Preserving Geometric Properties in Reconstructing Regions from Internal and Nearby Points

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Abstract

The problem of reconstructing a region from a set of sample points is common in many geometric applications, including computer vision. It is very helpful to be able to guarantee that the reconstructed region "approximates" the true region, in some sense of approximation. In this paper, we study a general category of reconstruction methods, called "locally-based reconstruction functions of radius α," and we consider two specific functions, $\mathcal{J}_\alpha(S)$ and $\mathcal{F}_\alpha(S)$, within this category. We consider a sample S, either finite or infinite, that is specified to be within a given Hausdorff distance δ of the true region R, and we prove a number of theorems which give conditions on R, δ that are sufficient to guarantee that the reconstructed region is an approximation of the true region. Specifically, we prove:

1. For any R, if F is any locally-based reconstruction method of radius α where α is small enough, and if the Hausdorff distance from S to R is small enough, then the dual-Hausdorff distance from $F(S)$ to R, the Hausdorff distance between their boundaries, and the measure of their symmetric difference are guaranteed to be small.
2. If R is r-regular, then for any $\epsilon, \phi > 0$, if α is small enough, and the Hausdorff distance from S to R is small enough, then each of the regions $\mathcal{J}_\alpha(S)$ and $\mathcal{F}_\alpha(S)$ is ϵ-similar to R and is an (ϵ, ϕ)-approximation in tangent of R.

Keywords: Shape reconstruction, locally-based reconstruction method, Hausdorff distance, ϵ-similar, approximation in tangent.

1 Introduction

The problem of inferring a geometric shape from a sample of points is a significant one in a number of applications, including computer vision, computer-aided manufacturing, geographical information systems, and robotic manipulation. The problem, as it stands, is obviously underdefined, and so different applications and circumstances may call for different solutions.

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One major desideratum of any reconstruction method is that, if a sample S is drawn from some underlying (presumably unknown) region R, then the reconstruction $F(S)$ should resemble or approximate R. Obviously, this cannot always be done (e.g. if S consists of a single point), but it seems reasonable to hope that, if S is a dense sampling of R, as compared to the size of significant features of R, then the reconstruction $F(S)$ should approximate R to some comparable accuracy. A weak form of this hope is an asymptotic statement: For any "well-behaved" region R, $F(S)$ can be guaranteed to be an arbitrarily good approximation of R if S is a sufficiently dense sampling of R. Again, there are a number of different forms that such a theorem can take, since there are numerous different geometric features of R that might be important to preserve in $F(S)$, and a number of different notions of what it means for $F(S)$ to "approximate" R. Which geometric features are important or which definitions of approximate are relevant depends on the application.

There are also different kinds of point samples. The points may be sampled from the boundary of the region R, or from the interior of R, or they may be be required only to be close to region R; not surprisingly, much stronger results can be obtained if the points are guaranteed to be on the surface [2]. The sample may or may not be extracted from a predetermined grid of test points. The sample may include only positive points, or both positive and negative points (that is, points specified to be outside the region). The points may be the result of a random sampling following some specified distribution (e.g. uniform), or they may be required to be sufficiently dense in R. The sample may be required to be finite or may be allowed to be infinite. In this paper, we consider S to be a sample of positive points that is within a specified Hausdorff distance δ of R — that is, every point in S is within δ of some point in R and vice versa. We do not consider random sampling, probabilistic issues, or negative points. A sample may be either finite or infinite.

All of our results and proofs apply in the same way in Euclidean space of arbitrary finite dimension.

In this paper we study a number of reconstruction methods and give sufficient conditions under which they preserve geometric properties or give accurate approximations. Specifically, we define a broad class of reconstruction methods called *locally-based region constructors with radius* α. We single out two particular reconstruction methods within this class, denoted $\mathcal{J}_\alpha(S)$, and $\mathcal{F}_\alpha(S)$, both parameterized by a positive distance α. We prove results that give conditions on region R, sample S, and constructor function F that are sufficient to ensure that:

1. The reconstructed region $F(S)$ is close to R in terms of the dual-Hausdorff distance, the Hausdorff distance between the boundaries, and the measure of the symmetric difference. (theorem 12 and corollary 13).

2. The particular reconstructors $\mathcal{J}_\alpha(S)$, and $\mathcal{F}_\alpha(S)$ are ϵ-similar to R and are approximations in tangent of R. (Theorems 28, 29, 44, and 45).

Section 2 briefly surveys related work on shape reconstruction from samples. Sections 3 and 4 presents basic terminology and notations. Section 5 gives the definition of a locally-based region constructor, and gives some examples and non-examples. Section 6 proves theorem 12, that a locally-based region constructor from a sufficiently dense sample gives a reconstruction that is accurate in the Hausdorff metric, and a partial converse (theorem 14). Section 7 presents the definition of an r-regular region and proves some basic properties. Section 8 presents two measures of similarity between regions, ϵ-similarity and (ϵ, ϕ)-approximation in tangent. Section 9 gives the proofs of theorem 28, that an α-ball reconstruction from a sufficiently fine sample of region R is ϵ-similar to R. and theorem 29 that such a reconstruction is an (ϵ, ϕ)-reconstruction in tangent. Section 10 proves the corresponding theorems (theorems 44 and 45) for the local convex hull constructor.

2 Related work

There is a substantial literature on reconstructing shapes from samples; [19] gives an extensive review. One well-known and widely used shape reconstruction method is the α-shape method introduced by Edelsbrunner. Roughly speaking, the α-shape reconstruction of sample S is computed by finding the Delaunay triangulation of S, then take the union of all the k-dimensional simplices that have circumradius $\leq \alpha$. This method was first proposed for two-dimensional shapes in [11], then extended to three dimensions in [13]. A further extension [12], called "weighted α-shapes" allows, in effect, different values of α at different points of the sample; this is appropriate when the density of sample points varies widely across the sample. These algorithms and variants of them are very widely used in practice; they give plausible and useful reconstructions, and the α-shape reconstruction can be computed for *all* α simultaneously by computing the Delaunay triangulation, for which fast algorithms are known.

Latecki et al. [16] give conditions under which the digitization of a region in a grid of square pixels preserves topological properties.

Stelldinger and Köthe [18] consider the "S-reconstruction" of a region from a subset S of a fixed grid of points G, which is the union of the cells that contain points of S in the Voronoi diagram of G. They show that in two dimensions the S-reconstruction of a region R can be guaranteed to be r-similar to R, but that this guarantee does not apply in dimensions 3 and higher.

Much of the more recent literature has focussed on the problem of reconstructing a surface from surface points rather than on reconstructing a region from interior points, as in this paper and in [11] and [13].

A number of papers have proven that the topology of a sampled region or sampled hypersurface is preserved under various forms of reconstruction with various conditions on the sample, including [1, 2, 3, 4, 5, 6, 7, 19, 20, 21].

Galton and Duckham present a number of different algorithms for shape reconstruction from sample points in two dimensions [10, 15].

Geometric and computational properties of regions constructed as the union of balls around specified centers, used here for the function $\mathcal{J}_\alpha(S)$, are analyzed in [14]

Reconstruction from points can alternatively be viewed as a form of learning from positive instances; for example, [17] adopts this viewpoint.

3 Basic definitions and terminology

Throughout this paper, k is the dimension of the geometric space.

A *point* is a point in $\mathbb{R}^k$. We will use boldface lower case letters such as $\mathbf{p}$ for points, and upper case italicized letters such as Q for sets of point.

If U and V are two sets, then we will write the set difference U minus V as $U \setminus V$.

For any region R, $\text{compl}(R)$ is the closure of the complement of R.

Let $\mathbf{p}$ be a point and let Q be a compact point set. Define $\Psi(\mathbf{p}, Q) = \text{argmin}_{\mathbf{q} \in Q} d(\mathbf{p}, \mathbf{q})$, the closest point to $\mathbf{p}$ in Q (ties broken arbitrarily). As usual $d(\mathbf{p}, Q) = \min_{\mathbf{q} \in Q} d(\mathbf{p}, \mathbf{q}) = d(\mathbf{p}, \Psi(\mathbf{p}, Q))$.

For any point $\mathbf{x}$ and $r > 0$, $B(\mathbf{x}, r)$ is the open ball of radius r centered at $\mathbf{x}$. $\bar{B}(\mathbf{x}, r)$ is the closure of $B(\mathbf{x}, r)$, the closed ball of radius r centered at $\mathbf{x}$.

The radius of a bounded point set Q, radius(Q), is the minimal value of r for which there exists an $\mathbf{x}$ such that $\bar{B}(\mathbf{x}, r) \supset Q$. The point $\mathbf{x}$ is denoted center(Q); clearly Q has exactly one center (the Chebyshev center).

Lemma 1 *Any bounded set Q has a unique Chebyshev center.*[1]

Proof: For any point $\mathbf{x}$, let $f_Q(\mathbf{x}) = \sup_{\mathbf{q} \in Q} d(\mathbf{x}, \mathbf{q})$. The lemma thus states that f_Q attains a minimum at a unique value. Let $C = \text{closure}(\text{convexHull}(Q))$.

Step 1: For any point $\mathbf{x}$, $f_C(\mathbf{x}) = f_Q(\mathbf{x})$. Proof: Let $\mathbf{w}$ be any point in C. Then $\mathbf{w}$ is the convex sum of a set S of points in closure(Q), so $d(\mathbf{x}, \mathbf{w}) \leq \max_{\mathbf{y} \in S} d(\mathbf{x}, \mathbf{y}) \leq f_Q(\mathbf{x})$. Thus $f_C(\mathbf{x}) \leq f_Q(\mathbf{x})$. The reverse inequality is immediate, since $C \supset Q$.

Step 2: $f_C(\mathbf{x})$ attains a minimum value over C, since f is continuous and C is compact.

Step 3: Let $\mathbf{y}$ be any point outside C. Let $\mathbf{z} = \Psi(\mathbf{x}, C)$. Let P be the hyperplane through $\mathbf{z}$ orthogonal to $\mathbf{zx}$. Since C is convex, no point in C is on the same side of $\mathbf{P}$ as $\mathbf{y}$. Thus, for every $\mathbf{c} \in C$, $d(\mathbf{c}, \mathbf{z}) \leq d(\mathbf{c}, \mathbf{y})$.

Therefore f_C has at least one global minimum in C.

Step 4: f_C has a unique minimum in C. Proof by contradiction. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are both minima of f_C in C. Let P be the perpendicular bisector of $\mathbf{a}$ and $\mathbf{b}$. and let $\mathbf{m}$ be the midpoint of $\mathbf{ab}$. Let $r = f_C(\mathbf{a})$ and let $\delta = d(\mathbf{a}, \mathbf{m})$. Then for any point $\mathbf{y} \in C$,

- if $\mathbf{y}$ is the same closed half-plane as $\mathbf{b}$, then since $d(\mathbf{a}, \mathbf{y}) \leq r$, $d(\mathbf{m}, \mathbf{y}) \leq \sqrt{r^2 - \delta^2}$.
- if $\mathbf{y}$ is the same closed half-plane as $\mathbf{a}$, then since $d(\mathbf{b}, \mathbf{y}) \leq r$, $d(\mathbf{m}, \mathbf{y}) \leq \sqrt{r^2 - \delta^2}$.

Thus $f_C(\mathbf{m}) \leq \sqrt{r^2 - \delta^2} < r$, but this contradicts the assumption that f_C was minimal at $\mathbf{a}$. ∎

The *diameter* of a bounded point set Q, diameter(Q) $= \sup_{\mathbf{p}, \mathbf{q} \in Q} d(\mathbf{p}, \mathbf{q})$. Obviously radius($Q$) $<$ diameter(Q) $\leq 2 \cdot$ radius(Q) (the lower bound is not tight).

For any point set Q, the (topological) boundary of Q denoted $\partial Q = \text{closure}(Q) \setminus \text{interior}(Q)$.

A region R is *regular* if R is equal to the closure of the interior of R.

The *Minkowski sum* of point sets P and Q, denoted $P \oplus Q = \{\mathbf{p} + \mathbf{q} \mid \mathbf{p} \in P, \mathbf{q} \in Q\}$

4 Metrics on regions, dilations, and erosions

Definition 1 *The one-sided Hausdorff distance from Q to R, $d_{H1}(Q, R) = \max_{\mathbf{q} \in Q} d(\mathbf{q}, R)$. The Hausdorff distance from Q to R, $d_H(Q, R) = \max(d_{H1}(Q, R), d_{H1}(R, Q))$. The dual-Hausdorff distance from Q to R, $d_{Hd}(Q, R) = \max(d_H(Q, R), d_H(\text{compl}(Q), \text{compl}(R)))$. [8, 9]*

The Hausdorff distance and the dual-Hausdorff distance are metrics over the space of compact regions.

If $d_{H1}(R, S) \leq p$ and $d_{H1}(S, R) \leq q$, then S is said to be a $p - q$ sampling of R [19].

[1] I presume that this easy result has been known since at least the time of Chebyshev, but I have found it surprisingly difficult to locate an exact statement in the literature, let alone a proof. A reviewer has pointed out to me that it is stated and proved as lemma 7 in the appendix of the long (technical report) version of [5].

Definition 2 *For any bounded region $R \subset \mathbb{R}^k$, let $\mathcal{M}^k(R)$ be the k-dimensional measure of R; e.g. the area of a planar region or the volume of a solid region. If R and Q are bounded regions, then define $d_M(R,Q) = \mathcal{M}^k(R \setminus Q \cup Q \setminus R)$, the measure of the symmetric difference of R and Q.*

This is a metric over the space of bounded regular regions.

Definition 3 *For any point set Q, and $r > 0$, the* dilation of Q by r, *denoted $D(Q,r)$, is defined as $D(Q,r) = \text{closure}(\bigcup_{\mathbf{x} \in Q} \bar{B}(\mathbf{x}, r))$. The* erosion of Q by r, *denoted $E(Q,r)$, is defined as $E(Q,r) = \{\mathbf{x} | B(\mathbf{x},r) \subset Q\}$.*

Note that if Q is compact then $D(Q,r)$ and $E(Q,r)$ are closed and bounded and hence likewise compact.

Lemma 2 *For any two bounded point sets Q, R and $r > 0$, $R \subset D(Q,r)$ if and only if $d_{H1}(R,Q) \leq r$.*

Proof: Immediate from the definition.

Lemma 3 *For any region R and $\alpha > 0$, $E(\text{compl}(R), \alpha) = \text{compl}(D(R,\alpha))$.*

Proof: For any point $\mathbf{x}$,

$$\begin{gathered}
\mathbf{x} \in E(\text{compl}(R), \alpha) \Leftrightarrow \\
B(\mathbf{x}, \alpha) \subset \text{compl}(R) \Leftrightarrow \\
[\forall_{\mathbf{y}}\ d(\mathbf{y}, \mathbf{x}) < \alpha \Rightarrow \mathbf{y} \in \text{compl}(R)] \Leftrightarrow \\
[\forall_{\mathbf{y}}\ \mathbf{y} \in R \Rightarrow d(\mathbf{y}, \mathbf{x}) \geq \alpha] \Leftrightarrow \\
[\forall_{\mathbf{y}}\ \mathbf{y} \in R \Rightarrow \mathbf{x} \notin B(\mathbf{y}, \alpha)] \Leftrightarrow \\
\mathbf{x} \notin \text{interior}(D(R, \alpha)) \Leftrightarrow \\
\mathbf{x} \in \text{compl}(D(R, \alpha)).
\end{gathered}$$

∎

Lemma 4 *For any regular regions Q and R,*
$d_{H1}(\partial Q, \partial R) \leq \max(d_{H1}(Q,R), d_{H1}(\text{compl}(Q), \text{compl}(R)))$.

Proof: Let $\mathbf{q} \in \partial Q$. Since $\mathbf{q} \in Q$ and $\mathbf{q} \in \text{compl}(Q)$, by definition of the Hausdorff distance, there exist $\mathbf{r}_1 \in R$, $\mathbf{r}_2 \in \text{compl}(R)$ such that $d(\mathbf{q}, \mathbf{r}_1) \leq d_{H1}(Q,R)$ and $d(\mathbf{q}, \mathbf{r}_2) \leq d_{H1}(\text{compl}(Q), \text{compl}(R))$. The line from $\mathbf{r}_1$ to $\mathbf{r}_2$ goes from R to $\text{compl}(R)$; hence it meets ∂R at some point $\mathbf{r}$. So

$$d(\mathbf{q}, \partial R) \leq d(\mathbf{q}, \mathbf{r}) \leq \max(d(\mathbf{q}, \mathbf{r}_1), d(\mathbf{q}, \mathbf{r}_2)) \leq \max(d_{H1}(Q,R), d_{H1}(\text{compl}(Q), \text{compl}(R)))$$

∎

Lemma 5 *Let R be a bounded regular region and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $d_H(R, E(R,\delta)) < \epsilon$.*

We define the function $\mathcal{L}_5(R,\epsilon)$ to be a value of δ satisfying the above condition.

In general in sections 3-6 of this paper, in order to simplify cross-references, when lemma or theorem n asserts, "For all x, y, z there exists w such that ..." we will define a function $\mathcal{L}_n(x,y,z)$ whose value is a value of w that satisfies the lemma.

Proof by contradiction: Suppose not. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, $d_H(R, E(R,\delta)) \geq \epsilon$. Since $E(R,\delta) \subset R$, this implies that for every $\delta > 0$ there exists a point $\mathbf{p}_\delta \in R$ such that $d(\mathbf{p}_\delta, E(R,\delta)) > \epsilon$. Consider the sequence $\mathbf{p}_{1/2}, \mathbf{p}_{1/3}, \mathbf{p}_{1/4} \ldots$. Since R is compact, these must have a cluster point $\mathbf{w} \in R$. Since R is regular, there exists a point $\mathbf{z} \in \text{interior}(R)$ such that $d(\mathbf{w},\mathbf{z}) < \epsilon/2$. Since $\mathbf{z} \in \text{interior}(R)$, there exists $\delta > 0$ such that $B(\mathbf{z},\delta) \subset \text{interior}(R)$; thus $\mathbf{z} \in E(R,\delta)$. Since $\mathbf{w}$ is a cluster point, there exists $\mathbf{p}_{1/M}$ such that $1/M < \delta$ and $d(\mathbf{w}, \mathbf{p}_{1/M}) < \epsilon/2$. But then

$$d(\mathbf{p}_{1/M}, E(R, 1/M)) \leq d(\mathbf{p}_{1/M}, E(R,\delta)) \leq d(\mathbf{p}_{1/M}, \mathbf{w}) + d(\mathbf{w},\mathbf{z}) < \epsilon$$

which contradicts the construction of $\mathbf{p}_{1/M}$. ∎

Lemma 6 *Let R be a bounded regular region and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $d_H(\text{compl}(R), E(\text{compl}(R),\delta)) < \epsilon$.*

We define the function $\mathcal{L}_6(R,\epsilon)$ to be a value of δ satisfying the above condition.

Proof: Let $Q = \text{closure}(D(R,\epsilon) \setminus R)$. Clearly Q is bounded and regular, so by lemma 5 there exists $\delta_0 > 0$ such that $d_H(Q, E(Q,\delta_0)) < \epsilon$. Let $\delta = \min(\delta_0, \epsilon)$.

Let $\mathbf{x}$ be any point in $\text{compl}(R)$. We need to show that there exists $\mathbf{y} \in E(Q,\delta)$ such that $d(\mathbf{x},\mathbf{y}) < \epsilon$. If $\mathbf{x} \in Q$ then there exists $\mathbf{y} \in E(Q,\delta) \subset E(\text{compl}(R),\delta)$ such that $d(\mathbf{x},\mathbf{y}) < \epsilon$. If $\mathbf{x} \notin Q$ then $\mathbf{x} \notin D(R,\epsilon)$. Since $\delta \leq \epsilon$, $\mathbf{x} \in E(\text{compl}(R),\epsilon)$ so we may choose $\mathbf{y} = \mathbf{x}$. ∎

Lemma 7 *Let R be a bounded regular region and let $\epsilon > 0$. Then there exists $\delta > 0$ such that, for any region Q, if $d_{Hd}(R,Q) < \delta$ then $d_M(R,Q) < \epsilon$.*

Proof: See [9] theorem 8.1.

5 Locally-based region constructors

In this section, we define the class of locally-based region constructors, and illustrate the category with some examples and non-examples.

Definition 4 *Let $\alpha > 0$. A function $G(S)$, mapping a set of points S to a region in $\mathbb{R}^k$ is a* local region constructor basis of maximal radius α *if it satisfies the following.*

A. For all S, either $G(S) = \emptyset$ or $\text{radius}(G(S) \cup S) \leq \alpha$.

B. For every point $\mathbf{y}$, there exist open regions $V, U_1 \ldots U_m$ such that $\mathbf{y} \in V$, and, for all $\mathbf{y}', \mathbf{x}_1 \ldots \mathbf{x}_m$, if $\mathbf{y}' \in V, \mathbf{x}_1 \in U_1 \ldots$ and $\mathbf{x}_m \in U_m$, then $\mathbf{y}' \in G(\{\mathbf{x}_1 \ldots \mathbf{x}_m\})$. (Here m may be equal to 1; $U_1 \ldots U_m$ are not necessarily disjoint; $\mathbf{x}_1 \ldots \mathbf{x}_m$ are not necessarily distinct.)

C. $G(\emptyset) = \emptyset$.

G is said to have diameter β if for all S either $G(S) = \emptyset$ or $\text{diameter}(G(S) \cup S) \leq \beta$.

Definition 5 *Let $G(S)$ be a local region constructor basis. The corresponding* locally-based region constructor *is the function $F(S)$ defined by $F(S) = \text{closure}(\bigcup_{S' \subset S} G(S'))$.*

The intuition behind condition (B) of definition 4 is the constructor function can cover each point $\mathbf{y}$ in space in a way that is somewhat robust with respect both to the position of $\mathbf{y}$ and to the sample; there is a set S of sample points such that $\mathbf{y} \in \text{interior}(G(S))$ and moreover $\mathbf{y} \in \text{interior}(G(S'))$ for any set S' close to S. We will show in theorem 14 that this is, in a certain sense, a necessary condition to reliably achieve a reconstruction that approximates the original. The examples below will further clarify the definition.

Example 1: Define $G(S)$ as follows: if $\text{radius}(S) \leq \alpha$ then $G(S) = \text{convexHull}(S)$ else $G(S) = \emptyset$. The corresponding region constructor is denoted $\mathcal{F}_\alpha(S) = \bigcup_{S' \subset S} G(S')$; it is called the *local convex hull constructor of radius* α.

Example 2: If $S = \{\mathbf{x}\}$ then $G(S) = \bar{B}(\mathbf{x}, \alpha)$; if $|S| > 1$ then $G(S) = \emptyset$. The corresponding function is denoted $\mathcal{J}_\alpha(S)$. This is known as the α-ball constructor [19]. Clearly $\mathcal{J}_\alpha(S) = D(S, \alpha) = S \oplus \bar{B}(\mathbf{o}, \alpha)$, where $\mathbf{o}$ is the origin.

Example 3: If $\text{radius}(S) \leq \alpha$ then $G(S) = \bar{B}(\text{center}(S), \text{radius}(S))$, the minimal ball containing S; else $G(S) = \emptyset$.

Example 4: $G(S)$ is the smallest box aligned with the coordinate axes containing S, if the span of that box in each dimension is at most $2\alpha/\sqrt{k}$. For $i = 1 \ldots k$ let $X_i(\mathbf{s})$ be the coordinate of point $\mathbf{s}$ in the ith dimension; let $U_i(S) = \max_{\mathbf{s} \in S} X_i(\mathbf{s})$; and let $L_i(S) = \min_{\mathbf{s} \in S} X_i(\mathbf{s})$. Define $G(S)$ as follows: If for all i, $U_i(S) - L_i(S) \leq 2\alpha/\sqrt{k}$ then $G(S) = \times_i [L_i(S), U_i(S)]$; else $G(S) = \emptyset$. (This is actually an analogue of example 3, using the $\mathcal{L}^\infty$ measure rather than the $\mathcal{L}^2$ measure.)

Example 5: Let $\mathcal{C}$ be any collection of open sets satisfying the conditions that

a. $\mathcal{C}$ covers the space; that is $\bigcup_{O \in \mathcal{C}} O = \mathbb{R}^k$.

b. for each region $O \in \mathcal{C}$, $\text{diameter}(O) \leq \alpha$.

Then for any set S, if $S = \{\mathbf{x}\}$ then let $G(S)$ be the union of all O such that $\mathbf{x} \in O \in \mathcal{C}$; if $|S| > 1$ then let $G(S) = \emptyset$. Note that since every point in $G(\{\mathbf{x}\})$ is within α of $\mathbf{x}$, the radius of $\{\mathbf{x}\} \cup G(\{\mathbf{x}\})$ is at most α.

Example 2 is a special case of example 5, with $\mathcal{C}$ being the set of all regions of diameter at most α.

Depending on the process collecting the sample points and the application using the reconstructed shapes, it may be reasonable to use reconstruction functions $G(S)$ that do not necessarily include all the sample points, or that are not convex.

Example 6: For some fixed m and α, if $|S| \geq m$ and $\text{radius}(S) \leq \alpha$ then
$G(S) = \bar{B}(\text{centroid}(S), \text{standardDeviation}(S))$; else $G(S) = \emptyset$. Imagine that one is trying to reconstruct the spatial range of a phenomena — for instance, the habitation area of a species — and that both the detection and the location of samples are uncertain. In that case, one might want to wait until m observations had been made within a radius of α before concluding that the location indeed belongs to the region, and to restrict the conclusion to the region within the standard deviation of the center of the observations.

Example 7: Suppose that there are detectors set up at points on the plane but that the detection process has a substantial error in the angle. In that case, for an observation at $\mathbf{s}$ by a sensor at $\mathbf{p}$, one might want to include the region of all points $\mathbf{x}$ such that $|d(\mathbf{x}, \mathbf{p}) - d(\mathbf{s}, \mathbf{p})| \leq \epsilon$ and such that $\angle \mathbf{xps} \leq \phi$. In this case $G(S)$ would be a non-convex region.

The proofs that each of the above examples satisfies definition 4 are trivial.

To further clarify the definition, let us give a few examples of functions $F(S)$ that are not local constructors with any basis. To prove these it is useful to observe two properties of locally-based constructors.

Definition 6 *Let $F(S)$ be a function mapping a set of points S to a set of points. F is* monotonic *if, for all S_1, S_2, if $S_1 \subset S_2$ then $F(S_1) \subset F(S_2)$. That is, adding more points to the sample does not take away points from the reconstruction.*

For any $\alpha > 0$, F is α-local *if the following holds: For any two samples S_1, S_2 and point $\mathbf{p}$, if $S_1 \cap \bar{B}(\mathbf{p}, \alpha) = S_2 \cap \bar{B}(\mathbf{p}, \alpha)$ then $\mathbf{p} \in F(S_1) \Leftrightarrow \mathbf{p} \in F(S_2)$. That is, if S_1 and S_2 are identical in the ball of radius α around $\mathbf{p}$ then either $\mathbf{p}$ is in both $F(S_1)$ and $F(S_2)$ or it is in neither.*

Lemma 8 *Let $G(S)$ be a local region constructor basis of diameter α and let $F(S)$ be the corresponding constructor. Then F is monotonic and α-local.*

Proof: Monotonicity is immediate from definition 5. To prove that F is α-local, let S_1 and S_2 be two samples, and let $\mathbf{p} \in F(S_1) \setminus F(S_2)$. Then there exists a set U such that $U \subset S_1$, $U \not\subset S_2$, and $\mathbf{p} \in G(U)$. Let $\mathbf{u}$ be a point in $U \setminus S_2$. Since G has diameter α, $d(\mathbf{u}, \mathbf{p}) \leq \alpha$ so $\mathbf{u} \in \bar{B}(\mathbf{p}, \alpha)$; thus S_1 and S_2 differ over $\bar{B}(\mathbf{p}, \alpha)$. ∎

We can now prove that various functions are not locally-based reconstructor functions because they do not satisfy one or the other of the above properties.

Non-example 1: $F(S) = \bar{B}(\mu(S), \sigma(S))$ where $\mu(S)$ is the mean and $\sigma(S)$ is the standard deviation does not have any local basis, because F is not local. The same holds for any other constructor based on weighted sums over all the points in S.

Non-example 2: The identity function $G(\{\mathbf{x}\}) = \{\mathbf{x}\}$, $F(S) = S$, is not a local constructor because G does not satisfy condition 1.B. That does not in itself prove that F might not have some other local basis; but it follows from theorem 12 below that it does not.

Non-example 3: If $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is the set of vertices of a perfect square of side $\leq \alpha$, then $G(S) =$ the interior of the square defined by S; else $G(S) = \emptyset$. This does not satisfy condition 1.B. Again one can show that the corresponding F does not satisfy theorem 12, and thus has no local basis.

Non-example 4: The α-shape function of [11, 13] is not a local constructor corresponding to any basis because it is not monotonic (see discussion below).

Monotonicity, it may be remarked, is not necessarily a desirable feature in a shape reconstruction function, and non-monotonicity not necessarily a failing; but the kind of non-monotonicity exhibited by α-shapes can be counter-intuitive. Figure 1 shows an example where the α-shape reconstruction of the sample $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is the triangle **abc** and thus includes point $\mathbf{p}$, but the α-shape reconstruction of the superset $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ is the union of the two triangles[2] **acd**, **bcd** and does not include $\mathbf{p}$, even though the new point $\mathbf{d}$ is very close to $\mathbf{p}$. α here is a value greater than the circumradius of **adc** and **bdc** but less than the circumradius of triangle **abd** so **abd** is not included in the α shape. If the application were, for example, reconstructing the habitat of a species from sightings, this would be anomalous behavior. (Keep in mind that the sample is of *interior* points, not of *boundary* points.)

It should be noted that any local region constructor $F(S)$ is defined for infinite S as well as finite S; for instance S itself could be a regular region. For example, one can imagine a circumstance in which it could be verified that each of a set of small squares lies inside a region R; the sample in that case would be the union of a set of squares rather than a finite set of points. Figure 2 shows the local convex hull reconstruction of one such sample. Note that the choice of the characteristic distance α depends primarily on the characteristics of the overall region R and of the density of the dots and not of the radius or shape of the individual dots.

[2]Under some definitions, the α-shape also includes the edge **ab**.

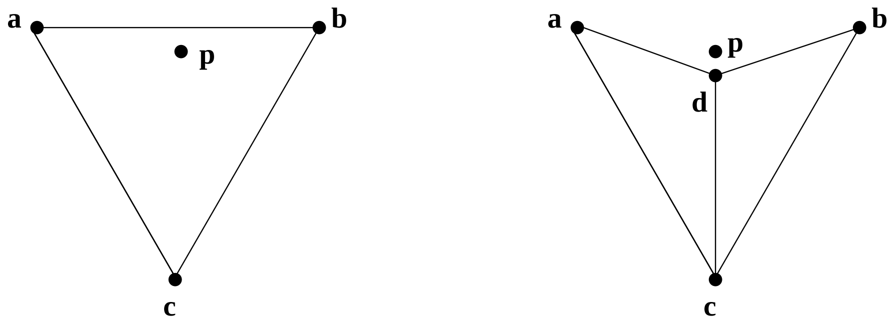


Figure 1: Non-monotonicity of α-shapes

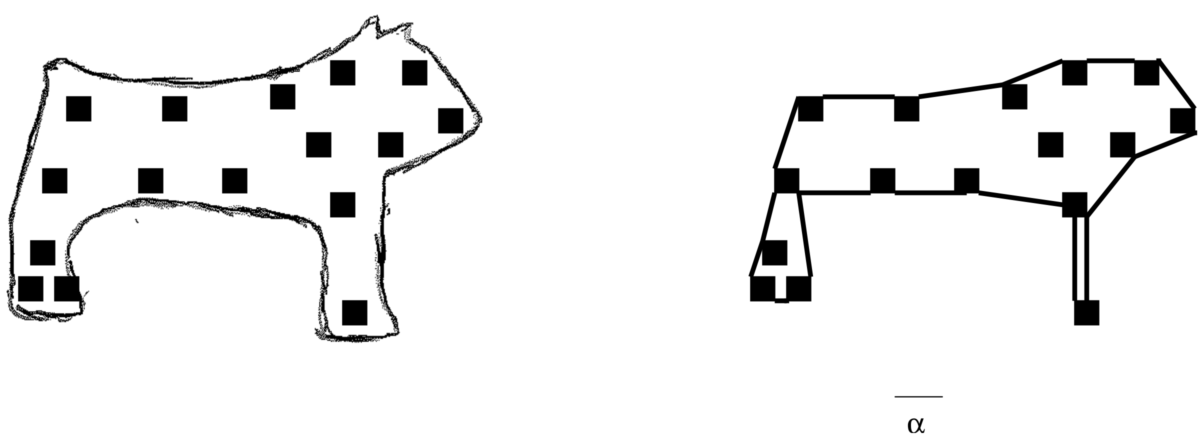


Figure 2: Reconstruction from a sample of squares

6 Locally-based region constructors approximate the original region

In this section we prove that, for any locally-based region constructor, if α is chosen small enough and the sample is chosen dense enough, then the constructor reconstructs the original region accurately, relative to the dual-Hausdorff distance.

Lemma 9 *Let F be a locally-based region constructor of diameter α. Then for any sample S, $F(S) \subset D(S, \alpha)$.*

Proof: Let G be a basis for F. By definition 4, for any set S' and point $\mathbf{x} \in S'$ all of $G(S')$ lies within distance α of $\mathbf{x}$. That is, $G(S') \subset D(\mathbf{x}, \alpha) \subset D(S', \alpha)$. Therefore,

$$F(S) = \bigcup_{S' \subset S} G(S') \subset D(S, \alpha)$$

∎

Lemma 10 *Let F be a locally-based region constructor of diameter α and let R and S be bounded point sets. Then $F(S) \subset D(R, \alpha + d_{H1}(S, R))$.*

Proof: Immediate from lemmas 2 and 9.

Lemma 11 *Let R be a bounded set and $\alpha > 0$. Let F be a locally-based region constructor of diameter α. Then there exists $\delta > 0$ such that, for any S if $d_{H1}(R, S) < \delta$ then $F(S) \supset E(R, \alpha)$.*

We define the function $\mathcal{L}_{11}(R, F, \alpha)$ to be a value of δ satisfying the above condition.

Proof: If $E(R, \alpha) = \emptyset$, then the claim is trivial. For each point $\mathbf{y} \in E(R, \alpha)$ choose open sets $V(\mathbf{y}), U_1(\mathbf{y}) \ldots U_m(\mathbf{y})$ satisfying condition 1.B. For any points $\mathbf{x}_1 \in U_1(\mathbf{y}) \ldots \mathbf{x}_m \in U_m(\mathbf{y})$, by condition 1.A, diameter($\{\mathbf{y}, \mathbf{x}_1 \ldots \mathbf{x}_m\}$) $\leq \alpha$; hence $d(\mathbf{y}, \mathbf{x}_i) < \alpha$. Therefore $U_i \subset R$. The collection $\{ \ V(\mathbf{y}) | \mathbf{y} \in E(R, \alpha) \ \}$ is an open covering of $E(R, \alpha)$, which is compact; therefore, it has a finite subcovering $\mathcal{V} = \{V(\mathbf{y}_1) \ldots V(\mathbf{y}_t)\}$. Consider the corresponding set of regions $U_1(\mathbf{y}_1) \ldots U_{m_1}(\mathbf{y}_1) \ldots U_1(\mathbf{y}_t) \ldots U_{m_t}(\mathbf{y}_t)$. Let δ be the minimal radius of all these. Now, let S be a point set such that $d_{H1}(R, S) < \delta$. By definition of δ, for any U_i there exists $\mathbf{u}_i$ such that $B(\mathbf{u}_i, \delta) \subset U_i$. By the constraint on the Hausdorff distance, there exists $\mathbf{s}_i \in S$ such that $d(\mathbf{u}_i, \mathbf{s}_i) < \delta$, thus $s_i \in B(\mathbf{u}_i, \delta) \subset U_i$.

Now let $\mathbf{y}$ be any point in $E(R, \alpha)$. Let $V(\mathbf{y}_j)$ be the open set in the collection $\mathcal{V}$ containing $\mathbf{y}$. Choose $\{\mathbf{s}_1 \ldots \mathbf{s}_m\} \subset S$ such that $\mathbf{s}_i \in U_i(\mathbf{y}_j)$. By condition 1.B, $\mathbf{y} \in G(\{\mathbf{s}_1 \ldots \mathbf{s}_m\})$. ∎

Theorem 12 *Let R be a bounded regular region and let $\epsilon > 0$. Then there exists $\alpha > 0$ such that, for any locally-based region constructor F of diameter α, there exists $\delta > 0$ such that for any S, if $d_H(S, R) < \delta$, then $d_{Hd}(F(S), R) \leq \epsilon$.*

Proof: Let $\alpha_1 = \mathcal{L}_5(R, \epsilon)$ and $\alpha_2 = \mathcal{L}_6(R, \epsilon)$. Thus $d_H(R, E(R, \alpha_1)) < \epsilon$ and $d_H(\text{compl}(R), E(\text{compl}(R), \alpha_2)) < \epsilon$.

Let $\alpha = \min(\alpha_1, \alpha_2/2, \epsilon/2)$. Let F be a locally-based region constructor of diameter α. Choose $\delta_1 = \mathcal{L}_{11}(R, F, \alpha)$. Let $\delta = \min(\delta_1, \alpha)$. Let S be any set of points such that $d_H(S, R) < \delta$.

By construction, and using lemma 10, we have

$$E(R,\alpha_1) \subset E(R,\alpha) \subset F(S,\alpha) \subset D(R,\alpha+\delta) \subset D(R,\epsilon)$$

Also $F(S,\alpha) \subset D(R,\alpha+\delta) \subset D(R,\alpha_2)$.

Since $F(S) \supset E(R,\alpha_1)$, we have $d_{H1}(R,F(S)) \leq d_{H1}(R,E(R,\alpha_1)) = d_H(R,E(R,\alpha_1)) < \epsilon$.

Since $F(S) \subset D(R,\epsilon)$ we have $d_{H1}(F(S),R) \leq d_{H1}(D(R,\epsilon),R) = d_H(D(R,\epsilon),R) \leq \epsilon$.

Thus $d_H(R,F(S,\alpha)) \leq \epsilon$.

Since $\mathrm{compl}(D(R,\alpha+\delta)) \subset \mathrm{compl}(F(S))$, we have $d_{H1}(\mathrm{compl}(R),\mathrm{compl}(F(S))) \leq d_{H1}(\mathrm{compl}(R),\mathrm{compl}(D(R,\alpha+\delta))) = d_H(\mathrm{compl}(R),\mathrm{compl}(D(R,\alpha+\delta))) \leq \epsilon$

Since $\mathrm{compl}(F(S)) \subset \mathrm{compl}(E(R,\alpha)) = D(\mathrm{compl}(R),\alpha) \subset D(\mathrm{compl}(R),\epsilon)$ we have $d_{H1}(\mathrm{compl}(F(S),\mathrm{compl}(R)) \leq d_{H1}(D(\mathrm{compl}(R),\epsilon),\mathrm{compl}(R)) \leq \epsilon$.

Thus $d_H(\mathrm{compl}(R),\mathrm{compl}(F(S,\alpha))) \leq \epsilon$.

Thus $d_{Hd}(R,F(S,\alpha)) \leq \epsilon$. ∎

Corollary 13 *Let R be a bounded regular region and let $\epsilon > 0$. Then*

- *There exists $\alpha > 0$ such that, for any locally-based reconstructor F of radius α, there exists $\delta > 0$ such that for any S, if $d_H(S,R) < \delta$, then $d_H(\partial F(S),\partial R) \leq \epsilon$.*
- *There exists $\alpha > 0$ such that, for any locally-based reconstructor F of radius α, there exists $\delta > 0$ such that for any S, if $d_H(S,R) < \delta$, then $d_M(F(S),R) \leq \epsilon$.*

Proof: The proof is immediate from theorem 12 and lemmas 4 and 7.

We can also show a partial converse to lemma 11: Any region constructor function that is local and monotonic and satisfies the conclusion of lemma 11 must correspond to some local basis. Note, however, that the condition of lemma 11 requires a diameter of α whereas the conclusion here guarantees only a radius of α, so there is a gap here of up to a factor of 2.

Theorem 14 *Let $\alpha > 0$. Let F be a function from a bounded point set S to a regular region, satisfying the following:*

i. $F(\emptyset) = \emptyset$.

ii. *F is monotonic.*

iii. *F is α-local.*

iv. *For any regular region R and $\epsilon > 0$ there exists $\delta > 0$ such that, for any sample S, if $d_{H1}(R,S) < \delta$ then $E(R,\epsilon) \subset F(S)$.*

Then F corresponds to a local reconstruction basis $G(S)$ of radius α.

Proof: The construction of $G(S)$ is the obvious one:

$$G(S) = \begin{cases} F(S) & \text{if } \mathrm{radius}(F(S) \cup S) \leq \alpha \\ \emptyset & \text{otherwise.} \end{cases}$$

Properties (A) and (C) of definition 4 are immediate.

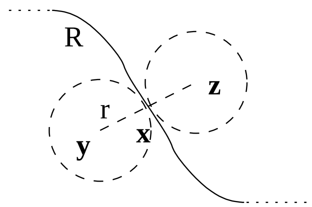


Figure 3: R-regular region

To show property (B), let $\mathbf{y}$ be any point. Let V be a neighborhood of $\mathbf{y}$. Let $\epsilon > 0$. Let $R = D(V, \epsilon)$. Using condition (iv), choose $\delta > 0$ such that, for any sample S, if $d_H(S, R) < \delta$ then $E(R, \epsilon) \subset F(S)$. Now, let $X = \{\mathbf{x}_1 \ldots \mathbf{x}_m\}$ be a finite sample such that $d_H(X, R) < \delta/2$. For $i = 1 \ldots m$, let $U_i = B(\mathbf{x}_i, \delta/2)$, and let $\mathbf{y}_i$ be any point in U_i. Let $Y = \{\mathbf{y}_1 \ldots \mathbf{y}_m\}$. Then $d_H(Y, R) \leq d_H(Y, X) + d_H(X, R) < \delta$ so by hypothesis $F(Y) \supset E(R, \epsilon) \supset V$.

Finally, let $H(S) = \cup_{S1 \subset S} G(S1)$; we need to show that $F(S) = H(S)$. By monotonicity $F(S) \supset H(S)$. To show the reverse, let $\mathbf{p}$ be any point in $F(S)$. Let $S1 = S \cap \bar{B}(\mathbf{p}, \alpha)$. Obviously $S \cap \bar{B}(\mathbf{p}, \alpha) = S1 \cap \bar{B}(\mathbf{p}, \alpha)$ so by locality $\mathbf{p} \in F(S1)$. Since radius$(S1) \leq \alpha$, $G(S1) = F(S1)$, so $\mathbf{p} \in G(S1) \subset H(S)$. ∎

7 R-regular regions

In this section, we define r-regular regions, and discuss some basic properties.

Definition 7 *Let $r > 0$ be a distance and let R be a topologically regular region. R is* r-regular *if, for every $\mathbf{x} \in \partial R$, there exist points $\mathbf{y}$ and $\mathbf{z}$ such that*

a. $\mathbf{x} \in \bar{B}(\mathbf{y}, r) \subset R$

b. $\mathbf{x} \in \bar{B}(\mathbf{z}, r) \subset \text{compl}(R)$

That is, $\mathbf{x}$ is on the boundary of a ball of radius r inside R, and on the boundary of a ball of radius r outside R (figure 3).

A number of slightly different definitions of "r-regularity" have been formulated in the literature; see [19] p. 18-20. The definition here is equivalent to the one used in [19].

We will denote the unit normal to ∂R at $\mathbf{x}$ directed outward from R as $\hat{N}_R(\mathbf{x})$ and denote the tangent (hyper)-plane to ∂R at $\mathbf{x}$ as $\pi_R(\mathbf{x})$. If R is a regular region, $\mathbf{x}$ is a point where ∂R is smooth, and d is a distance, then we define the function $\chi_R(\mathbf{x}, d) = \mathbf{x} + d \cdot \hat{N}_R(\mathbf{x})$. Note that χ_R is a continuous function of $\mathbf{x}$ and d at all smooth boundary points $\mathbf{x}$ of ∂R.

Lemma 15 *Let $r > 0$ and let R be an r-regular region. Let $\mathbf{x}$ be a point on ∂R and let $\mathbf{y}$ and $\mathbf{z}$ be points satisfying definition 7. Then*

a. ∂R is smooth at $\mathbf{x}$;

b. $\mathbf{y} = \chi_R(\mathbf{x}, -r)$ and $\mathbf{z} = \chi_R(\mathbf{x}, r)$.

Proof: See [16] theorem 1 p. 137. The proof given there is stated for two-dimensional space, but works for arbitrary dimension, as does the proof of lemma 16.

Lemma 16 *Let R be an r-regular region, and let $0 < q \leq r$. Then*

$$\partial E(R,q) = \{\chi_R(\mathbf{x}, -q) | \mathbf{x} \in \partial R\}$$

$$E(R,q) = R \setminus \{\chi_R(\mathbf{x}, -t) | \mathbf{x} \in \partial R, 0 \leq t < q\}$$

$$\partial D(R,q) = \{\chi_R(\mathbf{x}, q) | \mathbf{x} \in \partial R\}$$

$$D(R,q) = R \cup \{\chi_R(\mathbf{x}, t) | \mathbf{x} \in \partial R), 0 < t \leq q\}$$

Proof See [16] proposition 4, p. 139.

Lemma 17 *Let $\mathbf{p} \in \mathbb{R}^k$ be a point, and let $C \subset \mathbb{R}^k$ be a compact set. If there is a unique closest point to $\mathbf{p}$ in C, then $\Psi(\cdot, C)$ is continuous at $\mathbf{p}$.*

Proof of the contrapositive. Suppose that $\Psi(\cdot, C)$ is not continuous at $\mathbf{p}$. Let $\mathbf{z} = \Psi(\mathbf{p}, C)$. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists $\mathbf{x}_\delta$ such that $d(\mathbf{p}, \mathbf{x}_\delta) < \delta$ and $d(\Psi(\mathbf{x}_\delta, C), \mathbf{z}) > \epsilon$. For $i = 1, 2, \ldots$ let $\mathbf{y}_i = \Psi(\mathbf{x}_{1/i}, C)$. Since C is compact and since $\mathbf{y}_i \in C$ the sequence $\mathbf{y}_1, \mathbf{y}_2$ must have a cluster point; call this $\mathbf{y}$. Clearly, since $d(\mathbf{y}_i, \mathbf{z}) > \epsilon$, it follows that $d(\mathbf{y}, \mathbf{z}) \geq \epsilon$; thus $\mathbf{y} \neq \mathbf{z}$. Since $\mathbf{y}_i$ is the closest point in C to $\mathbf{x}_{1/i}$, it follows that $d(\mathbf{x}_{1/i}, \mathbf{y}_i) \leq d(\mathbf{x}_{1/i}, \mathbf{z}) \leq d(\mathbf{x}_{\mathbf{1/i}}, \mathbf{p}) + d(\mathbf{p}, \mathbf{z}) < d(\mathbf{p}, \mathbf{z}) + 1/i$. Therefore $d(\mathbf{p}, \mathbf{y}_i) \leq d(\mathbf{p}, \mathbf{x}_{1/i}) + d(\mathbf{x}_{1/i}, \mathbf{y}_i) \leq d(\mathbf{p}, \mathbf{z}) + 2/i$. Since $\mathbf{y}$ is a cluster point of the $\mathbf{y}_i$, one can choose large i such that $d(\mathbf{y}_i, \mathbf{y}) + 2/i$ is arbitrarily small. Hence $d(\mathbf{p}, \mathbf{y}) \leq d(\mathbf{p}, \mathbf{z})$. Hence $\mathbf{y}$ is a second point in C equally close to $\mathbf{p}$. ▮

Lemma 18 *Let R be an r-regular region, and let A be the open annulus* interior$(D(R,r) \setminus E(R,r))$. *Let $\mathbf{p}$ be a point in A. Then there is a point $\mathbf{x} \in \partial R$ which is strictly closer to $\mathbf{p}$ than any other point on ∂R; that is, $\Psi(\mathbf{p}, \partial R)$ is uniquely defined. Moreover $\mathbf{p} = \chi_R(\mathbf{x}, \pm d(\mathbf{p}, \mathbf{x}))$*

Proof: Since ∂R is compact, for fixed $\mathbf{p}$, $d(\mathbf{p}, \mathbf{x})$ attains a minimum at least one point $\mathbf{x} \in \partial R$. Let $Q = \bar{B}(\mathbf{p}, d(\mathbf{p}, \mathbf{x}))$. Since $\mathbf{x}$ is the closest point to $\mathbf{p}$ on ∂R, no part of ∂R lies inside Q. There are thus two cases: (1) $\mathbf{p} \in R$ and $Q \subset R$; (2) $\mathbf{p} \in$ compl(R) and $Q \subset$ compl(R). Choose $\mathbf{y}$ and $\mathbf{z}$ as in definition 7.

In case 1 $d(\mathbf{p}, \mathbf{x}) < r$; otherwise $\mathbf{p}$ would be in $E(R,r)$. Also Q is tangent to $\bar{B}(\mathbf{z}, r)$ at $\mathbf{x}$ since the former lies in R and the latter in compl(R). Hence the radius $\mathbf{px}$ lies along the line $\mathbf{zx}$ and thus along $\mathbf{yx}$. Thus $\mathbf{p} = \chi_R(\mathbf{x}, d(\mathbf{p}, \mathbf{x}))$. Moreover Q is tangent to $\bar{B}(\mathbf{y}, r)$ at $\mathbf{x}$; since none of ∂R is inside $\bar{B}(\mathbf{y}, r)$, the rest of ∂R must lie outside Q and thus be further from $\mathbf{p}$ than $\mathbf{x}$.

Case 2 is exactly analogous to case 1, switching the roles of $\mathbf{y}$ and $\mathbf{z}$; switching the roles of R and compl(R); and replacing $-\hat{N}_R(\mathbf{x})$ by $\hat{N}_R(\mathbf{x})$. ▮

Lemma 19 *Let R be an r-regular region, and let $\mathbf{x}, \mathbf{q} \in \partial R$. If $d(\mathbf{x}, \mathbf{q}) \leq r$ then $d(\mathbf{q}, \pi_R(\mathbf{x})) \leq d(\mathbf{x}, \mathbf{q})^2/2r$.*

Proof: (Figure 4.) If $\mathbf{q} \in \pi_R(\mathbf{x})$ this is trivial, so assume not. Let $\mathbf{y}, \mathbf{z}$ be as in definition 7. If the dimension of the space $k > 2$, then consider only the plane containing $\mathbf{x}, \mathbf{y}, \mathbf{z}$, and $\mathbf{p}$.

Assume that $\mathbf{q}$ lies on the same side of $\pi_R(\mathbf{x})$ as $\mathbf{y}$. Let S be the sphere of radius r centered at $\mathbf{y}$; thus, $\mathbf{x} \in \partial S$ and $\mathbf{q} \notin$ interior(S). Let $\mathbf{m}$ be the midpoint of $\mathbf{qx}$. Let Q be the perpendicular bisector of $\mathbf{qx}$ and let $\mathbf{y}'$ be the intersection of Q with $\mathbf{xy}$; thus $d(\mathbf{y}', \mathbf{x}) = d(\mathbf{y}', \mathbf{q})$. Let $r' = d(\mathbf{y}', \mathbf{x})$, and let S' be the sphere $\bar{B}(\mathbf{y}', r')$; thus $\mathbf{q} \in \partial S'$. Since $\mathbf{y}'$ is on the line $\mathbf{yx}$, S' and S are tangent at $\mathbf{x}$. Since $\mathbf{q} \notin$ interior(S), $S' \supset S$ and $r' \geq r$.

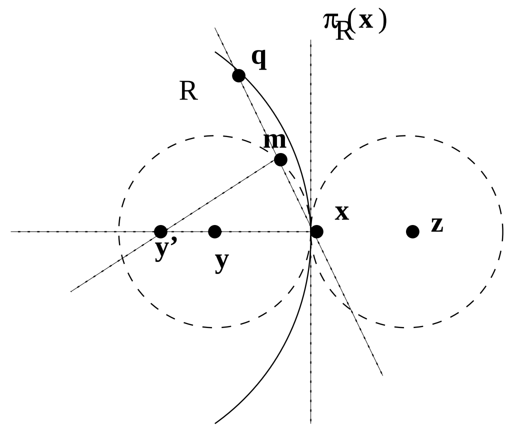


Figure 4: Construction for lemma 19

Let $\mathbf{n}$ be the projection of $\mathbf{m}$ onto $\mathbf{xy}$. Since $d(\mathbf{n}, \mathbf{x}) < d(\mathbf{m}, \mathbf{x}) < d(\mathbf{q}, \mathbf{x}) < r$, $\mathbf{n}$ is between $\mathbf{x}$ and $\mathbf{y}'$.

Since $\mathbf{xnm}$ and $\mathbf{xmy}'$ are similar right triangles, we have $d(\mathbf{x}, \mathbf{n})/d(\mathbf{x}, \mathbf{m}) = d(\mathbf{x}, \mathbf{m})/d(\mathbf{x}, \mathbf{y}')$. But $d(\mathbf{x}, \mathbf{n}) = d(\mathbf{q}, \pi_R(\mathbf{x}))/2$, $d(\mathbf{x}, \mathbf{m}) = d(\mathbf{x}, \mathbf{q})/2$ and $d(\mathbf{x}, \mathbf{y}') = r' \geq r$. Thus $d(\mathbf{q}, \pi_R(\mathbf{x})) = d(\mathbf{x}, \mathbf{q})^2/2r' \leq d(\mathbf{x}, \mathbf{q})^2/2r$.

If $\mathbf{q}$ is on the same side of $\pi_R(\mathbf{x})$ as $\mathbf{z}$ then the proof is exactly analogous, substituting $\mathbf{z}$ for $\mathbf{y}$ and using the fact that $\mathbf{q}$ is outside $B(\mathbf{z}, r)$. ∎

Lemma 20 *Let R be an r-regular region, let $\mathbf{x}, \mathbf{q} \in \partial R$, let $\mathbf{u}$ be the projection of $\mathbf{q}$ onto the plane $\pi_R(\mathbf{x})$. If $d(\mathbf{u}, \mathbf{x}) < r$ then $d(\mathbf{q}, \mathbf{u}) \leq r - \sqrt{r^2 - d^2(\mathbf{u}, \mathbf{x})}$*

Proof: Let points $\mathbf{y}, \mathbf{z}$ be as in definition 7. The line $\mathbf{qu}$ intersects the spheres $\bar{B}(\mathbf{y}, r)$ and $\bar{B}(\mathbf{z}, r)$ at points that are $r - \sqrt{r^2 - d^2(\mathbf{x}, \mathbf{u})}$ from $\mathbf{u}$, and $\mathbf{q}$ must lie between them, since it is not inside either sphere. ∎

Lemma 21 *Let R be an r-regular region, let $\mathbf{x} \in \partial R$, let $\mathbf{p} \in \pi_R(\mathbf{x})$, let $\mathbf{q} = \Psi(\mathbf{p}, \partial R)$. Then $d(\mathbf{p}, \mathbf{q}) \leq \sqrt{r^2 + d^2(\mathbf{x}, \mathbf{p})} - r < d^2(\mathbf{x}, \mathbf{p})/2r$.*

Proof: There are two cases.

Case 1: $\mathbf{p} \in \text{compl}(R)$. Let $\mathbf{c} = \chi_R(\mathbf{x}, -r)$. Then $B(\mathbf{c}, r) \subset R$, so the line from $\mathbf{c}$ to $\mathbf{p}$ crosses ∂R at some point $\mathbf{w}$ which is either equal to $\mathbf{p}$ or between $\mathbf{p}$ and $\mathbf{c}$. Since R is r-regular, $d(\mathbf{c}, \mathbf{w}) \geq r$. Since $\mathbf{q} = \Psi(\mathbf{p}, \partial R)$, $d(\mathbf{p}, \mathbf{q}) \leq d(\mathbf{p}, \mathbf{w})$. Since $\angle \mathbf{cxp}$ is a right angle we have

$$d(\mathbf{p}, \mathbf{q}) \leq d(\mathbf{p}, \mathbf{w}) = d(\mathbf{c}, \mathbf{p}) - d(\mathbf{c}, \mathbf{w}) \leq \sqrt{d^2(\mathbf{c}, \mathbf{x}) + d^2(\mathbf{x}, \mathbf{p})} - r = \sqrt{r^2 + d^2(\mathbf{x}, \mathbf{p})} - r < d^2(\mathbf{x}, \mathbf{p})/2r$$

The last inequality is a simple algebraic transformation.

Case 2: $\mathbf{p} \in R$. The argument is symmetric, using $\mathbf{c} = \chi_R(\mathbf{x}, r)$.

8 ϵ-similarity and (ϵ, ϕ)-approximation in tangent

In this section, we define three strong measures of similarity between regions, ϵ-similarity, simple ϵ-deformation, and and (ϵ, ϕ)-approximation in tangent; and we prove some basic properties.

Definition 8 *Let U and V be regions and let $\epsilon > 0$. U and V are* ϵ-similar *if there exists a homemorphism Γ from $\mathbb{R}^k$ to $\mathbb{R}^k$ such that $\Gamma(U) = V$ and such that, for all $\mathbf{p} \in \mathbb{R}^k$, $d(\mathbf{p}, \Gamma(\mathbf{p})) \leq \epsilon$ [18, 19]*

Lemma 22 *Let R be an r-regular region. Let Q be a region such that $E(R, r) \subset Q \subset D(R, r)$. For any point $\mathbf{x} \in \partial R$, the line $\{\chi_R(\mathbf{x}, t) | -\mu \leq t \leq \mu\}$ intersects ∂Q in at least one point $\mathbf{q}$.*

Proof: Immediate from the fact that point $\chi_R(\mathbf{x}, -r) \in E(R, r) \subset Q$ and point $\chi_R(\mathbf{x}, r) \in \partial D(R, r) \subset \text{compl}(Q)$.

Definition 9 *Let $\epsilon > 0$. A region Q is a* simple ϵ-deformation of region R *if*

- *For some $r > \epsilon$, R is r-regular.*
- *$E(R, \epsilon) \subset Q \subset D(R, \epsilon)$.*
- *For each $\mathbf{x} \in \partial R$, the line $\{\chi_R(\mathbf{x}, t) | -r \leq t \leq r\}$ intersects ∂Q in exactly one point $\mathbf{q}$.*

If this condition holds, then we write $\mathbf{q} = \mathit{Image}_{Q,R}(\mathbf{x})$. Note that $\mathbf{x} = \Psi(\mathbf{q}, \partial R)$.

Lemma 23 *Let Q be a simple ϵ-deformation of R. Then the function $\mathit{Image}_{Q,R}(\mathbf{x})$ is a continuous function from ∂R to ∂Q.*

Proof: Define the distance-valued function over ∂R, $H(\mathbf{x}) = (\text{Image}_{Q,R}(\mathbf{x}) - \mathbf{x}) \cdot \hat{N}_R(\mathbf{x})$, the signed distance from $\mathbf{x}$ to $\text{Image}_{Q,R}(\mathbf{x})$. Thus $\text{Image}_{Q,R}(\mathbf{x}) = \chi_R(\mathbf{x}, H(\mathbf{x}))$.

We claim that H is continuous. Proof by contradiction: Suppose that H is discontinuous at some point $\mathbf{b} \in \partial R$. Since H is bounded between $-r$ and r, H has a cluster point $h_1 \neq H(\mathbf{b})$ in the neighborhood of $\mathbf{b}$. Since ∂Q is closed, $\chi_R(\mathbf{b}, h_1) \in \partial Q$, but then the line $\{\chi_R(\mathbf{b}, t) | -r \leq t \leq r\}$ intersects ∂Q at two different points, contrary to assumption.

Since H is continuous and since χ_R is continuous, $\text{Image}_{Q,R}(\mathbf{x}) = \chi_R(\mathbf{x}, H(\mathbf{x}))$ is likewise continuous.

Lemma 24 *If Q is a simple ϵ-deformation of R, then Q is ϵ-similar to R.*

Proof: Choose $r > \epsilon$ so that R is r-regular. Define the homeomorphism $\Gamma(\mathbf{p})$ mapping $\mathbb{R}^k$ to itself as follows:

```
if (1) p ∈ E(R, r) then Γ(p) = p;
else (2) if p ∉ D(R, r) then Γ(p) = p;
else (3) if p ∈ Q \ E(R, r)
        let x = Ψ(p, ∂R);
            c = χ_R(x, −r); (Note that c ∈ ∂E(R, r)).
            y = Image_{Q,R}(x);
        Γ(p) = c + r · d(p, c)/d(y, c) N̂_R(x).
        end let;
else (4) if p ∈ D(R, r) \ Q
        let x = Ψ(p, ∂R);
            c = χ_R(x, r); (Note that c ∈ ∂D(R, r)).
            y = Image_{Q,R}(x);
            Γ(p) = c − r · d(p, c)/d(y, c) N̂_R(x).
        end let;
```

That is, Γ leaves the points in $E(R,r)$ and the points in $D(R,r)$ unchanged (1,2). The points in $Q \setminus E(R,r)$ are moved along the line normal to ∂R with a linear factor that maps the line from $\partial E(R,r)$ to ∂Q into the line from $\partial E(R,r)$ to ∂R (3). The points in $D(R,r) \setminus Q$ are moved along the line normal to ∂R with a linear factor that maps the line from $\partial D(R,r)$ to ∂Q into the line from $\partial D(R,r)$ to ∂R (4).

It is easily checked that Γ is consistent at the three boundaries; that is, (3) agrees with (1) on $\partial D(R,r)$ because $\mathbf{p} = \mathbf{c}$, so $\Gamma(\mathbf{p}) = \mathbf{p}$. (4) agrees with (2) on $\partial E(R,r)$ because $\mathbf{p} = \mathbf{c}$ so $\Gamma(\mathbf{p}) = \mathbf{p}$. (3) agrees with (4) on ∂Q because $\mathbf{p} = \mathbf{y}$ and $\Gamma(\mathbf{p}) = \mathbf{y}$.

The inverse of Γ is computed as follows:

if $\mathbf{p} \in E(R,r)$ then $\Gamma^{-1}(\mathbf{p}) = \mathbf{p}$;
else if $\mathbf{p} \notin D(R,r)$ then $\Gamma^{-1}(\mathbf{p}) = \mathbf{p}$;
else if $\mathbf{p} \in R \setminus E(R,r)$ then
    $\mathbf{c} = \chi_R(\mathbf{x}, -r)$
    $\mathbf{y} = \mathrm{Image}_{Q,R}(\mathbf{x})$;
    $\Gamma^{-1}(\mathbf{q}) = \mathbf{c} + d(\mathbf{y},\mathbf{c})d(\mathbf{q},\mathbf{c})/r\hat{N}_R(\mathbf{x})$.
else if $\mathbf{p} \in D(R,r) \setminus R$ then
    $\mathbf{c} = \chi_R(\mathbf{x}, r)$
    $\mathbf{y} = \mathrm{Image}_{Q,R}(\mathbf{x})$;
    $\Gamma^{-1}(\mathbf{q}) = \mathbf{c} - d(\mathbf{y},\mathbf{c})d(\mathbf{q},\mathbf{c})/r\hat{N}_R(\mathbf{x})$.

It follows from lemma 23 that Γ and Γ^{-1} are continuous. It is immediate that they are one-to-one, that Γ maps Q to R, and that for any $\mathbf{q} \in Q$, $d(\mathbf{q}, \Gamma(\mathbf{q})) \leq \epsilon$. ∎

Definition 10 *Let O be an open set of points. Let $\mathbf{q} \in \mathrm{closure}(O)$. A non-zero vector $\vec{v}$* points into O at $\mathbf{q}$ *if for some $\delta > 0$, for all $t \in (0,\delta)$, $\mathbf{q} + t\vec{v} \in O$. (For $\mathbf{q} \in O$, this holds for all $\vec{v}$; the condition is non-trivial for $\mathbf{q} \in \partial O$.)*

Lemma 25 *Let R be a smooth region and let Q be a closed set of points. Suppose that there exist points $\mathbf{y} \in \partial R$, $\mathbf{a} \in \mathrm{compl}(Q)$ (the closure of the complement of Q), and $\mathbf{b} \in Q$ such that $\mathbf{a} = \chi_R(\mathbf{y}, a)$ and $\mathbf{b} = \chi_R(\mathbf{y}, b)$ where $0 < a < b$. Then there exist points $\mathbf{x} \in \partial R$, $\mathbf{q} \in \partial Q$ such that $\mathbf{q} - \mathbf{x}$ is parallel to $\hat{N}_R(\mathbf{x})$ and such that $-\hat{N}_R(\mathbf{x})$ does not point into* interior(Q) *at* $\mathbf{q}$.

Proof (figure 5): There are three cases to consider.

Case 1: There exists c such that $a < c < b$ and $\mathbf{c} = \chi_R(\mathbf{y}, c) \notin Q$. Since the complement of Q is open, there is an open set around $\mathbf{c}$ that is outside Q. Let $h > c$ be the maximum value such that, for all $t \in (c, h)$ $\chi_R(\mathbf{y}, t) \notin Q$. Since Q is closed, the point $\mathbf{h} = \chi_R(\mathbf{y}, h)$ is on ∂Q. Then by construction $-\hat{N}_R(\mathbf{y})$ does not point into interior(Q) at $\mathbf{h}$.

Case 2: There exists c such that $a < c < b$ and $\mathbf{c} = \chi_R(\mathbf{y}, c) \in \mathrm{interior}(Q)$. Then for some $\delta > 0$, $B(\mathbf{c}, \delta) \subset \mathrm{interior}(Q)$. Since $\mathbf{a} \in \mathrm{compl}(Q)$ and the complement of Q is open, there exists a sequence $\mathbf{e}_1, \mathbf{e}_2 \ldots \notin Q$ that converge to $\mathbf{a}$. For $i = 1, 2 \ldots$ let $\mathbf{f}_i = \Psi(\mathbf{e}_i, R)$ and let $\mathbf{g}_i = \chi_R(\mathbf{f}_i, \mu)$. Since $\Psi(\cdot, R)$ and $\chi_R(\cdot, \mu)$ are continuous functions, the sequence $\mathbf{f}_1, \mathbf{f}_2 \ldots$ converges to $\Psi(\mathbf{a}, R) = \mathbf{y}$, and the sequence $\mathbf{g}_1, \mathbf{g}_2 \ldots$ converges to $\chi_R(\mathbf{y}, \mu) = \mathbf{c}$. Moreover since $d(\mathbf{x}, \mathbf{a}) < d(\mathbf{x}, \mathbf{c})$, for sufficiently large i, $d(\mathbf{f}_i, \mathbf{e}_i) < d(\mathbf{f}_i, \mathbf{g}_i)$. Thus for sufficiently large i, we have $\mathbf{f}_i \in \partial R$, $\mathbf{e}_i = \chi_R(\mathbf{f}_i, d(\mathbf{f}_i, \mathbf{e}_i)) \notin Q$, $\mathbf{g}_i = \chi_R(\mathbf{f}_i, \mu) \in B(\mathbf{c}, \delta) \subset Q$. So $\mathbf{f}_i, \mathbf{e}_i, \mathbf{g}_i$ satisfy the conditions on $\mathbf{y}, \mathbf{c}, \mathbf{b}$ respectively in case (1), so there exists an $\mathbf{h}$ on the line between $\mathbf{e}_i$ and $\mathbf{g}_i$ satisfying the condition, with $\mathbf{x} = \mathbf{f}_i$.

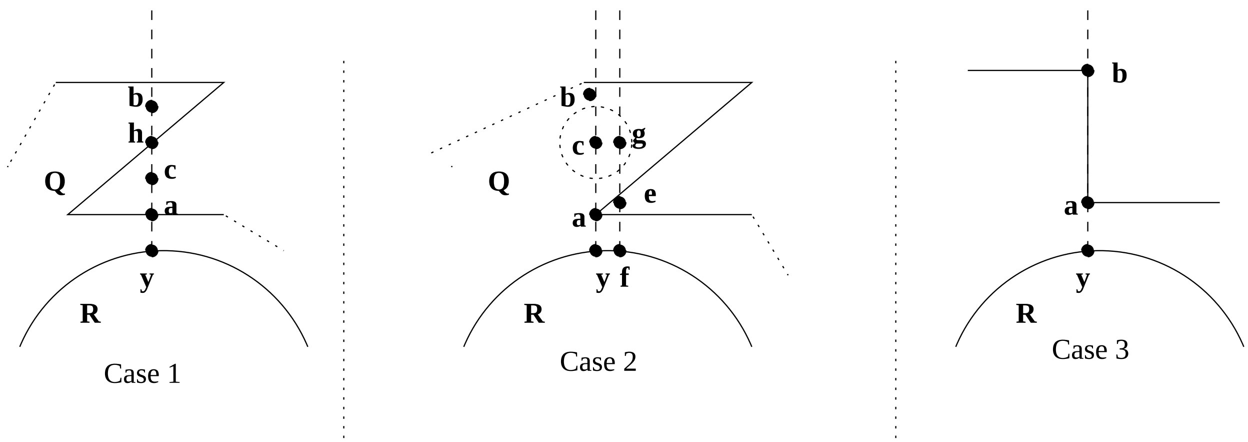


Figure 5: Cases in Lemma 25

Case 3: For every $c \in (a,b)$, $\chi_R(\mathbf{y},c) \in \partial Q$. Then the line **ab** lies in ∂Q, so $-\hat{N}_R(\mathbf{y})$ does not point into interior(Q) at **b**.

▮

Corollary 26 *Let R be an r-regular region. Let $\epsilon < r$. Let Q be a closed set of points such that $R \subset Q \subset D(R,\epsilon)$. If Q is not an ϵ-deformation of R, then there exist points $\mathbf{x} \in \partial R$, $\mathbf{q} \in \partial Q$ such that $\mathbf{q} - \mathbf{x}$ is parallel to $\hat{N}_R(\mathbf{x})$ and such that $-\hat{N}_R(\mathbf{x})$ does not point into* interior(Q) *at* $\mathbf{q}$.

Proof: By definition 9, if Q is not an ϵ-deformation of R, then there exists a point $\mathbf{y} \in \partial R$ such that the line from $\mathbf{y}$ to $\chi_R(\mathbf{y},\epsilon)$ intersects ∂Q in two points. Letting **a** be the nearer of these and **b** be the further, the conditions of lemma 25 are satisfied. ▮

We now define the strongest form of approximation considered in this paper, in which not only the topology must be correct and the region occupied nearly correct, but also the surface normal must be nearly correct, at all points of the surface where it is defined. (Many reconstruction methods, including both of those considered here, give a region with a piecewise smooth surface, rather than a universally smooth one, so the normal is not defined at the joins between surfaces.)

Definition 11 *Let Q and R be regular regions. Let $\epsilon > 0$ and $\phi > 0$. Q is said to be an (ϵ,ϕ)-*approximation in tangent *of R if there exists a homeomorphism Γ from $\mathbb{R}^k$ to $\mathbb{R}^k$ such that:*

i. $\Gamma(Q) = R$.

ii. For every point $\mathbf{q} \in Q$, $d(\mathbf{q},\Gamma(\mathbf{q})) \leq \epsilon$.

iii. For any point $\mathbf{q} \in \partial Q$, if ∂Q is smooth at $\mathbf{q}$ and ∂R is smooth at $\Gamma(\mathbf{q})$ then the angle between $\hat{N}_Q(\mathbf{q})$ and $\hat{N}_R(\Gamma(\mathbf{q}))$ is less than or equal to ϕ.

(This is a generalization of the definition given in [8].)

In sections 9 and 10 we will show that, given sufficiently strong conditions on the parameter α and the sample density δ, the two reconstructions $\mathcal{J}_\alpha(S)$ and $\mathcal{F}_\alpha(S)$ are guaranteed to be ϵ-similar to the original region R; and that with even stronger conditions, they can be guaranteed to be (ϵ,ϕ)-approximations in tangent. We illustrate the power of these results here by giving a natural

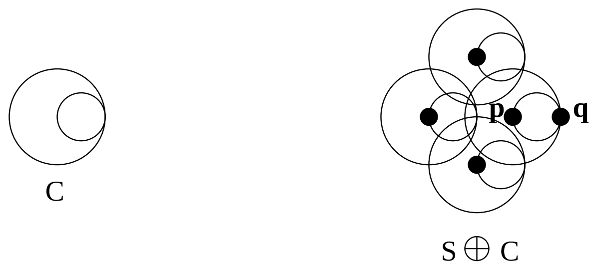


Figure 6: Non-Example 6

example of a local reconstruction method that cannot be guaranteed to be an approximation in tangent for any values of α and δ, and a contrived example of a method that cannot be guaranteed to be ϵ-similar.

Non-example 5: For $\alpha > 0$, let C_α be the cube $[-\alpha, \alpha]^k$. Let R be a smooth region, let S be a sample, and let $\phi < \pi/4$. Then the Minkowski sum $S \oplus C_\alpha$ is not an (ϵ, ϕ) approximation in tangent of R for any ϵ, because the normals to $S \oplus C_\alpha$ are all parallel to the coordinate axes and thus do not approximate the normals of R that lie at orientations in between.

Non-example 6 (figure 6): For $\alpha > 0$, let C_α be the crescent shape $\bar{B}(\mathbf{o}, \alpha) \setminus B((\alpha/2)\hat{x}, \alpha/2)$. Let R be a region homeomorphic to the unit disk, and let S be a finite sample. Let $F = S \oplus C_\alpha$. Then F is not homeomorphic to R. **Proof:** Let $\mathbf{p}$ be the point in S with maximal x-coordinate. Let $\mathbf{q} = \mathbf{p} + \alpha \cdot \hat{x}$. Then the distance from $\mathbf{q}$ to other point in S is greater than α, so the only points in F in a small neighborhood of F come from $\mathbf{p} \oplus C_\alpha$. Hence the topology of F in the neighborhood of $\mathbf{q}$ is the "double cusp", which is different from the topology of R at any point. Note that $S \oplus C_\alpha$ is a locally-based region constructor, and indeed satisfies theorem 12 for any $\delta < \alpha/4$.

9 Reconstruction by $\mathcal{J}_\alpha(S)$

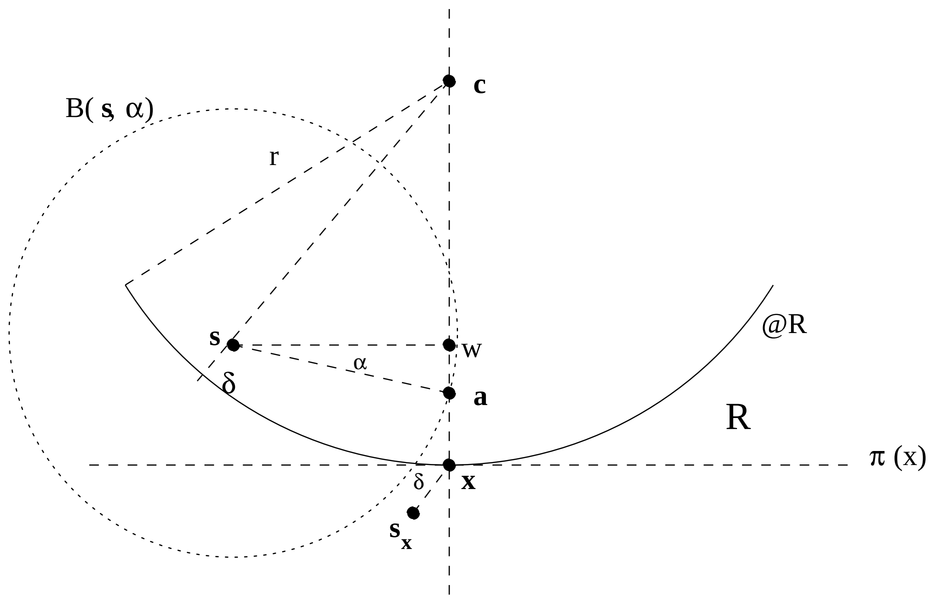
As noted in example 2 p. 7, the function $G(\{\mathbf{s}\}) = \bar{B}(\mathbf{s}, \alpha)$ is a locally-based region constructor basis; the corresponding region constructor $\mathcal{J}_\alpha(S) = D(S, \alpha) = S \oplus \bar{B}(\mathbf{o}, \alpha)$. This is known as "$\alpha$-ball reconstruction." [19]

Lemma 27 *Let R be an r-regular region. Let $\alpha < r$ and let $\delta < \alpha(r-\alpha)/(2r-\alpha)$. Then for any set of points S, if $d_H(S, R) \leq \delta$ then $\mathcal{J}_\alpha(S)$ is an $(\alpha+\delta)$-deformation of R.*

Proof by contradiction (Figure 7): Suppose not. Let $Q = \mathcal{J}_\alpha(S)$. Since $\alpha > \delta$, by lemma 2 $R \subset \text{interior}(Q)$. Any point in Q is within α of a point in S and thus within $\alpha + \delta$ of a point in R; hence $Q \subset D(R, \alpha + \delta)$. By lemma 26 if Q is not an $(\alpha + \delta)$-deformation of R, then there exist points $\mathbf{x} \in \partial R$ and $\mathbf{a} \in \partial Q$ such that $\mathbf{a} - \mathbf{x}$ is parallel to $\hat{N}_R(\mathbf{x})$ and $-\hat{N}_R(\mathbf{x})$ does not point into interior(Q) at $\mathbf{a}$.

Since $\mathbf{a} \in \partial Q$, there exists $\mathbf{s} \in S$ such that $d(\mathbf{a}, \mathbf{s}) = \alpha$. Since $\bar{B}(\mathbf{s}, \alpha) \subset Q$ and since $-\hat{N}_R(\mathbf{x})$ does not point into interior(Q) at $\mathbf{a}$, it must be the case that $(\mathbf{s} - \mathbf{a}) \cdot \hat{N}_R(\mathbf{x}) \geq 0$. Let $\mathbf{w}$ be the projection of $\mathbf{s}$ onto the line $\mathbf{xa}$. Then $\mathbf{a}$ is between $\mathbf{x}$ and $\mathbf{w}$; that is $d(\mathbf{a}, \mathbf{x}) \leq d(\mathbf{w}, \mathbf{x})$.

Let $\mathbf{c} = \chi_R(\mathbf{x}, r)$; since R is r-regular, $\bar{B}(\mathbf{c}, r) \subset \text{compl}(R)$. If $\mathbf{s} \in R$ then $d(\mathbf{c}, \mathbf{s}) \geq r$. If $\mathbf{s} \notin R$. then let $\mathbf{b} = \Psi(\mathbf{s}, \partial R)$; thus $d(\mathbf{s}, \mathbf{b}) \leq \delta$. Since $\mathbf{b} \in R$, we have $d(\mathbf{c}, \mathbf{b}) \geq r$; hence $d(\mathbf{c}, \mathbf{s}) \geq r - \delta$. Note that $d(\mathbf{s}, \mathbf{w}) \leq d(\mathbf{s}, \mathbf{a}) = \alpha$. Since $\mathbf{swc}$ is a right triangle, we have $d(\mathbf{c}, \mathbf{w})^2 = d(\mathbf{c}, \mathbf{s})^2 - d(\mathbf{s}, \mathbf{a})^2 \geq (r-\delta)^2 - \alpha^2$. So $d(\mathbf{x}, \mathbf{a}) < d(\mathbf{x}, \mathbf{w}) = d(\mathbf{c}, \mathbf{x}) - d(\mathbf{c}, \mathbf{w}) \leq r - \sqrt{(r-\delta)^2 - \alpha^2}$.



Note: R is outside the circle centered at **c**.

Figure 7: Construction for Lemma 27

I claim that the last expression is less than $\alpha - \delta$. Proof:
Since $\delta < \alpha \cdot (r - \alpha)/(2r - \alpha)$
we have $2r\delta - \alpha\delta < r\alpha - \alpha^2$
so $4r\delta - 2\alpha\delta < 2r\alpha - 2\alpha^2$
so $2r\delta + \alpha^2 - 2r\alpha - 2\alpha\delta < -2r\delta - \alpha^2$
so $r^2 + \delta^2 + 2r\delta + \alpha^2 - 2r\alpha - 2\alpha\delta < r^2 + \delta^2 - 2r\delta - \alpha^2$
so $(r + \delta - \alpha)^2 < (r - \delta)^2 - \alpha^2$
so $r + \delta - \alpha < \sqrt{(r - \delta)^2 - \alpha^2}$
so $r - \sqrt{(r - \delta)^2 - \alpha^2} < \alpha - \delta$. Thus $d(\mathbf{x}, \mathbf{a}) \leq r - \sqrt{(r - \delta)^2 - \alpha^2} < \alpha - \delta$.

Let $\mathbf{s_x}$ be a point in S such that $d(\mathbf{s_x}, \mathbf{x}) \leq \delta$. Then $d(\mathbf{s_x}, \mathbf{a}) \leq d(\mathbf{s_x}, \mathbf{x}) + d(\mathbf{x}, \mathbf{a}) < \alpha$, so $\mathbf{a} \in B(\mathbf{s_x}, \alpha) \subset \text{interior}(Q)$, but this contradicts the assumption that $\mathbf{a} \in \partial Q$. ∎

Theorem 28 *Let R be an r-regular region, and let $\epsilon > 0$. Let $\alpha < \min(r, \epsilon)$ and let $\delta < \min(\epsilon - \alpha, \alpha(r - \alpha)/(2r - \alpha))$. Let S be a set of points such that $d_H(S, R) < \delta$. Then $\mathcal{J}_\alpha(S)$ is ϵ-similar to R.*

Proof: By lemma 27, $\mathcal{J}_\alpha(S)$ is an $\alpha + \delta$ deformation of R. By lemma 24, $\mathcal{J}_\alpha(S)$ is ϵ-similar to R. ∎

It is easily calculated that the maximal value of δ/r consistent with the above constraint is $3 - 2\sqrt{2}$, attained when $\alpha/r = 2 - \sqrt{2}$. This same bound on δ is derived, in a somewhat different way and setting, in both [17] and [6].[3]

We now show that α-ball reconstruction gives accurate approximation in the much stronger sense that the surface normals on close points are close.

Theorem 29 *Let R be an r-regular region, let $\epsilon > 0$ and let $\phi > 0$. Let $\alpha < \min(\epsilon, r)$ and let $\delta < \min(\epsilon - \alpha, \alpha(r - \alpha)(1 - \cos\phi)/2r)$. Then for any set S, if $d_H(S, R) < \delta$ then $\mathcal{J}_\alpha(S)$ is an*

[3] I am grateful to the reviewer for drawing my attention, both to this bound, and to these papers.

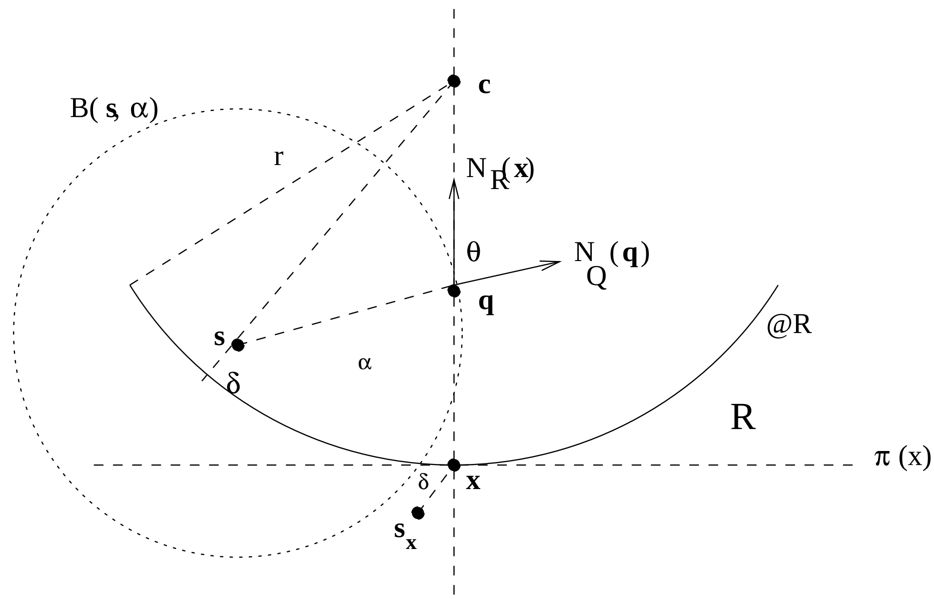


Figure 8: Construction for Theorem 29

approximation in tangent (ϵ, ϕ) of R.

Proof: (Figure 8). Let $Q = \mathcal{J}_\alpha(S)$. Since $\delta < \alpha(r-\alpha)(1-\cos\theta)/2r < \alpha(r-\alpha)/(2r-\alpha)$, theorem 28 holds, so Q is ϵ-similar to R.

Let point $\mathbf{q} \in \partial Q$ such that ∂Q is smooth at $\mathbf{q}$. Then there exists a point $\mathbf{s} \in S$ such that $d(\mathbf{s}, \mathbf{q}) = \alpha$; since ∂Q is smooth at $\mathbf{q}$, the normal $\hat{N}_Q(\mathbf{q})$ is parallel to the radius $\mathbf{q} - \mathbf{s}$. Let Γ be the homeomorphism mapping Q to R constructed in theorem 28, and let $\mathbf{x} = \Gamma(\mathbf{q})$; then by construction $\mathbf{x} \in \partial R$ and $\mathbf{q} = \chi_R(\mathbf{x}, d(\mathbf{x}, \mathbf{q}))$. As in the proof of lemma 27, let $\mathbf{c} = \chi_R(\mathbf{x}, r)$ and let $\mathbf{s_x}$ be a point in S such that $d(\mathbf{s_x}, \mathbf{x}) \leq \delta$. Since $\mathbf{q} \notin B(\mathbf{s_x}, \alpha) \subset \text{interior}(Q)$, we have $d(\mathbf{q}, \mathbf{x}) \geq \alpha - \delta$, so $d(\mathbf{c}, \mathbf{q}) = d(\mathbf{c}, \mathbf{x}) - d(\mathbf{q}, \mathbf{x}) \leq d(\mathbf{c}, \mathbf{x}) - r + \delta - \alpha$. As in the proof of lemma 27, $d(\mathbf{c}, \mathbf{s}) \geq r - \delta$.

The angle θ between $\hat{N}_Q(\mathbf{q})$ and $\hat{N}_R(\mathbf{x})$ is the complement of the angle $\angle\mathbf{sqc}$. Hence by the law of cosines $d(\mathbf{c}, \mathbf{s})^2 = d(\mathbf{s}, \mathbf{q})^2 + d(\mathbf{q}, \mathbf{c})^2 + 2d(\mathbf{s}, \mathbf{q})d(\mathbf{q}, \mathbf{c})\cos\theta$. But $d(\mathbf{c}, \mathbf{s}) \geq r - \delta$, $d(\mathbf{s}, \mathbf{q}) = \alpha$, and $d(\mathbf{q}, \mathbf{c}) \leq r + \delta - \alpha$, so we have

$$(r + \delta - \alpha)^2 + \alpha^2 + 2(r + \delta - \alpha)\alpha\cos\theta > (r - \delta)^2$$

so

$$r^2 + \delta^2 + \alpha^2 + 2r\delta - 2r\alpha - 2\delta\alpha + \alpha^2 + 2r\alpha\cos\theta + 2\delta\alpha\cos\theta - 2\alpha^2\cos\theta > r^2 - 2r\delta + \delta^2$$

so

$$4r\delta - 2\delta\alpha(1 - \cos\theta) > 2r\alpha(1 - \cos\theta) - 2\alpha^2(1 - \cos\theta)$$

so $4r\delta > 2r\alpha(1-\cos\theta) - 2\alpha^2(1-\cos\theta)$. Since $\delta < \alpha(r-\alpha)(1-\cos\phi)/2r$, we have $2\alpha(r-\alpha)(1-\cos\phi) > 2\alpha(r-\alpha)(1-\cos\theta)$, so $1 - \cos\phi > 1 - \cos\theta$, so $\theta < \phi$. ∎

10 Local convex hull constructor

In this section, we give conditions on the parameter α and the sample S sufficient to guarantee that the output of the local convex hull constructor $\mathcal{F}_\alpha(S)$ is ϵ-similar to R or an approximation in

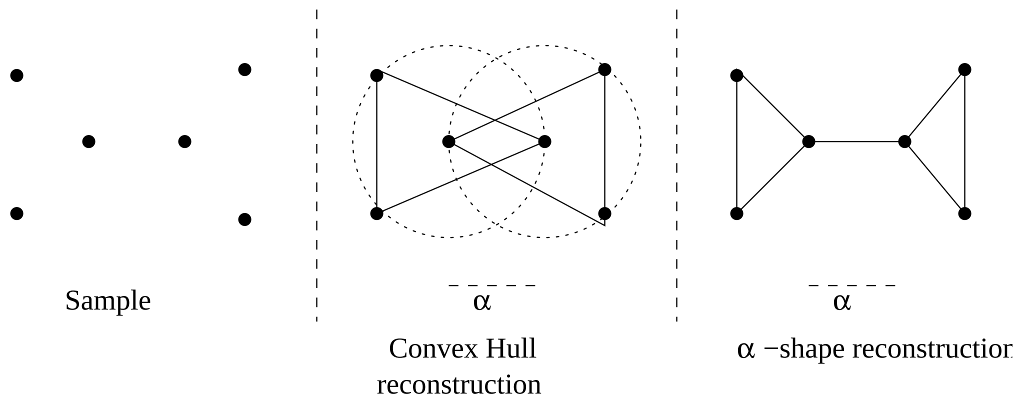


Figure 9: Convex Hull Reconstruction

tangent to R.

Though conceptually simple, this reconstruction is computationally awkward, among other reasons because it may require generating vertices that are not in S, by the intersection of two edges (Figure 9). Edelsbrunner's α-shapes [11] avoid this problem, by considering only simplices in the Delaunay triangulation.

Definition 12 *As in example 1, the* local convex hull constructor basis *of maximal radius α is the function $\mathcal{G}_\alpha(S)$ defined as follows: If* $\text{radius}(S) \leq \alpha$ *then* $\mathcal{G}_\alpha(S) = \text{convexHull}(S)$*, else* $\mathcal{G}_\alpha(S) = \emptyset$*. The local convex hull constructor of radius α is the function $\mathcal{F}_\alpha(S) = \cup_{S' \subset S} \mathcal{G}_\alpha(S')$.*

Lemma 30 *Let S be a set of points, and let $\mathbf{o}$ be a point. If $\bar{B}(\mathbf{o}, r) \subset D(S, r)$, then there is a subset $S' \subset S$ such that* $\text{radius}(S') \leq r$ *and* $\mathbf{o} \in \text{convexHull}(S')$*.*

Proof:[4] First we note that $\mathbf{o}$ $\in$convexHull(S). Proof by contradiction: If there is a plane P separating $\mathbf{o}$ from S, then let $\hat{n}$ be the normal to P pointing toward $\mathbf{o}$ and away from S; then the point $\mathbf{o} + r\hat{n}$ is distance r from $\mathbf{o}$ but more than r from any point in S.

Therefore, construct the Delaunay triangulation of S and let S' be the simplex containing $\mathbf{o}$. Let $\mathbf{p}$ be the circumcenter of S'. Thus $\mathbf{p}$ is equidistant from every vertex of S'; let r' be that distance. We claim that $r' \leq r$; proof by contradiction. Suppose that $r' > r$. By the property of the Delaunay triangulation, no point in $S - S'$ is inside $B(\mathbf{p}, r')$. Thus $\mathbf{p}$ is at least r' from every point in S; so $B(\mathbf{p}, r' - r)$ is disjoint from $D(S, r)$ and thus from $\bar{B}(\mathbf{o}, r)$. That is, $d(\mathbf{p}, \mathbf{o}) > r'$. However, since $\mathbf{o}$ is in the simplex of S' then necessarily $d(\mathbf{o}, \mathbf{p}) \leq r'$, which completes the contradiction. ▮

Corollary 31 *For any sets of points R and S, if $\alpha > \delta = d_{H1}(R, S)$ then $\mathcal{F}_\alpha(S) \supset E(R, \delta)$.*

Proof: If $\mathbf{o} \in E(R, \delta)$ then $\bar{B}(\mathbf{o}, \delta) \subset R \subset D(S, \delta)$. By lemma 30, there is a subset $S' \subset S$ such that $\text{radius}(S') \leq \delta < \alpha$ and $\mathbf{o}$ $\in$convexHull(S'). Thus $\mathbf{o} \in \mathcal{G}_\alpha(S') \subset \mathcal{F}_\alpha(S')$. ▮

Lemma 32 *Let R be a region; let $\delta > 0$; let S be a set of points such that $d_H(S, R) < \delta$; and let $\mathbf{o}$ be a point in $E(R, \delta)$. Then there exists a set of points $S' \subset S$ such that* $\text{radius}(S') < 2\delta$ *and* $\mathbf{o} \in \text{interior}(\text{convexHull}(S'))$.

[4] Thanks to Abhijit Guria for this elegant proof.

Proof: Let $\beta = d_H(S, R)$. Let $\mathbf{p}_1 \dots \mathbf{p}_{k+1}$ be any points such that $\mathbf{o} \in \text{interior}(\text{convexHull}(\{\mathbf{p}_1 \dots \mathbf{p}_{k+1}\}))$ and such that $d(\mathbf{o}, \mathbf{p}_i) < \delta - \beta$. Note that $B(\mathbf{p}_i, \beta) \subset B(\mathbf{o}, \delta) \subset R$; thus $\mathbf{p}_i \in E(R, \beta)$. By lemma 30 there exists $S_i \subset S$ such that $\mathbf{p}_i \in \text{convexHull}(S_i)$ and $\text{radius}(S_i) \leq \beta$. Note that each point in S_i is less than $\delta + \beta$ from $\mathbf{o}$. Let $S' = \cup_i S_i$. Then it is immediate that $\text{radius}(S') < 2\delta$ and $\mathbf{o} \in \text{interior}(\text{convexHull}(S'))$. ∎

Lemma 33 *Let R be an r-regular region and let $\zeta > 0$. Let R' be a subset of ∂R such that $\text{radius}(R') < 2\zeta r/(4 + \zeta^2)$. Then $\text{convexHull}(R') \subset D(R, \zeta \cdot \text{radius}(R'))$. Also, if $\mathbf{p}$ and $\mathbf{q}$ are any two points in $\text{convexHull}(R')$ and $\mathbf{u} = \Psi(\mathbf{q}, \partial R)$, then $d(\mathbf{p}, \pi_R(\mathbf{u})) \leq \zeta \cdot \text{radius}(R')$.*

Proof: Let $\rho = \text{radius}(R')$. First, note that since $\rho < 2\zeta r/(4 + \zeta^2)$ we have $4\rho + \zeta^2\rho < 2\zeta r$, so $-2\zeta\rho r + \zeta^2\rho^2 < -4\rho^2$ so $r^2 - 2\zeta\rho r + \zeta^2\rho^2 < r^2 - 4\rho^2$ so $r - \zeta\rho < \sqrt{r^2 - 4\rho^2}$ so $r - \sqrt{r^2 - 4\rho^2} < \zeta\rho$.

Let $\mathbf{r}'$ be any point in R' and let $\mathbf{v}$ be the projection of $\mathbf{r}'$ onto $\pi_R(\mathbf{u})$. Let W be the projection of R' onto $\pi_R(\mathbf{u})$. Then $\text{radius}(W) \leq \rho$ and $\mathbf{q}$ and $\mathbf{v}$ are both in $\text{convexHull}(W)$ so $d(\mathbf{u}, \mathbf{v}) \leq 2\rho$. By lemma 20 $d(\mathbf{r}', \mathbf{v}) \leq r - \sqrt{r^2 - 4\rho^2} < \zeta\rho$. Since this holds for all points $\mathbf{r}' \in R'$ and since $\mathbf{p} \in \text{convexHull}(R')$ we have $d(\mathbf{p}, \pi_R(\mathbf{u})) \leq \zeta\rho$. Since $\mathbf{q} \in \text{convexHull}(R')$ we have likewise $d(\mathbf{q}, \partial R)$ $= d(\mathbf{q}, \mathbf{u}) = d(\mathbf{q}, \pi_R(\mathbf{u})) \leq \zeta\rho$. but since $\mathbf{q}$ was an arbitrary point in $\text{convexHull}(R')$ this means that $\text{convexHull}(R') \subset D(R, \zeta\rho)$. ∎

Corollary 34 *Let R be an r-regular region and let $\zeta > 0$. Let R' be a subset of R such that $\text{radius}(R') < 2\zeta r/(4 + \zeta^2)$. Then $\text{convexHull}(R') \subset D(R, \zeta \cdot \text{radius}(R'))$.*

(This differs from lemma 33 in that R' can now be any subset of R, not just a subset of ∂R.)

Proof: Let $\mathbf{q}$ be a point in $\text{convexHull}(R')$; we wish to show that $d(\mathbf{q}, R) \leq \zeta \cdot \text{radius}(R')$. If $\mathbf{q} \in R$ this is trivial. If $\mathbf{q} \notin R$, then for each point $\mathbf{r}' \in R'$, draw the line from $\mathbf{q}$ to $\mathbf{r}'$. Each such line goes from outside R to inside R and thus must meet ∂R at some point $\mathbf{r}''$. Let R'' be the collection of all such points. It is immediate that $\text{radius}(R'') \leq \text{radius}(R')$ and that $\mathbf{q} \in \text{convexHull}(R'')$ (if a plane separates $\mathbf{q}$ from R'' then it likewise separates $\mathbf{q}$ from R'.) Therefore, R'' satisfies the conditions of lemma 33, so $d(\mathbf{q}, R) \leq \zeta \cdot \text{radius}(R')$. ∎

Lemma 35 *Let R be an r-regular region and let $\zeta > 0$. Let $\beta = 2\zeta r/(4 + \zeta^2)$. Let S be a point set and let $\delta = d_{H1}(S, R)$ and $\alpha = \text{radius}(S)$. If $\alpha + \delta \leq \beta$ then $\text{convexHull}(S) \subset D(R, (1 + \zeta)\delta + \zeta\alpha)$.*

Proof: Let $\mathbf{q}$ be a point in $\text{convexHull}(S)$. Let $R' = \{\Psi(\mathbf{s}, R) | \mathbf{s} \in S\}$. Thus for each $\mathbf{r} \in R'$, there exists $\mathbf{s} \in S$ such that $d(\mathbf{s}, \mathbf{r}) \leq d_{H1}(S, R) \leq \delta$. It is immediate that $\text{radius}(R') \leq \text{radius}(S) + d_{H1}(S, R) < \alpha + \delta < \beta$.

Since $\mathbf{q} \in \text{convexHull}(S)$ it can be expressed in the form $\mathbf{q} = \sum_i t_i \mathbf{s}_i$ where $t_i \in [0, 1]$ and $\sum_i t_i = 1$. Let $\mathbf{a} = \sum_i t_i \Psi(\mathbf{s}_i, R)$; since $d(\mathbf{s}_i, \Psi(\mathbf{s}_i, R)) \leq \delta$, we have $d(\mathbf{a}, \mathbf{q}) \leq \delta$. Since $\mathbf{a} \in \text{convexHull}(R')$, by corollary 34, we have $d(\mathbf{a}, R) \leq \zeta \cdot \text{radius}(R')$. Therefore

$$d(\mathbf{q}, R) \leq d(\mathbf{q}, \mathbf{a}) + d(\mathbf{a}, R) \leq d_{H1}(S, R) + \zeta \cdot \text{radius}(R') = d_{H1}(S, R) + \zeta \cdot (d_{H1}(S, R) + \text{radius}(S))$$

∎

Definition 13 *A set of points $S = \{\mathbf{s}_1 \dots \mathbf{s}_m\}$ is in* general position *if there is no affine space of dimension $m - 2$ containing S (equivalently, if the vectors $\{\mathbf{s}_2 - \mathbf{s}_1, \mathbf{s}_3 - \mathbf{s}_1, \dots, \mathbf{s}_m - \mathbf{s}_1\}$ are linearly independent).*

Definition 14 *Let $S = \{\mathbf{s}_1 \dots \mathbf{s}_m\}$ be a set of points in general position. A point $\mathbf{y}$ is in the* convex interior *of S if $\mathbf{y}$ is in the interior of convexHull(S), relative to the affine space containing S. Equivalently, there exist $t_1 \dots t_m$ such that $0 < t_i < 1$ for all i; $\sum_{i=1}^m t_i = 1$; and $\sum_{i=1}^m t_i \mathbf{s}_i = \mathbf{y}$.*

Lemma 36 *Let S be a set of points in general position with $|S| \leq k$, and let $\mathbf{a}$ and $\mathbf{b}$ be points in the convex interior of S. Let $\mathbf{c}$ be an arbitrary point. Let S^c be a set of $k+1$ points in general position such that $\mathbf{c}$ is in the convex interior of S^c. Then the vector $\mathbf{c} - \mathbf{a}$ points into* interior(convexHull($S^c \cup S$)) *at $\mathbf{b}$.*

Proof: Let $S = \{\mathbf{s}_1 \dots \mathbf{s}_m\}$. and let $S^c = \{\mathbf{c}_1 \dots \mathbf{c}_{k+1}\}$. Then there exist $u_1 \dots u_m$, $v_1 \dots v_m$, $w_1 \dots w_{k+1}$ such that

- $0 < u_i < 1$, $0 < v_i < 1$ for $i = 1 \dots m$; $0 < w_i < 1$ for $i = 1 \dots k+1$;
- $\sum_{i=1}^m u_i = 1$;
- $\sum_{i=1}^m v_i = 1$;
- $\sum_{i=1}^m w_i = 1$;
- $\mathbf{a} = \sum_{i=1}^m u_i \mathbf{s}_i$
- $\mathbf{b} = \sum_{i=1}^m v_i \mathbf{s}_i$
- $\mathbf{c} = \sum_{i=1}^{k+1} w_i \mathbf{c}_i$

Then $\mathbf{b} + t(\mathbf{c} - \mathbf{a}) = \sum_{i=1}^m v_i \mathbf{s}_i + t \cdot (\sum_{i=1}^{k+1} w_i \mathbf{c}_i - \sum_{i=1}^m u_i \mathbf{s}_i) = \sum_{i=1}^m (v_i - tu_i)\mathbf{s}_i + \sum_{i=1}^{k+1} tw_i \mathbf{c}_i$. If we choose $t < \min_i(v_i/u_i)$ then $v_i - tu_i > 0$ for all i. Also $\sum_{i=1}^m v_i - tu_i + \sum_{i=1}^{k+1} tw_i = 1$. Thus $\mathbf{b} \in$ interior(convexHull($S \cup S^c$)).

Lemma 37 *Let R be an r-regular region. Let $\alpha < r$ and let $\delta < \alpha/2$. Let S be a set of points such that $d_H(S, R) < \delta$. Let $\mathbf{y} \in S$ and let $\mathbf{c} = \Psi(\mathbf{y}, E(R, \alpha))$. Then* line$(\mathbf{y}, \mathbf{c}) \setminus \{\mathbf{y}\} \subset$ interior$(\mathcal{F}_\alpha(S))$.

Proof: By lemma 32, there exists $S^c \subset S$ such that $\mathbf{c} \in$ interior(convexHull(S^c)) and for all $\mathbf{s} \in S^c$, $d(\mathbf{s}, \mathbf{c}) \leq 2\delta < \alpha$. Let $S' = S^c \cup \{\mathbf{y}\}$. Note that every point in S' is within α of $\mathbf{c}$; hence radius$(S') \leq \alpha$, so convexHull$(S') \subset \mathcal{F}_\alpha(S)$. It is immediate that line$(\mathbf{c}, \mathbf{y}) \setminus \{\mathbf{y}\} \subset$ interior(convexHull(S')).

Lemma 38 *Let R be an r-regular region, let $\alpha < r$ and let $\delta < \alpha/4$. Let S be a set such that $d_H(S, R) < \delta$. Then $\mathcal{F}_\alpha(S)$ is a topologically regular set.*

Proof: Let $\mathbf{y}$ be any point in $\mathcal{F}_\alpha(S)$. Let S' be a subset of S such that $\mathbf{y} \in \mathcal{G}_\alpha(S')$. Let $\mathbf{o} =$ center(S'). Let $\mathbf{p} = \Psi(\mathbf{o}, E(R, 2\delta))$. By lemma 32 there exists $S^p \subset S$ such that $S^p \subset \bar{B}(\mathbf{p}, 2\delta)$ and $\mathbf{p} \in$ interior(convexHull(S^p)). Every point in S^p is within 4δ of $\mathbf{o}$; hence radius$(S' \cup S^p) \leq \alpha$. Thus convexHull$(S' \cup S^p) \subset \mathcal{F}_\alpha(S)$. But convexHull$(S' \cup S^p)$ is a regular set and contains $\mathbf{y}$. Thus, $\mathbf{y}$ is in the closure of the interior of convexHull$(S' \cup S^p)$ and therefore in the closure of the interior of $\mathcal{F}_\alpha(S)$; so $\mathcal{F}_\alpha(S)$ is regular.

Lemma 39 *Let S be a set of points and let $\mathbf{y} \in$* convexHull(S). *Then either $\mathbf{y} \in S$ or there is a finite subset $S' \subset S$ such that $\mathbf{y}$ is in the convex interior of S'.*

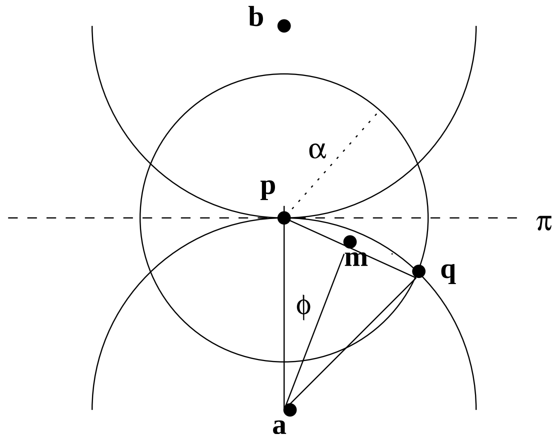


Figure 10: Construction for Lemma 40

Proof: Since $\mathbf{y} \in \text{convexHull}(S)$, there exists $\mathbf{s}_1 \ldots \mathbf{s}_{k+1}$ and $t_1 \ldots t_{k+1}$ such that $0 \leq t_i \leq 1$; $\sum_{i=0}^{k+1} t_i = 1$ and $\sum_{i=0}^{k+1} t_i \mathbf{s}_i = \mathbf{y}$. If any of the $t_i = 1$ then all the other t_i are equal to 0, and $\mathbf{y} = \mathbf{s}_i$. Otherwise, just extract the points $\mathbf{s}_i$ with non-zero coefficients, and then $\mathbf{y}$ is the positive sum of these.

Lemma 40 *Let R be an r-regular region. Let $\alpha \leq 2r$. Let $\mathbf{p}$ and $\mathbf{q}$ be points on ∂R such that $d(\mathbf{p}, \mathbf{q}) = \alpha$. Let $\phi = \sin^{-1}(\alpha/2r)$. Then the angle between $\pi_R(\mathbf{p})$ and $\mathbf{q} - \mathbf{p}$ is at most ϕ. The angle between $\hat{N}_R(\mathbf{p})$ and $\hat{N}_R(\mathbf{q})$ is at most 2ϕ.*

Proof (figure 10): Let $\mathbf{a} = \chi_R(\mathbf{p}, -r)$ and let $\mathbf{b} = \chi_R(\mathbf{p}, r)$. Since R is r-regular, $\mathbf{q}$ is outside the two disks $B(\mathbf{a}, r)$ and $B(\mathbf{b}, r)$. Restricting attention to the 2-dimensional plane containing $\mathbf{p}, \mathbf{q}, \mathbf{a}, \mathbf{b}$, and fixing $\mathbf{a}, \mathbf{b}$ and $\mathbf{p}$, the locus of points where $\mathbf{q}$ is α from $\mathbf{p}$ and not in these disks lies in the two arcs of the circle C of radius α around $\mathbf{p}$ bounded by the two disks; and it is obvious that the angle between $\hat{N}_R(\mathbf{p})$ and $\mathbf{q} - \mathbf{p}$ is furthest from $\pi/2$ when $\mathbf{q}$ is at any of the four intersections of the circle with one of the disks. Suppose that $\mathbf{q}$ is on an intersection of C with $\bar{B}(\mathbf{a}, r)$ (the other case is symmetric). Let $\mathbf{m}$ be the midpoint of $\mathbf{pq}$. Since $\angle\mathbf{amp}$ is a right angle, the angle $\angle\mathbf{map}$ is equal to ϕ and equal to the angle between $\pi_R(\mathbf{p})$ and $\mathbf{q} - \mathbf{p}$. Since both $\hat{N}_R(\mathbf{p})$ and $\hat{N}_R(\mathbf{q})$ form angles with $\mathbf{q} - \mathbf{p}$ between $\pi/2 - \phi$ and $\pi/2 + \phi$, the angle between the two normals is at most 2ϕ.

Lemma 41 *Let R be an r-regular region. Let $\mathbf{x}, \mathbf{z} \in \partial R$ be points such that $d(\mathbf{x}, \mathbf{z}) < r/4$. Let $\mu \leq r/2$, let $\nu \leq 2\mu$ and let $\mathbf{q} = \chi_R(\mathbf{z}, \nu)$. Let $\mathbf{v} = \mathbf{q} - (\nu + 2\mu)\hat{N}_R(\mathbf{x})$. Then $\mathbf{v} \in E(R, \mu)$.*

Proof (figure 11): By lemma 40 the angle ϕ between $\hat{N}_R(\mathbf{x})$ and $\hat{N}_R(\mathbf{z})$ is less than $2\sin^{-1}(1/8)$. Let $\mathbf{u} = \chi_R(\mathbf{z}, -2\mu)$. By lemma 16 $\mathbf{u} = \Psi(\mathbf{q}, E(R, 2\mu))$. Since $\mathbf{u}, \mathbf{q}, \mathbf{v}$ is an isoceles triangle with apex at $\mathbf{q}$ where the angle is ϕ and $d(\mathbf{u}, \mathbf{q}) = d(\mathbf{v}, \mathbf{q}) = 2\mu + d(\mathbf{q}, R) \leq 4\mu$, we have $d(\mathbf{u}, \mathbf{v}) = 2d(\mathbf{u}, \mathbf{q})\sin(\phi/2) \leq \mu$. Since $\mathbf{u} \in E(R, 2\mu)$, we have $\mathbf{v} \in E(R, \mu)$. ∎

Lemma 42 *Let R be an r-regular region. Let $\mathbf{x}, \mathbf{y} \in \partial R$. Let $\mathbf{p} = \chi_R(\mathbf{x}, p)$ and $\mathbf{q} = \chi_R(\mathbf{y}, q)$ where $p \leq r$ and $q \leq r$. Then $d(\mathbf{p}, \mathbf{q}) \geq d(\mathbf{x}, \mathbf{y})\sqrt{1 - 2\max(p, q)/r}$.*

Proof: Let $\beta = d(\mathbf{x}, \mathbf{y})$ and let $\mu = \max(p, q)$. Let $\hat{w} = (\mathbf{y} - \mathbf{x})/|\mathbf{y} - \mathbf{x}|$, $\hat{u} = \hat{N}_R(\mathbf{x})$, $\hat{v} = \hat{N}_R(\mathbf{y})$. Let $\phi = \sin^{-1}(\beta/2r)$. By lemma 40 the angles between $\hat{w}$ and $\hat{u}$ and between $\hat{w}$ and $\hat{v}$ are each within ϕ of $\pi/2$. So $\hat{w} \cdot \hat{u} \leq \cos(\pi/2 - \phi) = \sin(\phi)$.

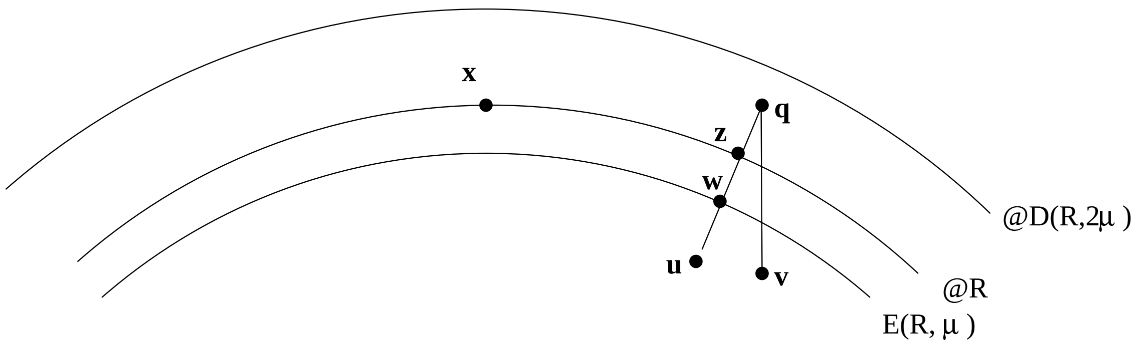


Figure 11: Construction for Lemma 41

Now $d^2(\mathbf{q},\mathbf{p}) = (\mathbf{q}-\mathbf{p})\cdot(\mathbf{q}-\mathbf{p}) = (\beta\hat{w}+q\hat{v}-p\hat{u})\cdot(\beta\hat{w}+q\hat{v}-p\hat{u}) = \beta^2-2\beta(\hat{w}\cdot(q\hat{v}-p\hat{u}))+|q\hat{v}-p\hat{u}|^2 > \beta^2-4\beta\mu\sin(\phi) = \beta^2(1-2\mu/r)$.
So $d(\mathbf{p},\mathbf{q}) > \beta\sqrt{1-2\mu/r}$.

Lemma 43 *Let R be an r-regular region. Let $\gamma < 1$, let $\zeta < \gamma$, let $\beta = 2\zeta/(4+\zeta^2)$, and let $\nu = (\sqrt{17}-1)/16 \approx 0.1952$. Let $\alpha < \min(\nu,\beta)\cdot r$ and let $\delta < \min(\beta r-\alpha, (1-\gamma)\alpha/3, (\gamma-\zeta)\alpha/(1+\zeta))$ Let S be any set such that $d_H(S,R) < \delta$. Let $\mathbf{y}$ be a point in $\partial\mathcal{F}_\alpha(S) - S$. Let $\mathbf{x} = \Psi(\mathbf{y},R)$. Then $-\hat{N}_R(\mathbf{x})$ points into* $\text{interior}(\mathcal{F}_\alpha(S))$ *at $\mathbf{y}$. Also $E(R,\delta) \subset \mathcal{F}_\alpha(S) \subset D(R,\gamma\alpha)$.*

Proof: By corollary 31, $\mathcal{F}_\alpha(S) \supset E(R,\delta)$

If S' is a subset of S such that $\text{radius}(S') \leq \alpha$ then the conditions of lemma 35 are satisfied,so $\text{convexHull}(S') \subset D(R,((1+\zeta)\delta+\zeta\alpha) = D(R,\gamma\alpha)$. Hence $\mathcal{F}_\alpha(S) \subset D(R,\gamma\alpha)$.

Let S^1 be a subset of S such that $\text{radius}(S^1) \leq \alpha$ and $\mathbf{y} \in \text{convexHull}(S^1)$. Let S^y be the subset of S^1 in general position such that $\mathbf{y}$ is in the convex interior of S^y. Let $\mathbf{x} = \Psi(\mathbf{y},R)$. Let $\mathbf{q}$ be the center of S^y; note that $\mathbf{q}$ is also in the convex interior of S^y. Since $\text{radius}(S^y) \leq \alpha$, we have $d(\mathbf{y},\mathbf{q}) \leq \alpha$.

There are now two cases to consider: (1) $\mathbf{q} \notin E(R,\delta)$: (2) $\mathbf{q} \in E(R,\delta)$.

Case 1: Suppose $\mathbf{q} \notin E(R,\delta)$. Let $\mathbf{z} = \Psi(\mathbf{q},R)$. Since $\mathbf{q} \in \text{convexHull}(S^y) \subset D(R,\gamma\alpha)$ we have $d(\mathbf{q},\mathbf{z}) \leq \gamma\alpha$. Since $d(\mathbf{y},\mathbf{q}) \leq \alpha$, $d(\mathbf{y},\mathbf{x}) \leq \gamma\alpha \leq \nu r$, and $d(\mathbf{q},\mathbf{z}) \leq \gamma\alpha \leq \nu r$, by lemma 42 we have $d(\mathbf{x},\mathbf{z}) \leq \nu r/\sqrt{1-2\nu} < r/4$. Let $\mathbf{v} = \mathbf{q} - (d(\mathbf{q},\mathbf{z})+2\delta)\cdot\hat{N}_R(\mathbf{x})$. By lemma 41 $\mathbf{v} \in E(R,\delta)$. By lemma 32 there exists a set $S^v \subset S$ such that $\text{radius}(S^v) \leq \delta$ and $\mathbf{v} \in \text{interior}(\text{convexHull}(S^v))$. Let $\mathbf{s}$ be a point in S^v. Then $d(\mathbf{s},\mathbf{q}) \leq d(\mathbf{s},\mathbf{v}) + d(\mathbf{v},\mathbf{q}) \leq \delta + 2\delta + \gamma\alpha \leq \alpha$. Thus all of $S^y \cup S^v$ is within α of $\mathbf{q}$, hence $\text{convexHull}(S^y \cup S^v) \subset \mathcal{F}_\alpha(S)$. Applying lemma 36 with $\mathbf{a}$ of lemma 36 being $\mathbf{q}$ here, $\mathbf{b}$ being $\mathbf{y}$ and $\mathbf{c}$ being $\mathbf{v}$, it follows that $\hat{N}_R(\mathbf{x})$ points into $\text{interior}(\mathcal{F}_\alpha(S))$ at $\mathbf{y}$.

Case 2: Suppose $\mathbf{q} \in E(R,\delta)$. By lemma 32 there exists a set $S^v \subset S$ such that $\text{radius}(S^v) \leq \delta$ and $\mathbf{q} \in \text{interior}(\text{convexHull}(S^v))$. Let $\mathbf{v}$ be a point on the ray $\{\mathbf{q} - t\vec{N}_R(\mathbf{x}) | t > 0\}$ such that $\mathbf{v} \in \text{interior}(\text{convexHull}(S^v))$. Continue as in case 1. ∎

Theorem 44 *Let R be an r-regular region and let $0 < \epsilon < r$. Let $\gamma < 1$, let $\zeta < \gamma$, let $\beta = 2\zeta/(4+\zeta^2)$, and let $\nu = (\sqrt{17}-1)/2 \approx 0.1952$. Let $\alpha < \min(\nu r, \beta r, \epsilon/\gamma)$ and let $\delta < \min(\beta r - \alpha, (1-\gamma)\alpha/3, (\gamma-\zeta)\alpha/(1+\zeta))$ Let S be any set such that $d_H(S,R) < \delta$. Then $\mathcal{F}_\alpha(S)$ and R are ϵ-similar.*

Proof: Let $F = \mathcal{F}_\alpha(S)$. By lemma 43, $E(R,\epsilon) \subset E(R,\delta) \subset F \subset D(R,\gamma\alpha) \subset D(R,\epsilon)$.

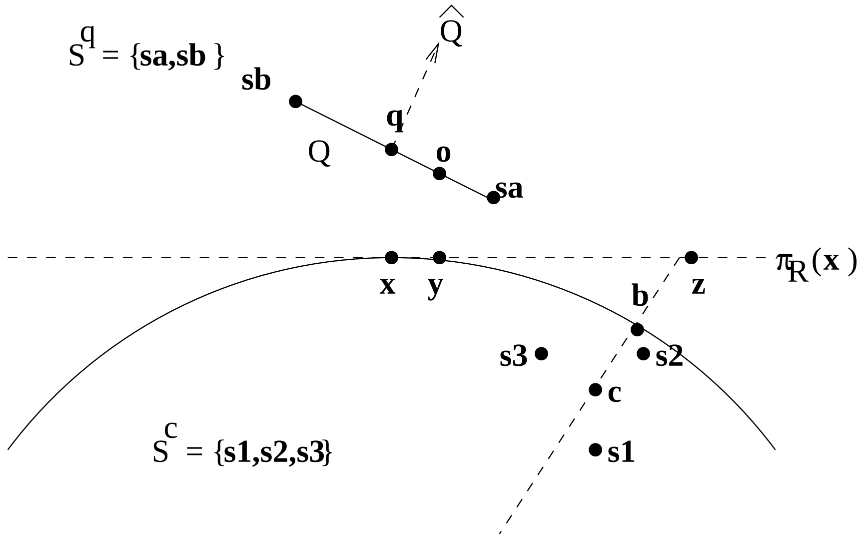


Figure 12: Construction for Theorem 45

Let $\mathbf{q}$ be any point on ∂F, and let $\mathbf{x} = \Psi(\mathbf{q}, R)$ Then the vector $\hat{N}_R(\mathbf{x})$ points into interior(F) at $\mathbf{q}$, by lemma 37, if $\mathbf{q} \in S$, and by lemma 43 if $\mathbf{q} \notin S$. By lemma 26, F is an ϵ-deformation of R, and by lemma 24, F is ϵ-similar to R. ∎

The maximum[5] possible value of δ/r consistent with the above constraints is $\delta/r = (\sqrt{5}-2)/10 = 0.0236$, achieved when $\alpha/r = 1/5$, $\gamma = 4 - 3\sqrt{5}/2 = 0.6459$, $\zeta = 2\sqrt{5} - 4 = 0.4721$, $\beta = \sqrt{5}/10 = 0.2236$.

For $\alpha \ll r$, δ is bounded by $\alpha/4 + o(\alpha)$, with $\gamma = 1/4$ and ζ chosen to be small. It is easily shown that Theorem 44 is false for any $\delta > \alpha$, so there is a gap of a factor of 4 in establishing a tight bound. The gap in the case where α is comparable to r is substantially larger.

We now show that, for α and δ small enough, the convex hull reconstruction α of a sample of radius δ accurately reconstructs the surface normal as well. Unlike the above proof of theorem 44, where we tried to formulate the conditions as weakly as the structure of the proof would allow, the collection of constraints involved here is so convoluted and in any case so far from a necessary condition, that we simply give one set of sufficient constraints.

Though the constant factors below are very overconservative, the order of magnitude dependence is correct. The radius α has to be bounded by $O(\phi r)$ since the normal to the region R may change by $\Theta(\alpha/r)$ within a face of radius α. The sample density δ has to be bounded by $O(\phi\alpha)$ since sample points may vary within δ of ∂R, "tipping" the surface of the convex hull by an angle $\Theta(\delta/\alpha)$. By contrast, note that in theorem 29 the choice of α is independent of ϕ, but δ is bounded by $O(r\phi^2)$.

Theorem 45 *Let R be an r-regular region, let $0 < \epsilon < r$ and let $0 < \phi \leq \pi/4$. Let $\alpha < \min(9\epsilon/\phi, \phi r/75)$ and let $\delta < \phi\alpha/45$. Let S be any set such that $d_H(S,R) < \delta$. Then $\mathcal{F}_\alpha(S)$ is an approximation in tangent (ϵ, ϕ) of R.*

Proof: (Figure 12). Throughout this proof, the verification that the various constraints are satisfied given the conditions are straightforward calculations that are omitted.

Let $F = \mathcal{F}_\alpha(S)$.

[5]Thanks to Sara Grundel for carrying out this calculation.

Let $\gamma = \phi/9$, $\eta = \phi/2$, $\zeta = \tan(\eta)/18$, $\beta = 2\zeta/(4+\zeta^2)$. Thus $\phi/72 < \beta < 0.01465\phi$. It is easily verified that the conditions of theorem 44 are satisfied, so there is an ϵ-similar homeomorphism Γ from F to R that maps each point $\mathbf{q} \in \partial F$ to $\Psi(\mathbf{q}, R)$.

Let $\mathbf{q}$ be any interior point in a face of $\partial\mathcal{F}_\alpha(S)$, let $\mathbf{x} = \Gamma(\mathbf{q}) = \Psi(\mathbf{q}, \partial R)$, and let $\hat{Q}$ be the normal to $\partial\mathcal{F}_\alpha(S)$ at $\mathbf{q}$. Let θ be the angle between $\hat{Q}$ and $\hat{N}_R(\mathbf{x})$. We need to show that $\theta < \phi$.

Let $\eta = \phi/2$. Let S^q be the subset of S such that $\mathbf{q}$ is in the convex interior of S^q. Let $\mathbf{o} = \text{center}(S^q)$, and let $\mathbf{y} = \Psi(\mathbf{o}, \pi_R(\mathbf{x}))$. By lemma 35 $d(\mathbf{o}, \mathbf{y}) < (1+\zeta)\delta + \zeta\alpha \leq \alpha\tan(\eta)/9$. Since the line $\mathbf{xy}$ is the projection of the line $\mathbf{qo}$ onto the plane $\pi_R(\mathbf{x})$, and since $d(\mathbf{q}, \mathbf{o}) \leq \alpha$ we have $d(\mathbf{x}, \mathbf{y}) \leq \alpha$. By lemma 40 the angle between $\hat{N}_R(\mathbf{x})$ and $\hat{N}_R(\mathbf{y})$ is at most $2\sin^{-1}(\alpha/2r) < \phi/2$ given the above constraints.

Let $\vec{W}$ be the projection of $\hat{Q}$ onto the plane $\pi_R(\mathbf{y})$: $\vec{W} = \hat{Q} - (\hat{Q} \cdot \hat{N}_R(\mathbf{y}))\hat{N}_R(\mathbf{y})$. If $\vec{W} = \vec{0}$, then $\hat{Q} = \hat{N}_R(\mathbf{y})$ which is within $\phi/2$ of $\hat{N}_R(\mathbf{x})$, so we are done. Otherwise, let $\hat{W}$ be the unit vector $\vec{W}/|\vec{W}|$. Let $\mathbf{z} = \mathbf{y} + (\alpha/2)\hat{W}$; thus $\mathbf{z} \in \pi_R(\mathbf{y})$. Let $\mathbf{b} = \Psi(\mathbf{z}, \partial R)$ and let $\mathbf{c} = \Psi(\mathbf{z}, \partial E(R, \delta))$. By lemma 21 $d(\mathbf{z}, \mathbf{b}) \leq \alpha^2/4r < \alpha/8$.

Using lemma 32, let S^c be a subset of S such that $S^c \subset \bar{B}(\mathbf{c}, \delta)$ and $\mathbf{c} \in$convexHull(S^c). Let $\mathbf{s}$ be a point in S^c. Then $d(\mathbf{o}, \mathbf{s}) \leq d(\mathbf{o}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) + d(\mathbf{z}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}) + d(\mathbf{c}, \mathbf{s}) \leq \alpha/8 + \alpha/2 + \alpha/8 + \delta + \delta \leq \alpha$. Thus $S^c \cup S^q$ all lies within α of $\mathbf{o}$. Let $G = \text{convexHull}(S^c \cup S^q)$; then $G \subset \mathcal{F}_\alpha(S)$. By lemma 37, the vector $\vec{C} = \mathbf{c} - \mathbf{o}$ points inward into G from $\mathbf{q}$ and thus points inward into $\mathcal{F}_\alpha(S)$. Therefore, $(\mathbf{c} - \mathbf{o}) \cdot \hat{Q} < 0$.

Since $d(\mathbf{y}, \mathbf{b}) < \alpha/2$, by lemma 40 the angle between $\hat{N}_R(\mathbf{b})$ and $\hat{N}_R(\mathbf{y})$ is at most $2\sin^{-1}(\alpha/4r) < \eta$. Hence $|(\mathbf{c} - \mathbf{z}) \cdot \hat{W}| < d(\mathbf{c}, \mathbf{z})\sin\eta \leq (d(\mathbf{c}, \mathbf{b}) + d(\mathbf{b}, \mathbf{z}))\sin\eta \leq (\delta + \alpha^2/4r)\sin\eta < \alpha/6$ given the above constraints.

Note that $\vec{C} = \mathbf{c} - \mathbf{o} = (\mathbf{c} - \mathbf{z}) + (\mathbf{z} - \mathbf{y}) + (\mathbf{y} - \mathbf{o})$, and that $\mathbf{z} - \mathbf{y})$ is parallel to $\hat{W}$ and orthogonal to $\hat{N}_R(\mathbf{y}$ while that $\mathbf{y} - \mathbf{o})$ is anti-parallel to $\hat{N}_R(\mathbf{y}$ and orthogonal to $\hat{W}$. Therefore $\vec{C} \cdot \hat{W} = (\mathbf{z} - \mathbf{y}) \cdot \hat{W} + (\mathbf{c} - \mathbf{z})\hat{W} \geq \alpha/2 - \alpha/6 = \alpha/3$. The projection $\vec{C} \cdot \hat{N}_R(\mathbf{x})$ has length at most $d(\mathbf{o}, \mathbf{y}) + d(\mathbf{z}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}) \leq \alpha\tan(\eta)/9 + \alpha\tan(\eta)/9 + \delta < \alpha\tan(\eta)/3$. Let $\vec{C'} = \vec{C} - ((\hat{C} \cdot \hat{W})\hat{W} + (\hat{C} \cdot \hat{N}_R(\mathbf{x}))\hat{N}_R(\mathbf{x}))$; thus $\vec{C'}$ is normal to both $\hat{W}$ and $\hat{N}_R(\mathbf{x})$. Thus we have

$$0 > \vec{C} \cdot \hat{Q} = [(\hat{C} \cdot \hat{W})\hat{W} + (\hat{C} \cdot \hat{N}_R(\mathbf{y})) + C'] \cdot [(\hat{Q} \cdot \hat{W})\hat{W} + (\hat{Q} \cdot \hat{N}_R(\mathbf{y}))] = (\hat{C} \cdot \hat{W})(\hat{Q} \cdot \hat{W}) + (\hat{C} \cdot \hat{N}_R(\mathbf{y}))(\hat{Q} \cdot \hat{N}_R(\mathbf{y}))$$

Since $(\hat{C} \cdot \hat{W})$, $(\hat{Q} \cdot \hat{W})$ and $(\hat{Q} \cdot \hat{N}_R(\mathbf{y}))$ are all positive, we have $(\hat{Q} \cdot \hat{W})/(\hat{Q} \cdot \hat{N}_R(\mathbf{y})) < |\hat{C} \cdot \hat{N}_R(\mathbf{y})|/(\hat{C} \cdot \hat{W}) \leq \tan(\eta)$, so the angle between $\hat{Q}$ and $\hat{N}_R(\mathbf{y})$ is less than $\eta < \phi/2$. Since the angle between $\hat{N}_R(\mathbf{y})$ and $\hat{N}_R(\mathbf{x})$ is also less than $\phi/2$, the angle between $\hat{Q}$ and $\hat{N}_R(\mathbf{x})$ is less than ϕ. ∎.

11 Conclusion

We have studied the formulation of conditions under which reconstruction of regions from samples can be guaranteed to approximate the original region, under a number of measures of approximation. Specifically, we have considered six measures of approximation: Hausdorff distance, Hausdorff distance between boundaries, measure of the symmetric difference, dual-Hausdorff distance, ϵ-similarity and (ϵ, ϕ)-approximation in tangent. We have defined a broad class of reconstruction methods, called *locally based reconstruction methods* that are guaranteed to achieve close approximation under the first four metrics, given a sufficiently dense and accurate sample (theorem 12 and corollary 13). Conversely, we have shown that any local, monotonic reconstruction method that does always achieve accurate approximation in these senses must satisfy the conditions of a locally-based approximation metric (theorem 14). For two particular reconstruction methods, α-ball reconstruction $\mathcal{J}_\alpha(S)$ and local convex hull reconstruction $\mathcal{F}_\alpha(S)$, we have given one set of conditions on the parameter α and on the density δ of the sample S sufficient to guarantee that the reconstruction of an r-regular region is ϵ-similar (theorems 28 and 44), and a stronger set sufficient to guarantee that the reconstruction is an (ϵ, ϕ)-approximation in tangent (theorems 29 and 45).

Two questions stand out as particular interesting for further research in this direction:

1. Can one generalize the last set of theorems, and show that there is a general class of reconstruction methods, of which $\mathcal{J}$ and $\mathcal{F}$ are instances, that are guaranteed to achieve ϵ-similarity and (ϵ, ϕ) approximation in tangent?
2. Both theorems 29 and 45 require that the density δ of the sample S be proportional to ϕ^2 where ϕ is the desired accuracy of the reconstructed surface normal, and one can show that, for these methods, this is necessary. It is not obvious that there could not be a reconstruction method that achieved as accurate an approximation of surface normal for a much less dense sample. It would be interesting either to formulate such a method, or to prove that no such method exists.

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