Geometry review, part I

Geometry review I

Vectors and points

- points and vectors
- Geometric vs. coordinate-based (algebraic) approach
- operations on vectors and points

Lines

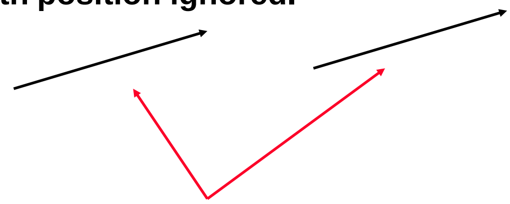
- implicit and parametric equations
- intersections, parallel lines

Planes

- implicit and parametric equations
- intersections with lines

Geometry vs. coordinates

<u>Geometric view:</u> **a vector is a directed line segment, with position ignored.**



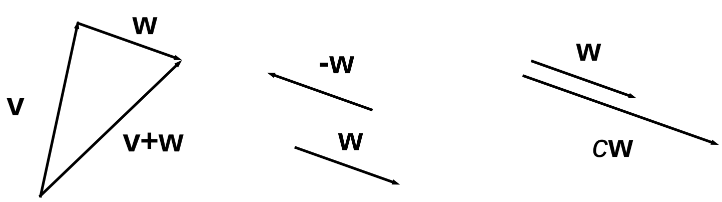
different line segments (but with the same length and direction) define the same vector

A vector can be thought of as a translation.

<u>Algebraic view:</u> **a vector is a pair of numbers**

Vectors and points

Vector = directed segment with position ignored



addition **negation** **multiplication by number**

Operations on points and vectors:

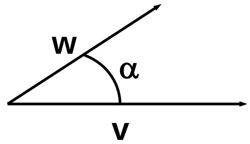
point - point = vector

point + vector = point

Dot product

Dot product: used to compute projections, angles and lengths.

Notation: (w·v) = dot product of vectors w and v.



$(w\cdot v) = |w|\,|v| \cos\alpha$,

$|v|$ = length of v

Properties:

- if w and v are perpendicular, $(w\cdot v) = 0$
- $(w\cdot w) = |w|^2$
- angle between w and v: $\cos\alpha = (w\cdot v)/|w||v|$
- length of projection of w on v: $(w\cdot v)/|v|$

Coordinate systems

For computations, vectors can be described as pairs (2D), triples (3D), … of numbers.

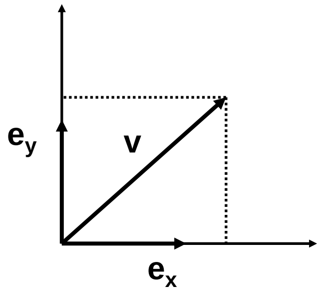
Coordinate system (2D) =
point (origin) + 2 basis vectors.

Orthogonal coordinate system:
basis vectors perpendicular.

Orthonormal coordinate system:
basis vectors perpendicular and of unit length.

Representation of a vector in a coordinate system:
2 numbers equal to the lengths (signed) of projections on basis vectors.

Operations in coordinates



$$v = \underbrace{(v\cdot e_x)}_{v_x}e_x + \underbrace{(v\cdot e_y)}_{v_y}e_y$$

works only for orthonormal coordinates!

$$v = v_x\, e_x + v_y\, e_y = [\, v_x, v_y\,]$$

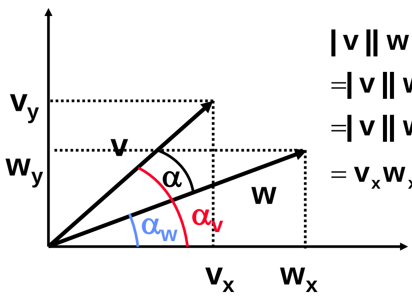
Operations in coordinate form:

$v + w = [\, v_x, v_y\,] + [\, w_x, w_y\,] = [v_x + w_x, v_y + w_y]$

$-w = [\, -w_x, -w_y\,]$

$\alpha\, w = = [\alpha\, w_x, \alpha\, w_y\,]$

Dot product in coordinates



$$|\, v\,||\, w\,|\cos\alpha = |\, v\,||\, w\,|\cos(\alpha_v - \alpha_w)$$
$$= |\, v\,||\, w\,|(\cos\alpha_v\cos\alpha_w + \sin\alpha_v\sin\alpha_w)$$
$$= |\, v\,||\, w\,|(v_x w_x / |\, v\,||\, w\,| + v_y w_y / |\, v\,||\, w\,|)$$
$$= v_x w_x + v_y w_y$$

Linear properties become obvious:

((v+w)·u) = (v·u) + (w·u)

(*a*v·w) = *a*(v·w)

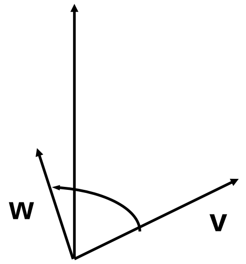
3D vectors

Same as 2D (directed line segments with position ignored), but we have different properties.

- In 2D, the vector perpendicular to a given vector is unique (up to a scale).
- In 3D, it is not.
- Two 3D vectors in 3D can be multiplied to get a vector (vector or cross product).
- Dot product works the same way, but the coordinate expression is

  $(v \cdot w) = v_x w_x + v_y w_y + v_z w_z$

Vector (cross) product



$\mathbf{v} \times \mathbf{w}$ has length $|\mathbf{v}||\mathbf{w}| \sin\alpha$

= area of the parallelogram with two sides given by v and w, and is perpendicular to the plane of v and w.

Direction (up or down) is determined by the right-hand rule.

$(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$

$(c\mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \times \mathbf{w})$

$\boxed{\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}}$ unlike a product of numbers or dot product, vector product is not commutative!

Vector product

Coordinate expressions

$\mathbf{v} \times \mathbf{w}$ **is perpendicular to v, and w:**

$$(\mathbf{u}\cdot\mathbf{v}) = 0 \qquad (\mathbf{u}\cdot\mathbf{w}) = 0$$

the length of u is $|\mathbf{v}||\mathbf{w}|\,\mathbf{sin}\,\alpha$ **:**

$$(\mathbf{u}\cdot\mathbf{u}) = |\mathbf{v}|^2|\mathbf{w}|^2\,\mathbf{sin}^2\alpha = |\mathbf{v}|^2|\mathbf{w}|^2(1-\mathbf{cos}^2\alpha)$$

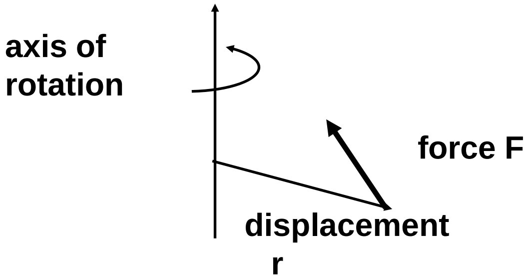
$$= |\mathbf{v}|\ |\mathbf{w}|\ \left(|\mathbf{v}||\mathbf{w}| - (\mathbf{v},\mathbf{w})\right)$$

Solve three equations for u_x, u_y, u_z

Vector product

Physical interpretation: torque

torque = $\mathbf{r} \times \mathbf{F}$



Vector product

Coordinate expression:

$$\det\begin{bmatrix} e_x & e_y & e_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = \left[\det\begin{bmatrix} v_y & v_z \\ w_y & w_z \end{bmatrix}, -\det\begin{bmatrix} v_x & v_z \\ w_x & w_z \end{bmatrix}, \det\begin{bmatrix} v_x & v_y \\ w_x & w_y \end{bmatrix} \right]$$

Notice that if $v_z = w_z = 0$, that is, vectors are 2D, the cross product has only one nonzero component (z) and its length is the determinant

$$\det\begin{bmatrix} v_x & v_y \\ w_x & w_y \end{bmatrix}$$

Vector product

More properties

$$(a \cdot (b \times c)) = b(a \cdot c) - c(a \cdot b)$$

$$((a \times b) \cdot (c \times d)) = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d)$$