. Political democracy does not require that all men savor the universe at identical levels of intensity. [1]

The analytic program, then, is a prosthetic device, acting as a surrogate for the "real thing." The unit circle as perceived by the eye and acted on by the brain is a very different thing from the symbol string $x^2 + y^2 = 1$. The two sensations are interrelated and each can be considered as an "analytical continuation" of the other and each is on an even intellectual basis with the other. The eye "perceives" many things about the circle which may be difficult or impossible to mimic via the analytic symbols. The visual circle is the carrier of an unlimited number of theorems which are instantly perceived. The perceived gestalt of the

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[1] Attempts to translate theorems in one sense perception to theorems in a second sense perception can lead to analytical mathematics of the highest interest and difficulty. See M. Kac [2].

circle is at once the formulation of these theorems and their proof. (In connection with some work on approximation theory, I once had to demonstrate the visually-obvious theorem that a circle cannot be filled up by a finite number of nonoverlapping circles of smaller radius. I was lucky in that I found a simple analytic proof. What if I had been confronted with something as difficult as the Jordan curve theorem and my analytic standards were high?)

The regular isocahedron sitting on my desk and perceived as a three-dimensional object is a different thing from a list of the coordinates of its vertices. It is a different thing from the abstract group of rotations that move it into itself. It is a gestalt, complete in itself, self-vindicating, rejoicing in its uniqueness, the carrier for an unlimited number of "theorems of perceived type" that are grasped or intuited and *do not even have to be stated.*

Chilton's Drawing of {5, 3, 3}

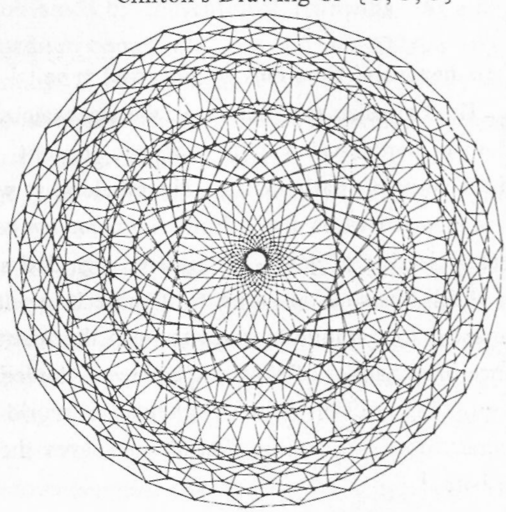

FIGURE 1

(From "Introduction to Geometry", H. S. M. Coxeter, Wiley, 1961)

Take a look at Figure 1. This is a two-dimensional projection of the polytope with Schläfli symbol {5, 3, 3}. The first thing about the figure that catches my eye is that it seems to split up into a number of consecutive rings (at least seven), each of which has a different mesh-pattern or density characteristic. You may seek a conventional proof of this fact if you like, having previously introduced a satisfactory definition of what a mesh-pattern means. I could probably go on for an hour telling what I saw in this fairly complicated image and exceed by far the number of formal theorems in the literature about the polytope $\{5, 3, 3\}$.

Take a look at Figure 2. This was obtained by computing the function $\{|x^3 + y^3| \div 10\} \bmod 3$, and plotting the resultant values 0, 1, 2 as three grey

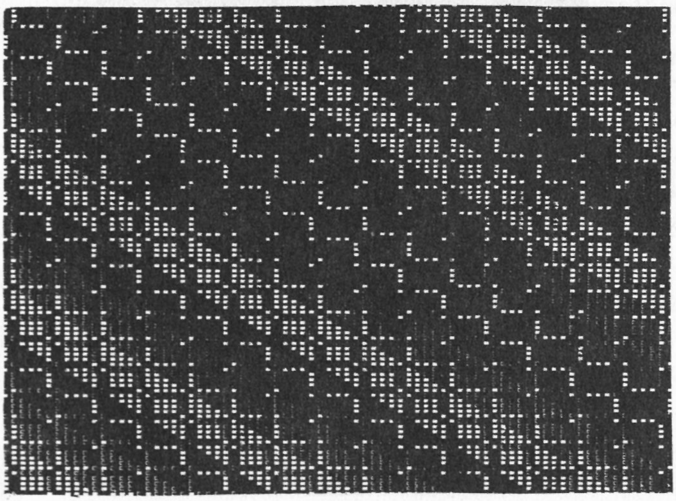

FIGURE 2

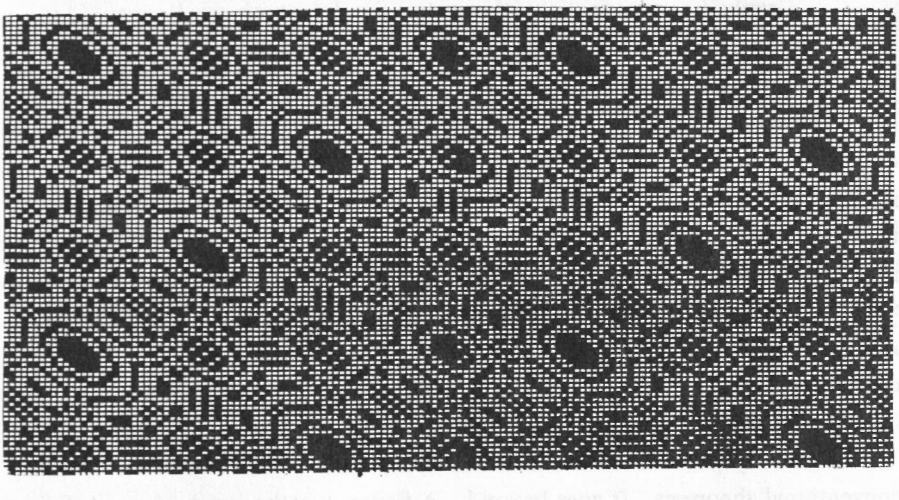

FIGURE 3

Patterns defined by black, grey, and white areas determined by reducing a mathematical function modulo 3. With x, y origin at the center, the top picture is from $\{|x^3 + y^3| \div 10\} \bmod 3$ the bottom one from $[(x^2 + xy + y^2) \div 30] \bmod 3$. (Courtesy K. Knowlton, T. Rainer.)

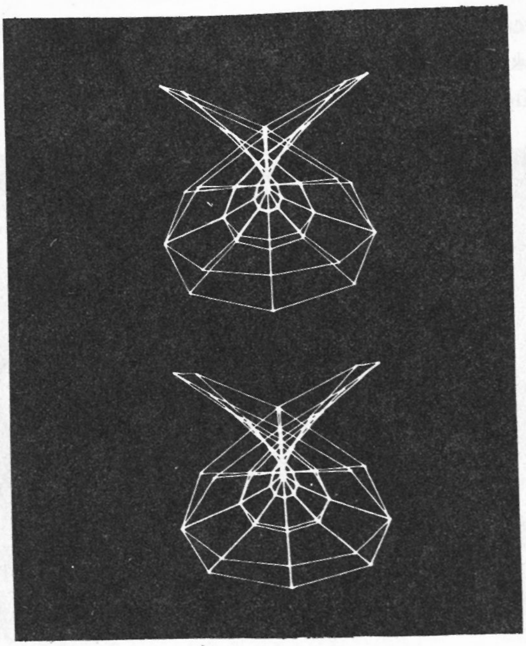

FIGURE 4

Stereo Pair of $(z^2, z^3 + \epsilon z)$

Graphed as a projection of a surface in E^4, with small negative ϵ. (Banchoff & Strauss)

levels; white, grey, black. The resulting herringbone figure of fairly intricate texture, with its symmetries, periodicities, accidentals, is certainly part of the theory of cubic residues. One might formulate theorems in number theory to account for what one sees. Some might prove difficult, others trivial. On another level, though, there is no need for this reduction. One sees what one sees: a characteristic pattern which is the carrier of a mélange of number theorems of the conventional type, but which has an integrity of its own and does not require conventional interpretation. We are seeing a theorem of "the perceived type." In view of the possibility of such figures, the paucity of geometrical illustrations in books on number theory is absolutely incredible.

Again my point is not–what we all know–that a good figure can suggest conventional theorems. It goes beyond. A figure, together with its rule of generation, is automatically and without further ado a definition, theorem and proof of "the perceived type."

8. What is mathematics? I return at last to the question in the introductory paragraph. I would suggest that mathematics is the program, the execution, the output; the gestalt perceived and interpreted in the light of experience and tradition. Analytical mathematics can be accommodated into this scheme by identifying program with proof. Within the methodology of conventional mathematics, an output is very often guessed or intuited and the program (proof) is sought. In computer graphics the output is self-vindicating.

Though I am arguing that the concept of mathematics should be broadened, one must of course draw boundaries somewhere. Does a toy kaleidoscope generate theorems of perceived type? Is, for example, a loaf of bread put out by an automated bakery and generated from raw materials by means of a recipe a theorem of the perceived type? Additional considerations will obviously enter and provide limitations.

Given the stochastic or fuzzy nature of the universe, with the possibilities of erroneous programs, erroneous execution, round-off error, etc., the theorems of the perceived type must be regarded as having validity only in a probabilistic sense. However, I believe that conventional "hand-crafted" theorems likewise have only probabilistic validity. This point of view was explained in some detail in Davis **[1]**.

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Supplementary remarks.

§3. *The role of the visual image in mathematical discovery.* About the turn of the century, Poincaré divided mathematicians into two types: the geometers and the analysts. Geometers think about mathematical objects in pictures while analysts operate with formulas. Occasionally the same results have been obtained independently. Thus both Riemann (the geometer) and Weierstrass (the analyst) developed a theory of integrals of algebraic functions. In more recent times, the Bergman-Weil generalization of Cauchy's formula to several complex variables was probably developed by Bergman from geometric and by Weil from analytic considerations.

However, it appears to me that whatever the path taken in these investigations, the goal and the final formulation was essentially analytic. I look forward to a situation where the geometric element becomes more independent and marches less to the tune of the analytic.

René Thom **[11]** argues for the restoration of geometry from a pedagogical point of view. He puts forward the claim that "any question in algebra is either trivial or impossible to solve. By contrast the classic problems of geometry present a wide range of challenges."

The historical problem of the decline of the visual image in mathematics is one that is worthy of serious study. A mathematical Gibbon should undertake it. I do not believe it is a phenomenon limited to mathematics, but extends (even!) to the graphic arts. It is related to a general tendency of breaking up and recombination (e. g., cubist art) which emerged in the industrial age and has continued up through the current post-industrial age.

Discussions of this historical tendency with R. B. Kelman put him in mind of a pathological condition of dyslexia attendant upon some sorts of brain damage. This appears to be due in part to improper communication between brain areas. The spatial (geometric) functions may be performed in one area while the symbolic (algebraic) functions may be performed in another area. Within the mathematical culture we seem to be dealing with a dysfunction which has largely shut off the operation of the "geometrizing" area.

§6. *Relationship between computer language and visual effect.* This can be profound, as every language or collection of subroutines sets up limitations. *Example.* Two fonts of capital letters were created by two almost identical processes. In the first process, however, trigonometric interpolation was used while the second process used interpolation by cubic splines. The stylistic differences were sufficiently strong to be picked up by the eye. *Example.* A conventionally-trained artist produced a recognizable portrait of a colleague with a CALCOMP plotter using a certain subroutine that was available. What, I asked him, distinguished the result from a freehand drawing? He replied that the CALCOMP was producing strokes that were impossible by ordinary wrist and arm movements, so that the overall effect was different.

§7. Admittedly, the idea of the "theorem of perceived type" is somewhat vague and mysterious. Perhaps an analogy will dispel some of the fog by showing that the mysterious is, in fact, a commonplace experience within the psychology of perception.

The score of Mozart's Symphony No. 40 is a program. When the score is translated into sound by an orchestra playing in a standardized way it becomes the G Minor Symphony as commonly understood. The score and the music, though not physically identical, are aspects of the same thing. This symphony with its own musical themes, texture, tonalities, nuances, rhythms and patterns is unique. It is identifiable by many people. It is fairly stable (a few bad notes here and there will not make much difference), but nonetheless it is an aleatory process operating at a reasonably high probability level. It is capable of having judgments of various sorts applied to it. It is capable of having mathematical statements made about it, e.g., the average pitch is such and such, or certain parts are invariant under time translations. But it is self-vindicating in the sense that it needs no further intellectual amplification or retranslation into other non-aural modes in order to establish its integrity or to be appreciated. The G Minor Symphony represents a unique experience and, stretching a point, the passage from the score to the music might be said to constitute a "theorem of the perceived type."

It is interesting to note that the word "theorem" is derived from the verb "θεωρειν" which means "to look at."

The point of view advocated here is related to that developed by M. Polanyi in his book *Personal knowledge.*

§8. Apropos of the question of whether an automated loaf of bread or Mozart's Symphony No. 40 is a "theorem," in an essay written a number of years ago, James Bryant Conant once posed the problem of whether cooking is a branch of chemistry and, if it is, why is it not taught at Harvard. Conant's conclusion was that this is largely a matter of convention.

If one considers attempts to create computer music (admittedly not very successful, though Mozart himself was one of the first to write on the topic), then one may be more prone to accept the G Minor Symphony as defining a mathematical theorem or structure.

Probabilistic validity. The point made in Davis [1] is, briefly, that the verification of a mathematical proof requires examination of long symbol strings to see whether they follow the canons of mathematical deduction. As verification errors are inevitable, and are part of the real world, even within simple arithmetic, the theorems which emerge have only probabilistic validity. The longer the strings are the greater is the likelihood of error.

One sympathetic but traditional correspondent, reasserting the position of Platonic mathematics, writes:

> Absolute, universally accepted proof is an ideal and one which we may never attain. But ideals keep us striving in a definite direction. Justice is an ideal which is certainly not realized in our society but it does have value.

To this I add that prudent societies, while yearning for ideal justice, do well to provide themselves with courts of law which dispense pragmatic justice. It is therefore misleading to promulgate Platonism as the sole philosophy operative within mathematics. On this and on probabilistic validity see R. Thom [**11**].

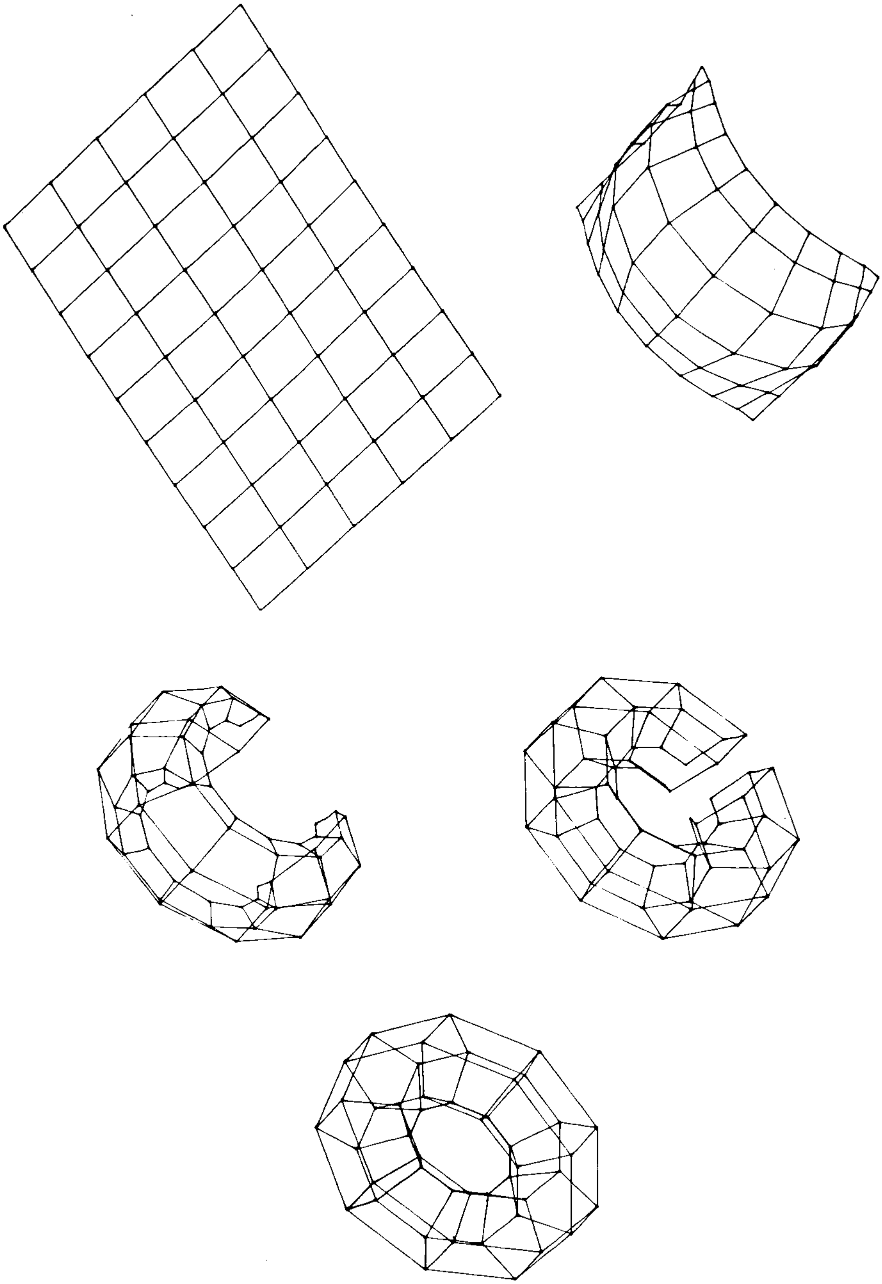

FIGURE 5

Stills from a computer-made movie: wrapping a rectangle to form a torus. (Courtesy T. Banchoff and C. M. Strauss.)

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