Homework 4, due Thursday May 1.

1. (textbook, problem 8 from ch. 3). Consider the function $g(x) = x^2 + 3/16$.

   (a) This function has two fixed points. What are they?

   (b) Consider the fixed point iteration $x_{k+1} = g(x_k)$ for this g. For which of the points you found in (a) can you be sure that the iterations will converge (to the fixed point)? Briefly justify your answer. You may assume that the initial guess is sufficiently close to the fixed point.

   (c) For the point or points you found in (b), roughly how many itera- tions will be required to reduce the convergence error by a factor of 10?

2. (based on textbook, problem 9).

   (a) Derive a cubically convergent method for solving the scalar equation $f(x) = 0$ in a way similar to the derivation of Newton's method, using evaluations of $f(x_n)$, $f'(x_n)$ and $f''(x_n)$. The following remarks may be helpful in constructing the algorithm:

   - Use Taylor's expansion with three terms plus a remainder term.
   - Show that in the course of derivation a quadratic equation arises, and therefore two distinct schemes can be derived.

   (b) Show that the rate of convergence (under the appropriate conditions) is indeed cubic.

   (c) Can you speculate what makes this method less popular than Newton's method, despite its cubic convergence? Give two reasons.

3. Express the Newton iteration for solving each of the following systems of nonlinear equations.

   (a) $x_1^2 + x_2^2 = 1, \quad x_1^2 - x_2 = 0$;

   (b) $x_1^2 + x_1 x_2^3 = 9, \quad 3x_1^2 x_2 - x_2^3 = 4$.

4. In this problem and the next problem, we consider a 2d spring-mass system with bending forces. It consists of masses m_i at positions $\mathbf{p}_i$, $i = 0 \dots N$, connected sequentially by springs of stiffness k_s that is, i is connected to $i+1$. The first and the last masses are fixed. In addition to the usual spring forces we have considered before, we add bending forces and gravity. The bending force acting on mass i is $F_i^{bend} = k_b(\mathbf{p}_{i+1} - 2\mathbf{p}_i + \mathbf{p}_{i-1})$ for $i = 1 \dots N-1$. The gravity is just $-m_i g \mathbf{e}_2$, where $\mathbf{e}_2 = [0, 1]$, and $g = 9.81 m/s^2$.

   (a) Write the Newton method formulation for finding the equilibrium of this system, that is, for solving the system of equations

   $$F_i^{total} = F_i^{bend} + F_i^{spring} + F_i^{gravity} = 0, \; i = 1 \dots N-1$$

   .

   (b) Write equations for forward Euler, backward Euler, and Verlet integration schemes for the dynamic version of this system, that is,

   $$\frac{d^2\mathbf{p}_i}{dt^2} = F_i^{total}, \; i = 1 \dots N-1$$

5. (a) Implement the Newton iteration for the equilibrium system; display the solution at every step. Plot the y position of the point in the center of your chain (use $N = 20$, $k_s = 10$, $k_b = 1$, $m_i = 1$ for all i) as function of the iteration number.

   (b) Implement the forward Euler and Verlet integration for the dynamic system. Plot the y position of the point in the center of your system as function of time. Choose time steps when Euler converges and diverges, and compare to the Verlet method. In the initial position, all points are on the horizontal line $y = 0$, with the first point at 0 and the last at 1.

For this problem, I recommend that you use parts of the demo code that compute forces and blocks of the Jacobian matrix for spring forces only, and add the forces and Jacobian matrices for bending to the assembled matrix. For gravity, only forces need to be updated, as the force is constant, its Jacobian matrix is zero.