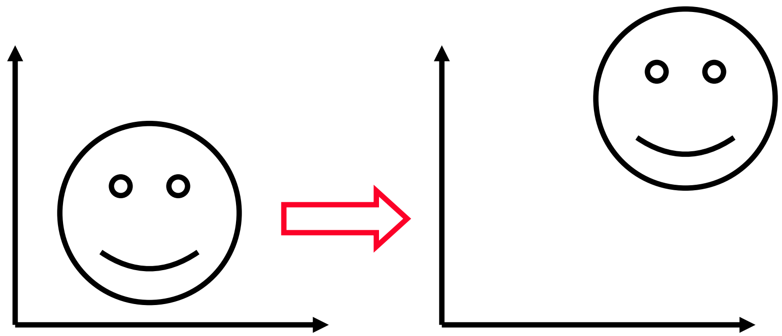
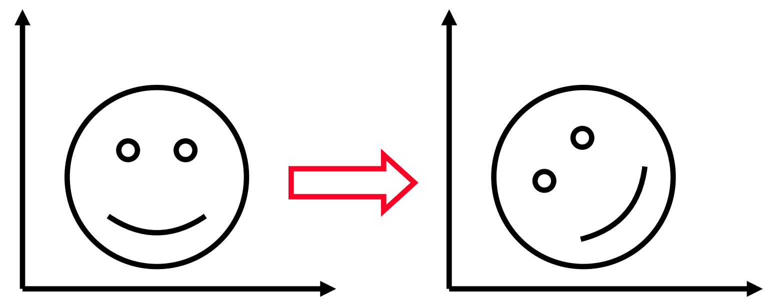
Transformations

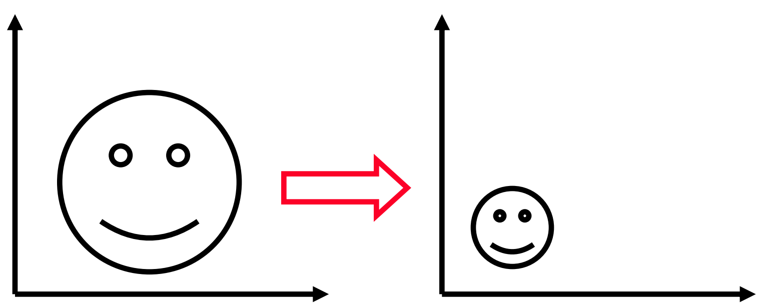
Examples of transformations:



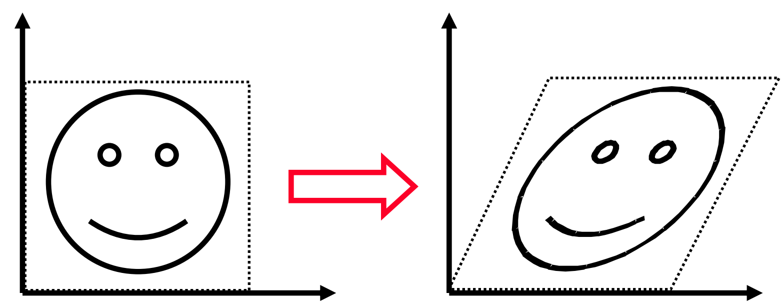
translation



rotation



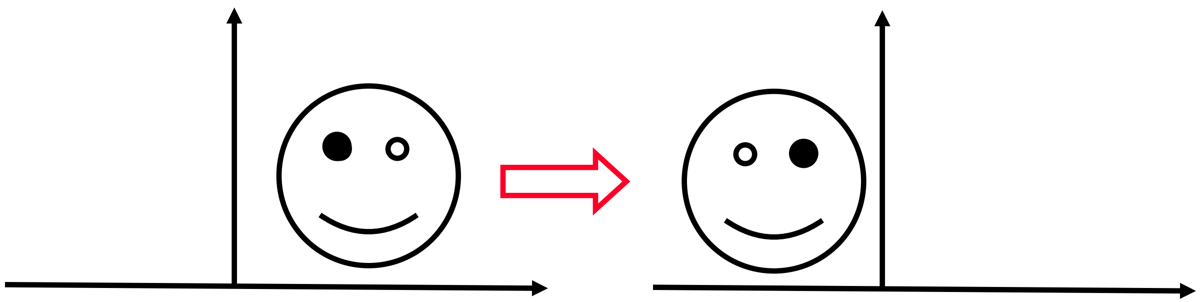
scaling



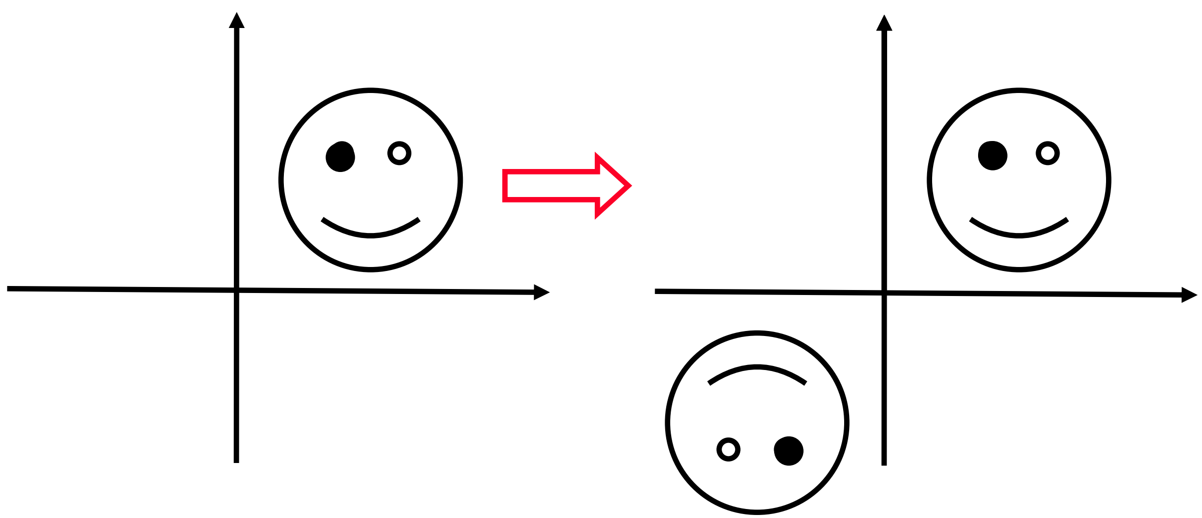
shear

Transformations

More examples:



reflection with respect to the y-axis



reflection with respect to the origin

Transformations

Linear transformations: take straight lines to straight lines.

All of the examples are linear.

Affine transformations: take paralllel lines to parallel lines.

All of the examples are affine,

an example of linear non-affine is perspective projection.

Orthogonal transformations: preserve distances, move all objects as rigid bodies.

rotation, translation and reflections are affine.

Composition of transformations

- **Order matters! (rotation * translation $\neq$ translation * rotation)**
- **Composition of transformations = matrix multiplication:**
  if T is a rotation and S is a scaling, then applying scaling first and rotation second is the same as applying transformation given by the matrix TS (note the order).
- **Reversing the order does not work in most cases**

Transformations and matrices

Any affine transformation can be written as

$$\begin{pmatrix} \mathbf{x'} \\ \mathbf{y'} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \qquad \mathbf{p'} = \mathbf{Ap}$$

Images of basis vectors under affine transformations:

$$\mathbf{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(column form of writing vectors)

$$\mathbf{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{Ae}_x = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{11} \\ \mathbf{a}_{21} \end{pmatrix} \qquad \mathbf{Ae}_y = \begin{pmatrix} \mathbf{a}_{21} \\ \mathbf{a}_{22} \end{pmatrix}$$

Transformations and matrices

Matrices of some transformations:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ **shear**

$$\begin{pmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{pmatrix}$$ **scale by factor s**

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$ **rotation**

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ **reflection with respect to the origin**

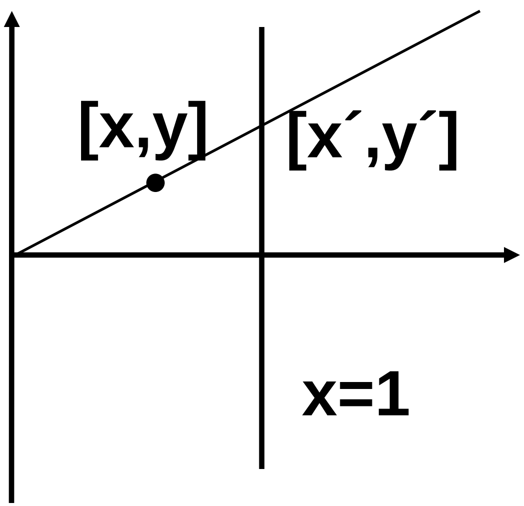
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$ **reflection with respect to y-axis**

Homogeneous coordinates

Problem

Even for affine transformations we cannot write them as a single 2x2 matrix; we need an additional vector for translations.

We cannot write all linear transformations even in the form Ax +b where A is a 2x2 matrix and b is a 2d vector. Example: perspective projection



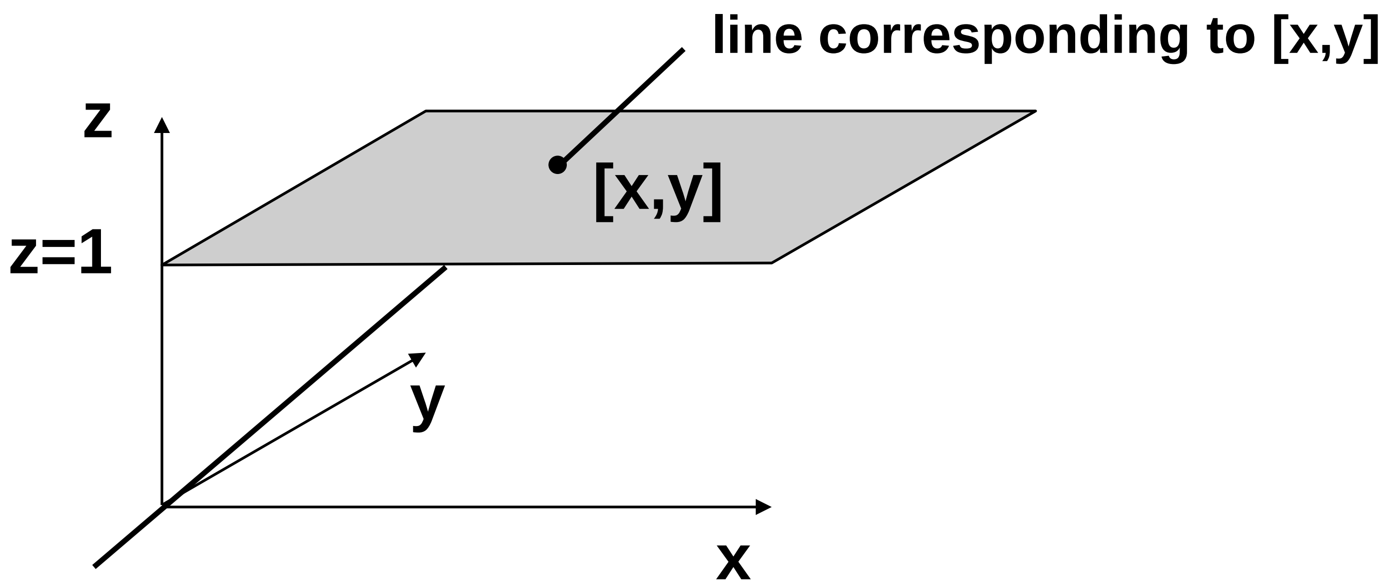
$$x' = 1$$
$$y' = y/x$$

equations not linear!

Homogeneous coordinates

- replace 2d points with 3d points, last coordinate 1
- for a 3d point (x,y,w) the corresponding 2d point is (x/w,y/w) if w is not zero
- each 2d point (x,y) corresponds to a line in 3d; all points on this line can be written as [kx,ky,k] for some k.
- (x,y,0) does not correspond to a 2d point, corresponds to a direction (will discuss later)
- Geometric construction: 3d points are mapped to 2d points by projection to the plane z =1 from the origin

Homogeneous coordinates



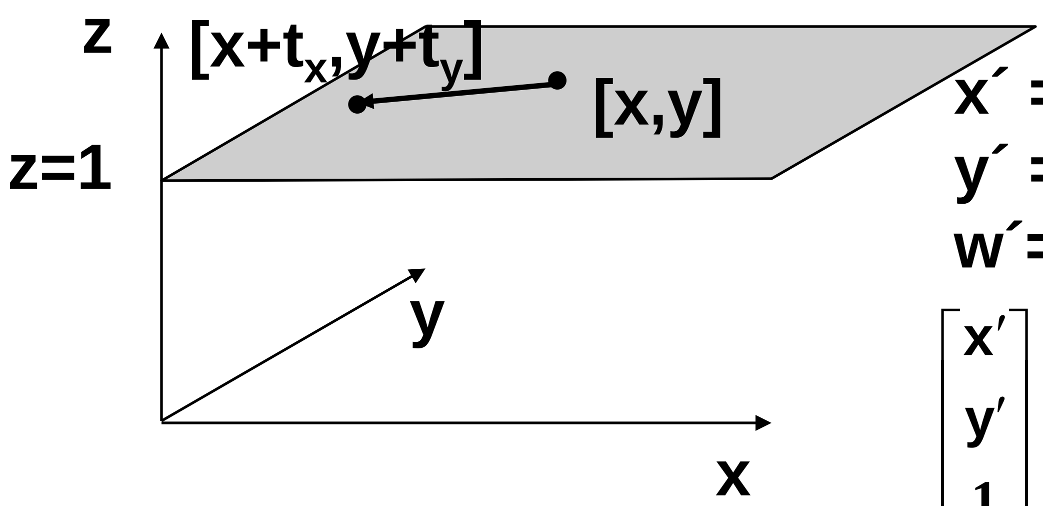
From homogeneous to 2d: [x,y,w] becomes [x/w,y/w]
From 2d to homogeneous: [x,y] becomes [kx,ky,k]
(can pick any nonzero k!)

Homogeneous transformations

Any linear transformation can be written in matrix form in homogeneous coordinates.

Example 1: translations

[x,y] in hom. coords is [x,y,1]



$x' = x+t_x = x+ t_x \cdot 1$

$y' = y+t_y = y+t_y \cdot 1$

$w' = 1$

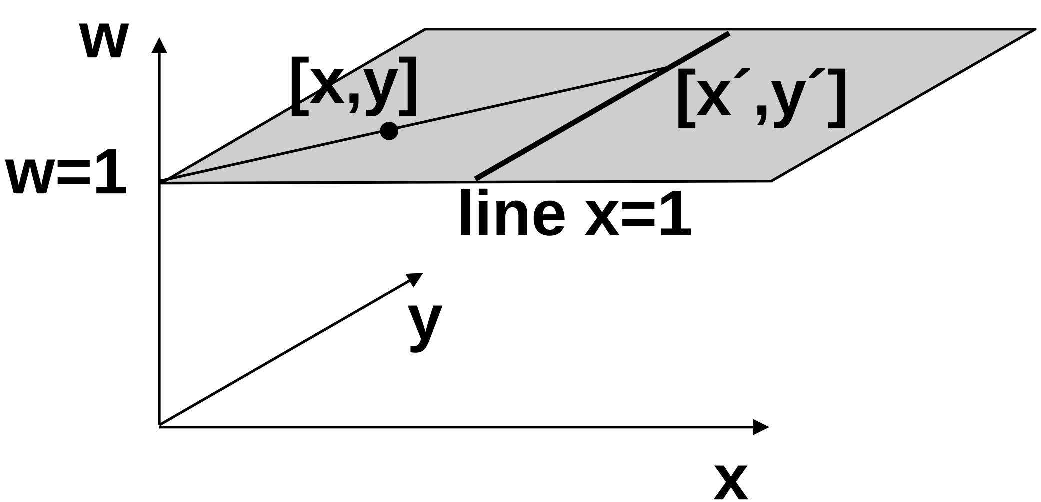
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous transformations

Example 2: perspective projection

x´ = 1
y´ = y/x
w´= 1

Can multiply all three components by the same number -- the 2D point won't change! Multiply by x.



x´ = x
y´ = y
w´= x

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Matrices of basic transformations

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ rotation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$ translation

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ scaling

$$\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ skew

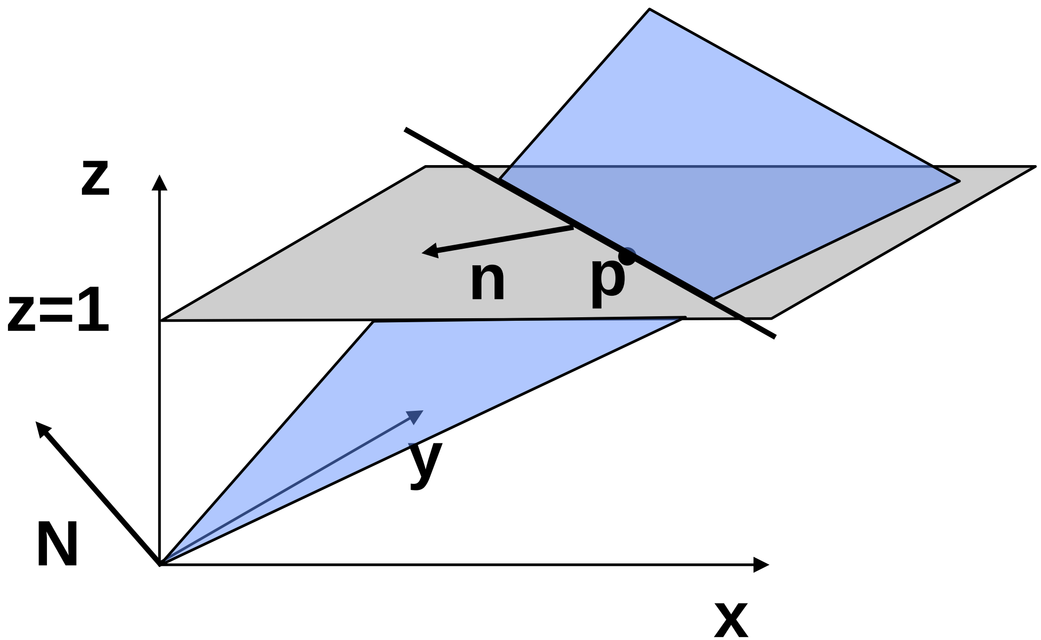
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$ general affine transform

Homogeneous line equation

Implicit line equation in 2D: $(n \cdot (q-p)) = 0$,

n = 2D vector, p = 2D point on the line.

Goal: rewrite in homogeneous coordinates.



2D point corresponds to a 3D line through origin; 2D line corresponds to a plane through origin

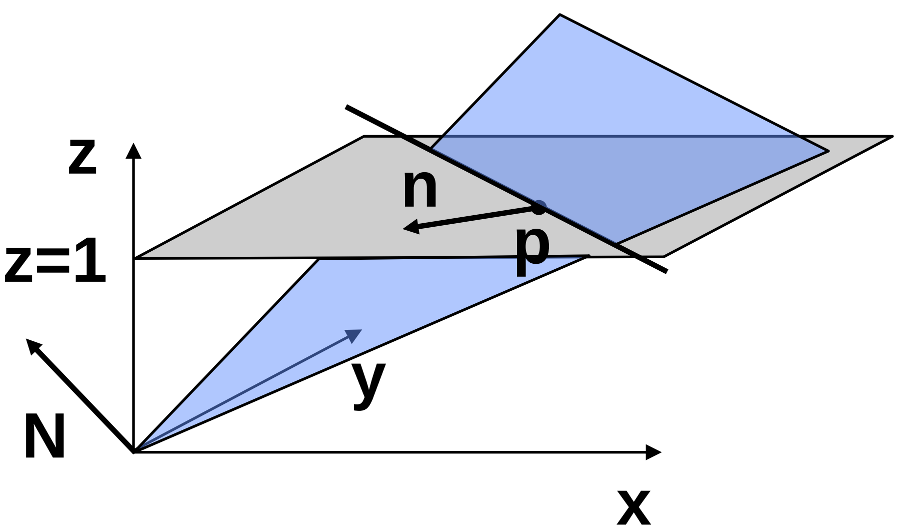
In other words, the 2D line is intersection of a plane through origin with the plane z=1.

Homogeneous line equation

Rewrite the line equation:

$$(n,q-p) = n_x x + n_y y + (n,-p) = ([n_x, n_{y,} - (n,p)],[x,y,1]) = (N,\overline{q})$$

where $N=[n_x,n_y,-(n,p)]$ is the normal to the plane corresponding to the line, and $\overline{q}$ is the homogeneous form of $q=[x,y]$: $\overline{q}=[x,y,1]$



Homogeneous form of the line equation:

$$(N \cdot \overline{q}) = 0$$

Homogeneous coordinates

regular 3D point to homogeneous:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \longrightarrow \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

homogeneous point to regular 3D:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ p_w \end{pmatrix} \longrightarrow \begin{pmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \end{pmatrix}$$

Translation and scaling

Similar to 2D; translation by a vector

$$t = [t_x, t_y, t_z]$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Nonuniform scaling in three directions

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations around coord axes

angle θ, around X axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

around Y axis:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

note where the minus is!

around Z axis:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$