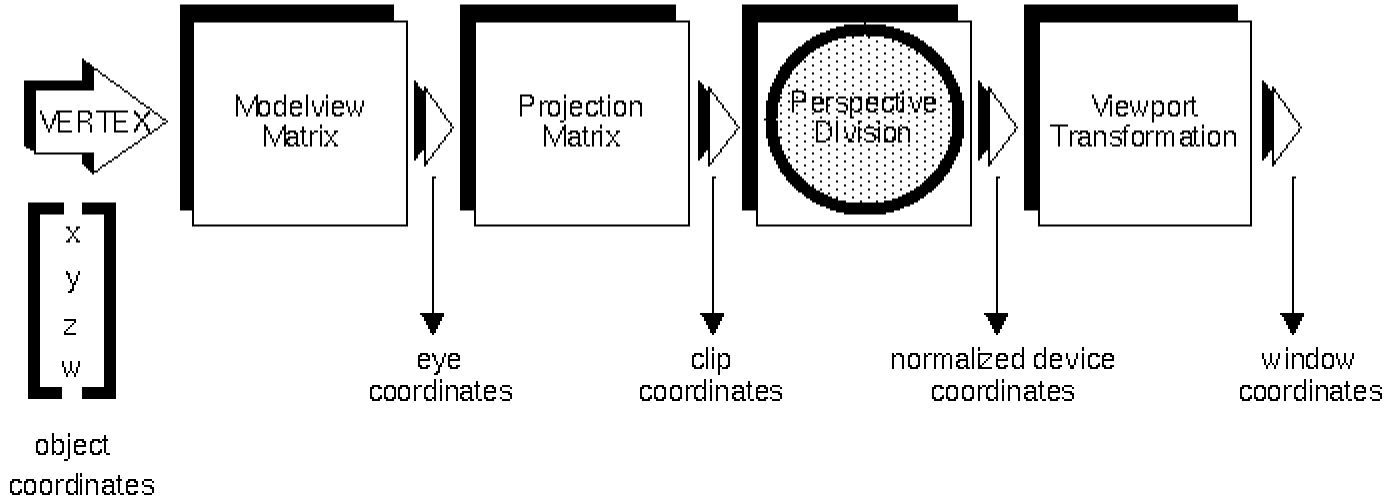
Perspective transformations

Transformation pipeline



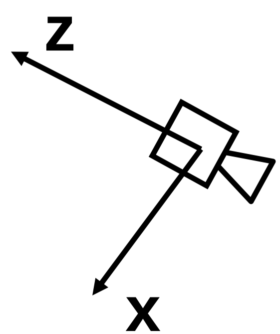
Modelview: model (position objects) + view (position the camera)

Projection: map viewing volume to a standard cube

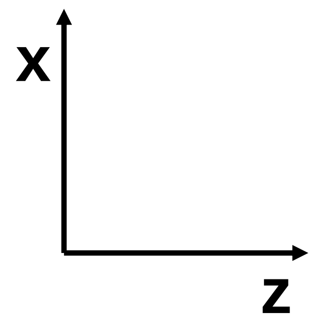
Perspective division: project 3D to 2D

Viewport: map the square [-1,1]x[-1,1]
in normalized device coordinates to the screen

Coordinate systems



Eye coordinates



World coordinates

World coordinates - fixed initial coord system; everything is defined with respect to it

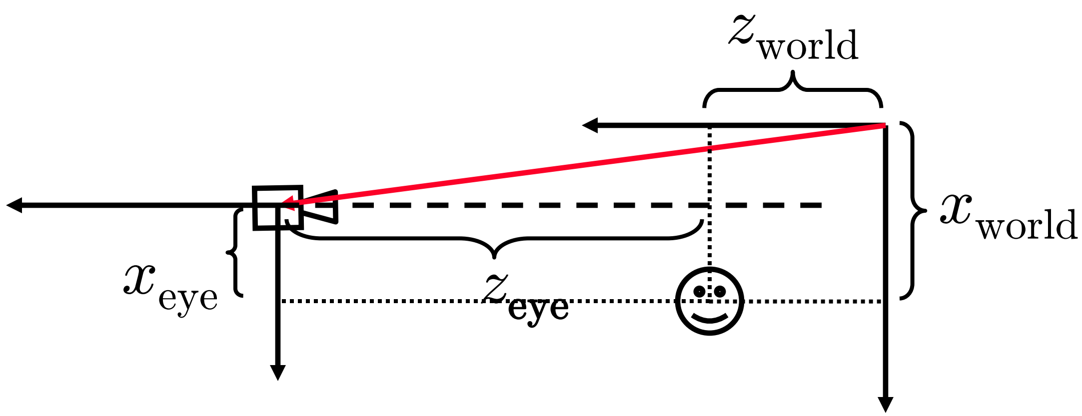
Eye coordinates - coordinate system attached to the camera; in this system camera looks down negative Z-axis

Positioning the camera

- Modeling transformation: reshape the object, orient the object, position the object with respect to the world coordinate system
- Viewing transformation: transform world coordinates to eye coordinates
- Viewing transformation is the *inverse* of the camera positioning transformation
- Viewing transformation should be rigid: rotation + translation
- Steps to get the right transform: first, orient the camera correctly, then translate it

Positioning the camera

Viewing transformation is the *inverse* of the camera positioning transformation:



Camera positioning: translate by (t_x, t_z)

Viewing transformation (world to eye):

$$x_{\text{eye}} = x_{\text{world}} - t_z$$

$$z_{\text{eye}} = x_{\text{world}} - t_x$$

Look-at positioning

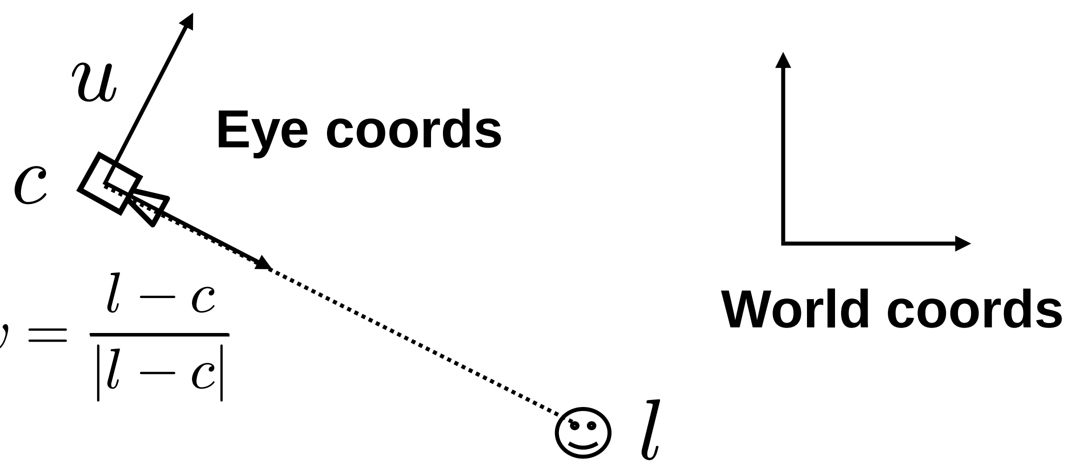
Find the viewing transform given the eye (camera) position, point to look at, and the up vector

- Need to specify two transforms: rotation and translation.
- translation is easy
- natural rotation: define implicitly using a point at which we want to look and a vector indicating the vertical in the image (*up vector*)

can easily convert the eye point to the direction vector of the camera axis; can assume up vector perpendicular to view vector

Look-at positioning

Problem: given two pairs of perpendicular unit vectors, find the transformation mapping the first pair into the second



$$v = \frac{l - c}{|l - c|}$$

Look-at positioning

Determine rotation first,
looking how coord vectors change:

$$R \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = v \qquad R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v \times u \qquad R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = u$$

$$R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R = [v \times u, u, -v]$$

Look-at positioning

Recall the matrix for translation:

$$T = \begin{bmatrix} 1 & 0 & 0 & c_x \\ 0 & 1 & 0 & c_y \\ 0 & 0 & 1 & c_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we have the camera positioning matrix, TR

To get the viewing transform, invert: $(TR)^{-1} = R^{-1}T^{-1}$

For rotation the inverse is the transpose!

$$R^{-1} = [v \times u \; u \; -v]^T = \begin{bmatrix} (v \times u)^T \\ u^T \\ -v^T \end{bmatrix}$$

Look-at viewing transformation

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = [e_x \; e_y \; e_z \; -c]$$

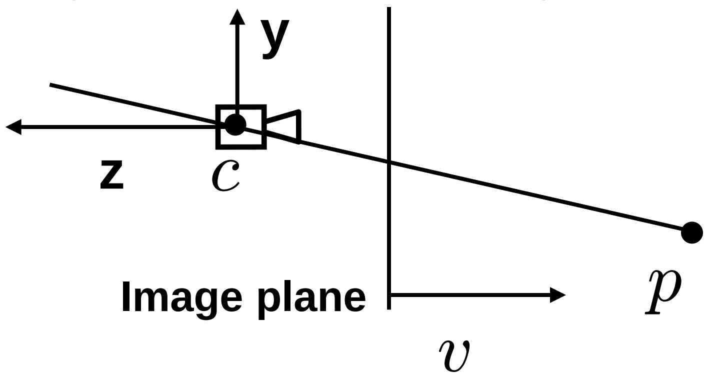
$$\boxed{V = R^{-1}T^{-1} = \begin{bmatrix} (v \times u)^T & -(v \times u \cdot c) \\ u^T & -(u \cdot c) \\ -v^T & (v \cdot c) \\ [0,0,0] & 1 \end{bmatrix}}$$

Positioning the camera in OpenGL

- imagine that the camera is an object and write a sequence of rotations and translations positioning it
- change each transformation in the sequence to the opposite
- reverse the sequence
- Camera positioning is done in the code *before* modeling transformations
- OpenGL does not distinguish between viewing and modeling transformation and joins them into the modelview matrix

Space to plane projection

In eye coordinate system



$$c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

projecting to the plane
z = -1

$$\text{Proj}(p) = \begin{bmatrix} -p_x/p_z \\ -p_y/p_z \\ -1 \end{bmatrix}$$

$$\text{Proj}(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Visibility

Objects that are closer to the camera occlude the objects that are further away

- All objects are made of planar polygons
- A polygon typically projects 1 to 1
- idea: project polygons in turn ; for each pixel, record distance to the projected polygon
- when writing pixels, replace the old color with the new one only if the new distance to camera for this pixel is less then the recorded one

Z-buffering idea

- *Problem:* need to compare distances for each projected point
- *Solution:* convert all points to a coordinate system in which (x,y) are image plane coords and the distance to the image plane increases when the z coordinate increases
- In OpenGL, this is done by the projection matrix

Z buffer

Assumptions:

- **each pixel has storage for a z-value, in addition to RGB**
- **all objects are "scanconvertible" (typically are polygons, lines or points)**

Algorithm:

```
initilize zbuf to maximal value

for each  object
    for each pixel (i,j) obtained by scan conversion
        if znew(i,j) < zbuf(i,j)
            zbuf(i,j) = znew(i,j) ;
            write pixel(i,j)
```

Z buffer

What are z values?

Z values are obtained by applying the projection transform, that is, mapping the viewing frustum to the standard cube.

Z value increases with th distance to the camera.

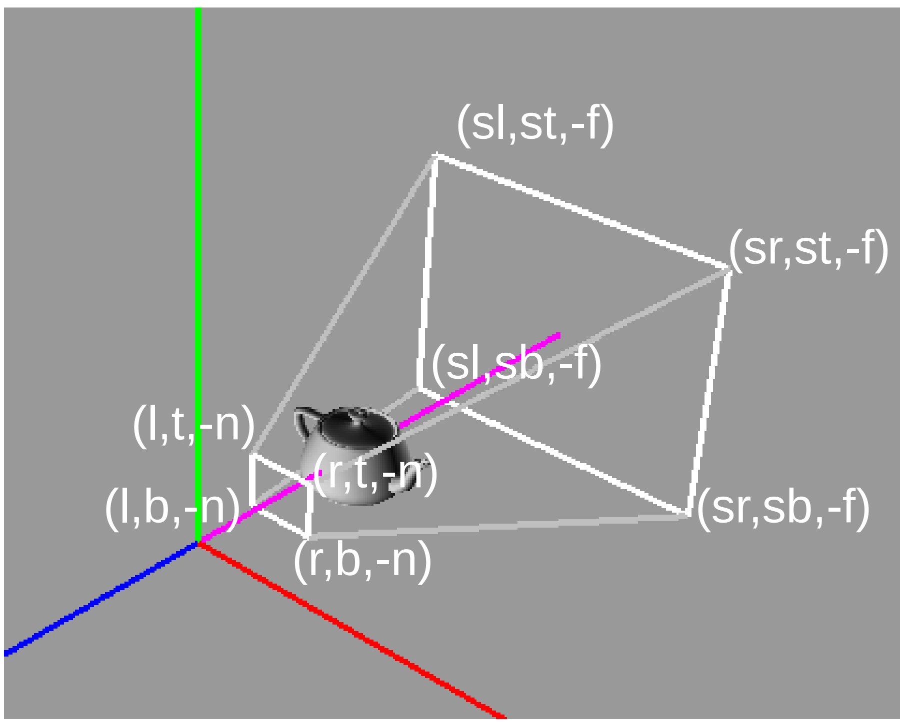
Z values for each pixel are computed for each pixel covered by a polygon using linear interpolation of z values at vertices.

Typical Z buffer size: 24 bits (same as RGB combined).

Camera specification

Define the dimensions of the viewing volume (frustum)

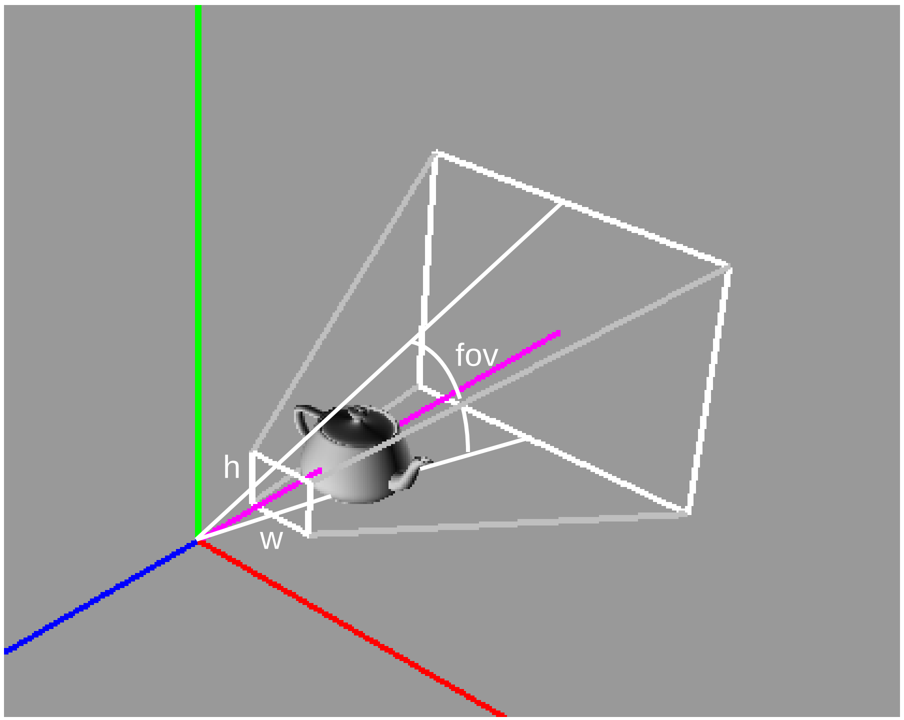
- **most general glFrustum(left,right,bottom,top,near,far)**



In the picture:
l = left
r = right
b = bottom
t = top
n = near
f = far
s = far/near

Camera specification

Less general but more convenient:
gluPerspective(field_of_view,aspect_ratio, near,far)



In the picture:
fov = field of view,
h/w = a=aspect ratio

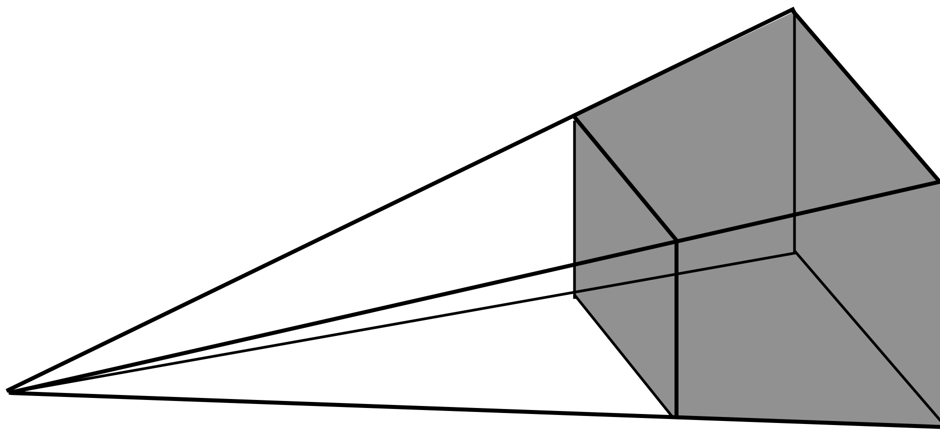
Relationship to frustum:

left	= -a*near*tan(fov/2)
right	= a*near*tan(fov/2)
bottom	= -a*near*tan(fov/2)
top	= a*near*tan(fov/2)

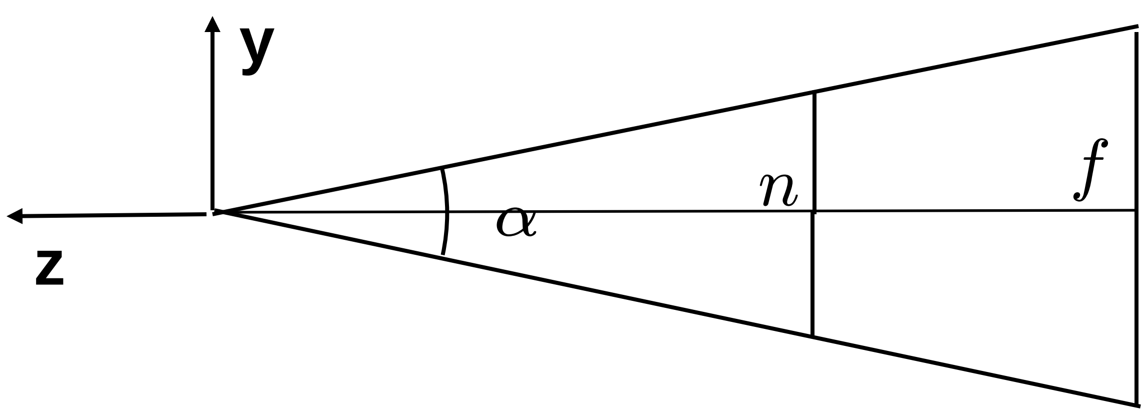
gluPerspective requires fov in degrees, not radians!

Viewing frustum

Volume in space that will be visible in the image



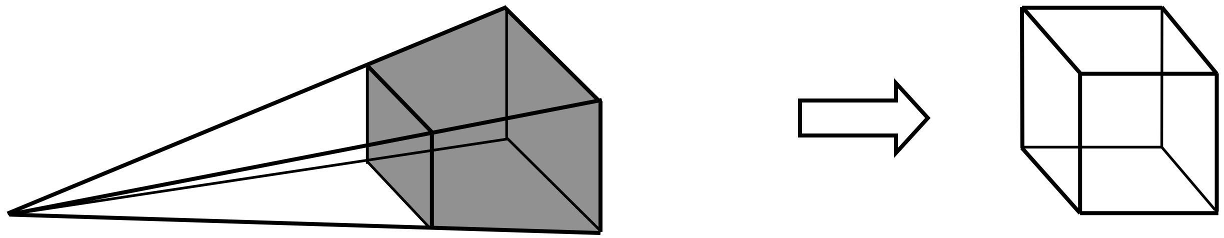
r is the aspect ratio of the image width/height



Projection transformation

Maps the viewing frustum into a standard cube
extending from -1 to 1 in each coordinate

(normalized device coordinates)



3 steps:

change the matrix of projection to keep z:

result is a parallelepiped

translate:

parallelepiped centered at 0

scale in all directions:

cube of of size 2 centered at 0

Projection transformation

change $\mathrm{Proj}(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$ **so that we keep z:**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \\ -p_z \end{bmatrix}$$

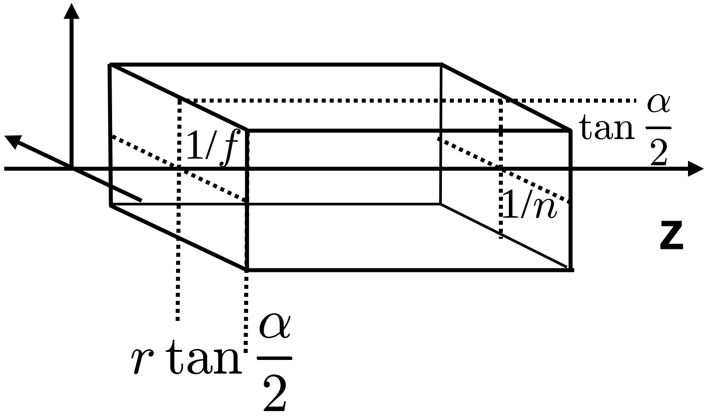
the homogeneous result corresponds to $\begin{bmatrix} -p_x/p_z \\ -p_y/p_z \\ -1/p_z \end{bmatrix}$

the last component increases monotonically with z!

Projection transformation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

maps the frustum to an axis-aligned parallelepiped



already centered in (x,y), center in z-direction and scale:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2}\left(\frac{1}{f}+\frac{1}{n}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{1}{r\tan\frac{\alpha}{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\frac{\alpha}{2}} & 0 & 0 \\ 0 & 0 & \frac{2}{\left(\frac{1}{n}-\frac{1}{f}\right)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection transformation

Combined matrix, mapping frustum to a cube:

$$P = ST \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{r \tan \frac{\alpha}{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan \frac{\alpha}{2}} & 0 & 0 \\ 0 & 0 & \frac{f+n}{n-f} & 2\frac{fn}{n-f} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

To get normalized image plane coordinates (valid range [-1,1] both) , just drop z in the result and convert from homogeneous to regular.

To get pixel coordinates, translate by 1, and scale x and y (Viewport transformation)