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# FORMAC MEETS PAPPUS

## SOME OBSERVATIONS ON ELEMENTARY ANALYTIC GEOMETRY BY COMPUTER

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**1. Introduction.** One of the truly great advances in mathematics was the algebraization of geometry via the notion of a coordinate system. The broad outlines of this program were indicated in the "Discours de la méthode" of René Descartes (1637) while the essential features were grasped, although not made explicit, by Pierre de Fermat. There is no doubt that Descartes regarded his invention as a universal method, and he wrote that it removed geometry as much from its previous condition as the orations of Cicero were removed from simple ABC's.

The method of Descartes is frequently regarded by students and by teachers as a "machine" into which one feeds the hypotheses of certain geometric situations and which is guaranteed to "grind out" the desired conclusions given sufficient patience on the part of the problem solver. However, it is no denigration of Descartes to assert what also has long been known: that many elementary situations give rise to impossibly long and tedious algebraic computations, and hence the universal method which replaces brains by brawn founders upon the rock of limited human patience and endurance. Ways around are then sought; these include clever coordinate systems, special transformations, determinants, other methods of abridged notation, special devices, constructions, tricks, etc., etc. Several of these devices have subsequently become of prime importance in their own right.

The object of the present paper is to describe what happens when these difficulties are deliberately met broadside and overcome by making use of the symbolic manipulation possibilities of electronic computers. The problem to which we have applied Descartes' method is a classic theorem of Pappus. The language in which we tackled the problem was FORMAC, and the machine was an IBM 360/50 at Brown University with 256 K (K = 1,024) bytes of core storage. As of Summer 1968, this is considered to be medium-sized storage.

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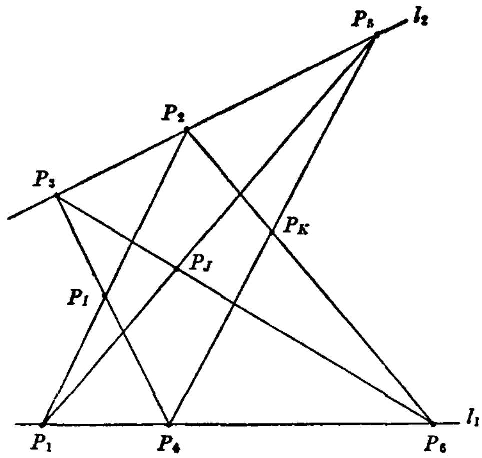


FIG. 1

**2. The Theorem of Pappus.** Pappus of Alexandria (c. 320 A.D.) was one of the last significant mathematicians of antiquity. There are a number of theorems which bear his name, but the one we have in mind is as follows.

*Let $l_1$ and $l_2$ be two straight lines in the plane. On $l_1$ take three points $P_1$, $P_4$, $P_6$ arbitrarily and on $l_2$ take three points $P_3$, $P_2$, $P_5$ arbitrarily. Now connect up the points in the criss-cross fashion indicated in the figure. Let the points of intersection of the three criss-crosses be designated by $P_I$, $P_J$, $P_K$, respectively. Then, $P_I$, $P_J$, and $P_K$ are collinear.*

This beautiful theorem (see Figure 1), it turns out, is basic to certain investigations in the foundations of projective geometry. (If Pappus' theorem holds in a projective plane, then the plane is isomorphic to a projective plane over a field.) The interested reader can find information on ancient methods of proof (see p. 289 in [7]). A modern analytic proof can be found on p. 81 of [4].

The program of the present paper is to assign coordinates to $P_1, \cdots, P_6$, to solve for the intersections $P_I$, $P_J$, $P_K$ in terms of those coordinates and then simply to verify by algebra that the points $P_I$, $P_J$, $P_K$ are, in fact, collinear.

**3. Details of the Method.** We shall assign general (letter) coordinates to the points. We shall try insofar as possible not to take advantage of the projective group nor of the group of rigid motions or dilations. This, following a remark of Professor Ulf Grenander, can be described as "the method of artificial stupidity" and is to be contrasted with current studies in Computer Science called "artificial intelligence." We did not wholly succeed in this. For reasons explained later we used a rigid motion to place the configuration in a simple position. We shall employ the usual rectangular coordinates, although, effectively, we will be using homogeneous coordinates in that we have arranged our computations so that no divisions occur.

The first formula we need to work out is the point of intersection of a simple criss-cross (Figure 2).

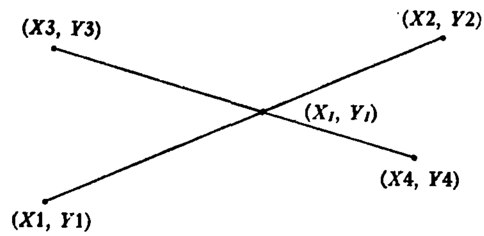


FIG. 2

Let two lines be determined by (X1, Y1), (X2, Y2) and (X3, Y3), (X4, Y4). The point of intersection is given by

$$X_I = \mathrm{NI/DI}, \qquad Y_I = \mathrm{MI/DI}, \tag{3.1}$$

where

(3.2)*
$$\begin{aligned}\mathrm{NI} &= \mathrm{Y2\,X3\,X1 - Y4\,X3\,X1 - X4\,Y2\,X1 + X4\,Y3\,X1 - X3\,X2\,Y1} \\ &\quad \mathrm{+ X4\,X2\,Y1 + Y4\,X3\,X2 - X4\,Y3\,X2}, \\ \mathrm{MI} &= \mathrm{Y3\,Y2\,X1 - Y4\,Y2\,X1 - Y3\,X2\,Y1 + Y4\,X2\,Y1 - Y4\,X3\,Y1} \\ &\quad \mathrm{+ X4\,Y3\,Y1 + Y4\,Y2\,X3 - X4\,Y3\,Y2}, \\ \mathrm{DI} &= \mathrm{Y3\,X1 - Y4\,X1 - X3\,Y1 + X4\,Y1 - Y3\,X2 + Y4\,X2 + Y2\,X3} \\ &\quad \mathrm{- X4\,Y2}.\end{aligned}$$

Similar formulas pertain to the points $P_J$ and $P_K$.

*Formulas listed with an asterisk (*) were derived by the computer.* The interested reader is invited to check them by whatever means he has at his disposal.

Notice that each coordinate $X_I$, $Y_I$ is the ratio of two sums of 8 monomials in the variables X1, Y1, etc.

The second formula we need is the condition that three points $P_I$: $(X_I, Y_I)$, $P_J$: $(X_J, Y_J)$, $P_K$: $(X_K, Y_K)$ be collinear. This condition is

$$\begin{vmatrix} X_I & Y_I & 1 \\ X_J & Y_J & 1 \\ X_K & Y_K & 1 \end{vmatrix} = 0. \tag{3.3}$$

*In this paper, determinants are only employed as a shorthand for their brute expansions.*

We can use (3.1) to rewrite this as

$$\begin{vmatrix} \mathrm{NI/DI} & \mathrm{MI/DI} & 1 \\ \mathrm{NJ/DJ} & \mathrm{MJ/DJ} & 1 \\ \mathrm{NK/DK} & \mathrm{MK/DK} & 1 \end{vmatrix} = 0, \tag{3.4}$$

or as

$$\text{(3.5)} \qquad \frac{1}{(\mathrm{DI})(\mathrm{DJ})(\mathrm{DK})}\begin{vmatrix} \mathrm{NI} & \mathrm{MI} & \mathrm{DI} \\ \mathrm{NJ} & \mathrm{MJ} & \mathrm{DJ} \\ \mathrm{NK} & \mathrm{MK} & \mathrm{DK} \end{vmatrix} = 0.$$

Call the determinant part of (3.5) DE. Thus, DE is the sum of six terms of the form (NI)(MJ)(DK), etc. Each of the NI, MJ, etc. is the sum of eight monomials in the Xi, Yi, and so (NI)(MJ)(DK) will consist of (at most) $8\times 8\times 8 = 512$ monomials. The determinant DE will consist of (before possible reductions) $6\times 512 = 3{,}072$ monomials. (To put this figure in some perspective, recall that the complete expansion of an $n\times n$ determinant consists of $n!$ terms.) It now should be clear why a broadside attack on Pappus' Theorem is tedious.

**4. Machine Proof of Pappus' Theorem.** FORMAC is a computing system that provides the capability of doing nonnumerical manipulation as well as numerical calculation. The interested reader may consult references [**8**] and [**9**] for details. The system consists of a preprocessor program and the PL/I compiler. The preprocessor translates the FORMAC program into a PL/I program which in turn calls various FORMAC routines at execution time. The PL/I compiler is that program which translates the PL/I language into machine language.

Numerical calculation can be done either in floating point arithmetic or in rational arithmetic. For example, $2\times(3/10)$ can either be computed as .6 or as the rational number 3/5. Some of the algebraic capabilities of FORMAC are expansion of products of sums, substitution of one expression for another, symbolic differentiation, and automatic simplification. Simplification of symbolic expressions by computer is by no means a trivial task. It requires explicit programming of such simple transformations as $x^1\to x$, $y+0\to y$, $xy-yx\to 0$. In addition, the program must run through every expanded algebraic expression and combine like terms. A special version of FORMAC would be required to deal with noncommutative multiplication.

The nonnumeric features of FORMAC that were most essential in the Pappus program were expansion of products and automatic simplification of products. This simplification is crucial to the economy of memory space. The computer form of these expansions required a considerable amount of core storage. In our memory of 256 K bytes, the Pappus program itself required only 35 statements or about 11,000 bytes. The FORMAC system required about 134,000 bytes leaving approximately 117,000 bytes for the algebraic paper work. In the IBM 360, one byte will hold one symbolic (i.e., alphabetic or numeric) character.

A FORMAC program was written which accepted the symbolic coordinates of the points $P_1, \cdots, P_6$ as input and criss-crossed them in the following order: $P_1P_2$ with $P_3P_4$, $P_1P_5$ with $P_3P_6$, $P_4P_5$ with $P_2P_6$. Call the three points of intersection $P_I$, $P_J$, $P_K$ the *Pappus points* for $P_1, \cdots, P_6$. The program then computed the determinant DE in terms of the symbolic coordinates of $P_1, \cdots, P_6$.

We now assume that the lines $l_1$ and $l_2$ are parametrized as follows:

$$l_1: \begin{cases} x = t \\ y = 0 \end{cases} \qquad l_2: \begin{cases} x = ct \\ y = at + b. \end{cases} \tag{4.1}$$

The input to our Pappus program was therefore

$$\begin{array}{ll} \mathrm{X1} = \mathrm{T1} & \mathrm{Y1} = 0 \\ \mathrm{X2} = \mathrm{CT2} & \mathrm{Y2} = \mathrm{B} + \mathrm{AT2} \\ \mathrm{X3} = 0 & \mathrm{Y3} = \mathrm{B} \\ \mathrm{X4} = \mathrm{T4} & \mathrm{Y4} = 0 \\ \mathrm{X5} = \mathrm{CT3} & \mathrm{Y5} = \mathrm{B} + \mathrm{AT3} \\ \mathrm{X6} = \mathrm{T6} & \mathrm{Y6} = 0. \end{array} \tag{4.2}$$

The above simplifications were adopted after it was found that using six general points on two arbitrary lines caused core space to be exceeded. The output (after 4.52 minutes of execution time which included compile and preprocessor time) was

$$\mathrm{DE} = 0. \tag{4.3)*}$$

As a curiosity, we have reproduced in (4.4)* one of the six terms in the determinant DE. It should be observed that special selection of the coordinates and other cancellations and simplifications have reduced the number of monomials to 41 from a possible 512.

$$\begin{aligned} \mathrm{P1} = {} & \mathrm{B^3\ A\ T2\ C\ T1\ T6\ T3\ T4 + B^2\ A^2\ T2^2\ C\ T1\ T6\ T3\ T4} \\ & + \mathrm{2\ B^2\ A^2\ T2\ T1^2\ T6\ T3\ T4 - B^4\ T2\ C^2\ T6\ T3\ T4} \\ & + \mathrm{B^3\ A\ T2^2\ C^2\ T6\ T3\ T4 + B^4\ T2\ C^2\ T1\ T3\ T4} \\ & - \mathrm{B^3\ A\ T2^2\ C^2\ T1\ T3\ T4 - 2\ B^2\ A^2\ T2\ T1\ T6^2\ T3\ T4} \\ & - \mathrm{2B^3\ A\ T2\ C\ T6^2\ T3\ T4 + B^3\ A\ T2\ C\ T1^2\ T3\ T4} \\ & - \mathrm{B^2\ A^2\ T2^2\ C\ T1^2\ T3\ T4 + B^4\ T2\ C\ T1\ T6\ T4} \\ & + \mathrm{B^3\ A\ T2^2\ C\ T1\ T6\ T4 + B^3\ A\ T2\ T1^2\ T6\ T4 + B^4\ T2^2\ C^2\ T6\ T4} \\ & - \mathrm{B^4\ T2^2\ C^2\ T1\ T4 + B\ A^3\ T2\ T1^2\ T6\ T3^2\ T4} \\ & - \mathrm{B^3\ A\ T2\ C^2\ T6\ T3^2\ T4 + B^3\ A\ T2\ C^2\ T1\ T3^2\ T4} \\ & - \mathrm{B\ A^3\ T2\ T1\ T6^2\ T3^2\ T4 - B^2\ A^2\ T2\ C\ T6^2\ T3^2\ T4} \\ & + \mathrm{B^2\ A^2\ T2\ C\ T1^2\ T3^2\ T4 - B^3\ A\ T2\ T1\ T6^2\ T4 - B^4\ T2\ C\ T6^2\ T4} \\ & - \mathrm{B^3\ A\ T2^2\ C\ T1^2\ T4 + B^2\ A^2\ T2\ T1\ T6\ T3\ T4^2} \\ & + \mathrm{B\ A^3\ T2^2\ T1\ T6\ T3\ T4^2 + B^3\ A\ T2\ C\ T6\ T3\ T4^2} \\ & + \mathrm{B^2\ A^2\ T2^2\ C\ T6\ T3\ T4^2 - B^3\ A\ T2\ C\ T1\ T3\ T4^2} \\ & - \mathrm{B^2\ A^2\ T2^2\ C\ T1\ T3\ T4^2 - B^2\ A^2\ T2\ T1^2\ T3\ T4^2} \\ & - \mathrm{B\ A^3\ T2^2\ T1^2\ T3\ T4^2 + B^3\ A\ T2\ T1\ T6\ T4^2} \\ & + \mathrm{B^2\ A^2\ T2^2\ T1\ T6\ T4^2 + B^4\ T2\ C\ T6\ T4^2} \\ & + \mathrm{B^3\ A\ T2^2\ C\ T6\ T4^2 - B^4\ T2\ C\ T1\ T4^2 - B^3\ A\ T2^2\ C\ T1\ T4^2} \\ & - \mathrm{B^3\ A\ T2\ T1^2\ T4^2 - B^2\ A^2\ T2^2\ T1^2\ T4^2.} \end{aligned} \tag{4.4)*}$$

**5. Pascal's Theorem.** The theorem of Pappus is a special case of the more general theorem of Blaise Pascal: *if a hexagon is inscribed in a conic then the intersections of opposite sides of the hexagon are collinear.* This theorem was discovered in 1640 when Pascal was 16. An immense literature has grown up around this so-called "mystic hexagram." For example, six given points will (in some order) determine sixty different hexagrams. If these points lie on a conic, sixty Pascal lines will be determined. These lines fall into twenty groups of three, each group passing through a common point. These twenty points lie by fours on fifteen lines, three of the lines going through each point. See, e.g., G. Salmon, Appendix. Pascal was himself reputed to have derived four hundred other theorems from his theorem.

A broadside attack on Pascal by FORMAC might go like this: Parametrize the conic in some way, e.g., take the ellipse $x=a\cos t$, $y=b\sin t$ and then take the six points $P_i$ on the ellipse as $x_i=a\cos t_i$, $y_i=b\sin t_i$, $i=1, 2, \cdots, 6$. Now use the program to form the Pappus points for $P_i$ and then form DE. FORMAC has the capability of dealing with sin and cos symbolically, and can be instructed to reduce by using $\sin^2 x=1-\cos^2 x$.

A second possibility is to use $y=(b/a)\sqrt{a^2-x^2}$ and take $P_i$: $(x_i, y_i)$ where $y_i=(b/a)\sqrt{a^2-x_i^2}$. FORMAC also has fractional power capabilities.

Neither of these approaches succeeded with our 256K memory. (In Summer, 1968, the storage of the Brown computer was increased to 512 K. This was still insufficient.) The message "no more free list space available" was received before the second intersection point PK was computed. An indirect machine approach to Pascal will be indicated shortly.

**6. New Geometrical Theorems by Machine.** By leaving a little slack in the situation, one can come up with new theorems or generalizations of old theorems. For example, let us *not* require that $P_1, \cdots, P_6$ lie on two straight lines but compute, quite generally, DE and DI, DJ, DK for arbitrary positions of $P_1, \cdots, P_6$. Since DE/DIDJDK $=2$ times the signed area of the triangle $P_IP_JP_K$, we can obtain a complete formula for this area and hence the possibility of deriving theorems. The following theorem was obtained after an inspection of the machine print out.

THEOREM.* *Let $P_1, \cdots, P_6$ be six points in the plane and let $P_I$, $P_J$, $P_K$ be their three Pappus points in the order previously adopted. Consider $P_1, \cdots, P_5$ to be fixed while $P_6$ is variable. The locus of points $P_6$ such that the signed area $\sigma$ of the Pappus triangle $P_IP_JP_K$ is a constant is a conic. If $\sigma=0$ then the conic passes through $P_1, \cdots, P_5$. As $\sigma$ varies, the conic varies in a pencil of conics.*

*Proof.* The analytic condition for the constancy of the Pappus area is

$$\text{DE/DIDJDK} = \sigma = \text{constant}, \quad \text{or} \tag{6.1}$$

$$\text{DE} - \sigma\text{DIDJDK} = 0. \tag{6.2}$$

We used the following input to the Pappus program: (a simple rigid motion does it)

$$X1 = 0,\ Y1 = 0;\quad X2 = F,\ Y2 = G;\quad X3 = P,\ Y3 = E;$$

$$X4 = A,\ Y4 = 0;\quad X5 = H,\ Y5 = K;\quad X6 = B,\ Y6 = C$$

Single letter variables are preferable to subscripted variables insofar as the storage requirements are less. Now we have

$$\begin{aligned}
\mathrm{DE} = &- E^2 F^2 K H A C + P^2 G^2 K H A C + E^2 F^2 B K A C\\
&- P^2 G^2 B K A C - E^2 G B H^2 A C + E G^2 B H^2 A C\\
&+ E^2 G F H^2 A C - E P G^2 H^2 A C + P^2 G B K^2 A C\\
&- E F^2 B K^2 A C - P^2 G F K^2 A C + E P F^2 K^2 A C\\
&+ E^2 F K H A^2 C - P G^2 K H A^2 C + E^2 G B H A^2 C\\
&- E G^2 B H A^2 C - E^2 G F H A^2 C + E P G^2 H A^2 C\\
&- E^2 F B K A^2 C + P G^2 B K A^2 C + E F B K^2 A^2 C\\
&- P G B K^2 A^2 C + P G F K^2 A^2 C - E P F K^2 A^2 C\\
&+ E^2 G B^2 K H A - E G^2 B^2 K H A - E^2 G F B^2 K A\\
&+ E P G^2 B^2 K A + E G F B^2 K^2 A - E P G B^2 K^2 A\\
&- P^2 G K H A C^2 + E F^2 K H A C^2 + P^2 G F K A C^2\\
&- E G F H^2 A C^2 + E P G H^2 A C^2 - E P F^2 K A C^2\\
&- E F K H A^2 C^2 + P G K H A^2 C^2 + E G F H A^2 C^2\\
&- E P G H A^2 C^2 - P G F K A^2 C^2 + E P F K A^2 C^2\\
&- E^2 G B K H A^2 + E G^2 B K H A^2 + E^2 G F B K A^2\\
&- E P G^2 B K A^2 - E G F B K^2 A^2 + E P G B K^2 A^2
\end{aligned} \tag{6.3*}$$

$$\begin{aligned}
\mathrm{DIDJDK} = &\ 2 G B K H A C - G F K H A C - P G K H A C - E G F H A C\\
&- E^2 F H A C + E P G H A C + P G^2 H A C - E F B K A C\\
&+ P G B K A C + E P F K A C - P^2 G K A C + E G H^2 A C\\
&+ G^2 H^2 A C + 2 E F B K H C - 2 P G B K H C + P G F K H C\\
&- E P F K H C + P^2 G K H C - E F^2 K H C - E G H A^2 C\\
&- G^2 H A^2 C - G B K A^2 C + P G K A^2 C + E G F H^2 C\\
&+ E^2 F H^2 C - E P G H^2 C - P G^2 H^2 C - E G B K H A\\
&- G^2 B K H A + E G F K H A + P G^2 K H A + E^2 G F H A\\
&- E P G^2 H A + E G F B K A - P G^2 B K A - E P G F K A\\
&+ P^2 G^2 K A - E G^2 H^2 A + G F B K^2 A + P G B K^2 A\\
&- P G F K^2 A - G B^2 K^2 A - E G F B K H\\
&- E^2 F B K H + E P G B K H + P G^2 B K H + E^2 F^2 K H\\
&- P^2 G^2 K H + E F H A C^2 - P G H A C^2 - G H^2 A C^2\\
&+ G H A^2 C^2 - E F H^2 C^2 + P G H^2 C^2 + E G^2 H A^2 + G^2 B K A^2\\
&- P G^2 K A^2 - E^2 G F H^2 + E P G^2 H^2 - P G F B K^2\\
&+ E P F B K^2 - P^2 G B K^2 + E F^2 B K^2 + P^2 G F K^2\\
&- E F B^2 K^2 + P G B^2 K^2 - E P F^2 K^2.
\end{aligned} \tag{6.4*}$$

Now note that for fixed $P_1, \cdots, P_5$ and variable $P_6$, $\mathrm{DE}-\sigma\mathrm{DIDJDK}$ is a linear combination of two quadratic forms in the coordinates of $P_6$.

This theorem was derived after an inspection of a machine print out and this process can be described as *computer assisted theorem derivation.*

Notice that the quadratic form DE is such that when two points $P_i$, $P_j$ coincide, the form reduces to 0. Therefore, DE = 0 represents the condition that $P_i$ $i=1, \cdots, 6$ lie on a conic. But DE is the computed collinearity determinant of the three Pappus points. Hence, this computation demonstrates plainly that collinearity of the Pappus points is equivalent to the six original points $P_i$ lying on a conic. Thus we have a form of Pascal's theorem. DE is, of course, the $6\times 6$ determinant

$$\text{DE} = -\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \text{F}^2 & \text{G}^2 & \text{FG} & \text{F} & \text{G} & 1 \\ \text{P}^2 & \text{E}^2 & \text{PE} & \text{P} & \text{E} & 1 \\ \text{A}^2 & 0 & 0 & \text{A} & 0 & 1 \\ \text{H}^2 & \text{K}^2 & \text{HK} & \text{H} & \text{K} & 1 \\ \text{B}^2 & \text{C}^2 & \text{BC} & \text{B} & \text{C} & 1 \end{vmatrix}. \tag{6.5}$$

We consider yet another problem: what are the conditions that the Pappus points $P_I$, $P_J$, $P_K$ form a right angled triangle with right angle at $P_J$? The conditions are

$$\frac{(\text{MJ}/\text{DJ}) - (\text{MI}/\text{DI})}{(\text{NJ}/\text{DJ}) - (\text{NI}/\text{DI})} = -\frac{(\text{NJ}/\text{DJ}) - (\text{NK}/\text{DK})}{(\text{MJ}/\text{DJ}) - (\text{MK}/\text{DK})} \tag{6.6}$$

or

$$\begin{aligned} S \equiv (\text{MJ}\cdot\text{DI} - \text{DJ}\cdot\text{MI})(\text{MJ}\cdot\text{DK} - \text{DJ}\cdot\text{MK}) \\ + (\text{NJ}\cdot\text{DI} - \text{DJ}\cdot\text{NI})(\text{NJ}\cdot\text{DK} - \text{NK}\cdot\text{DJ}) = 0. \end{aligned} \tag{6.7}$$

From (3.2), the number of monomials implicit in the left hand side of (6.7) is $2\cdot(8\cdot 8+8\cdot 8)(8\cdot 8+8\cdot 8)=2^{15}=32768$, which again indicates the enormous build-up of the formal algebra corresponding to very simple geometrical operations. This kind of storage requirement may saturate memories of moderate size, and some simplifications may be in order. Again, we take $P_1$ at the origin and $P_4$ on the $x$-axis by means of the input

$$\begin{aligned} \text{X1} = 0, \text{Y1} = 0; \text{X2} = \text{F}, \text{Y2} = \text{G}; \text{X3} = \text{D}, \text{Y3} = \text{E}; \text{X4} = \text{A}, \text{Y4} = 0; \\ \text{X5} = \text{H}, \text{Y5} = \text{K}; \text{X6} = \text{B}, \text{Y6} = \text{C}. \end{aligned}$$

The introduction of the three zeros will reduce the Pappus points as follows: NI, MI = 1 monomial, DI = 3 monomials; NJ, MJ = 2 monomials; DJ, MK = 4 monomials; NK, DK = 6 monomials.

The number of monomials implicit in the left hand side of (6.7) is therefore reduced to

$$(2\cdot 3 + 4\cdot 1)(2\cdot 6 + 4\cdot 4) + (2\cdot 3 + 4\cdot 1)(2\cdot 6 + 6\cdot 4) = 640.$$

*Even with a computer at one's disposal, transformations and shorthand notations may therefore be sought to reduce storage requirements and to interpret the output.* The race against the $n!$ buildup of determinants cannot be won by the computer alone operating in the crude mode outlined.

The final computation output (combined and simplified) was

(6.8)* $S$ = approximately 300 monomials each of degree 10.

The first three monomials listed were

$$-\mathrm{E^2DF^2BKHAC} + \mathrm{D^3G^2BKHAC} + \mathrm{E^2F^2B^2KHAC} - \cdots.$$

A (human) scan of the output for $S$ yields the following computer assisted theorem.

THEOREM.* *Let $P_1, \cdots, P_6$ be six points in the plane and $P_I$, $P_J$, $P_K$ be their Pappus points. Let $P_1, \cdots, P_5$ be fixed while $P_6$ is variable. The locus of points $P_6$ such that $P_IP_J$ is perpendicular to $P_JP_K$ is a cubic curve.*

**7. What Constitutes a Proof in Mathematics?** The reader who is not used to thinking about mathematics in terms of machine work may object to the claim that the printout

$$\mathrm{DE} = 0$$

constitutes a proof of Pappus' Theorem. What if the programming was erroneous? What if the initial data were false? What if there was a machine malfunction? What if the programmer, in a moment of pique, simply programmed the computer to type out $\mathrm{DE} = 0$, and let it go at that?

These are certainly valid objections. Similar objections, however, can be raised with conventional proofs. One aspect of a mathematical proof is that it consists of a finite string of symbols which must be recognized one by one and processed either by a person or by a machine or by both. Now symbols must have physical traces on paper, in the brain, or elsewhere, and cannot be reproduced and recognized with perfect fidelity. Human processing is subject to such things as fatigue, limited knowledge or memory, and to the psychological desire to force a particular result to "come out."

The splicing together of several theorems may cause difficulty. A colleague tells the following story. He received a paper for refereeing which was written by a competent mathematician. The conclusion of the paper seemed to our colleague to be intuitively erroneous. He therefore checked the details of the proof. The details seemed to be in order. He was forced to conclude, therefore, that there was probably an error in one of the theorems used in the paper, but not proved in the paper. The author's references led him to a theorem in a well-known book in probability theory. In this book, the cited theorem was printed erroneously. The words 'closed set' and 'open set' had inadvertently been interchanged.

This sort of story is unfortunately common. A former editor of the *Math-*

*ematical Reviews* once remarked (in a moment of jest) that he thought that 50% of the mathematical papers published contained theorems that were in some degree in error. Suppose the figure were only 1%?

What recourse do we have? For machine proofs as well as conventional proofs, one can (a) run the program several times, (b) inspect the program, (c) invite other people to inspect the program or to write and run similar programs. In this way, if a common result is repeatedly obtained, one's degree of belief in the theorem goes up. Iterated over an interval of years, this process can converge to theorems whose degree of credibility is high.

These considerations lead us to a position—which is rarely discussed in works on the philosophy of mathematics and which is very unpopular—that a mathematical proof has much in common with a physical experiment; that its validity is not absolute, but rests upon the same foundation of repeated experimentation.

**8. Prospects for Computer Discovery in Elementary Analytical Geometry.** It seems clear from the present work that all one's favorite theorems of elementary geometry are capable of brute analytic proof by computer within reasonable computing times, given high-speed memories of moderate capacities. We have also given several examples of theorem formation with an assist from the computer. The theorems derived are of the conventional sort in elementary geometry; whether or not they appeal to the reader or whether they can be regarded as a minor exercise which can be given simple conventional proof is somewhat beside the point here.

A more interesting question is whether by means of the present method, the computer-mathematician combination can generate new theorems in elementary geometry which will ultimately be of historical significance. While voyages into the future are a risky parlor game, we should like to suggest that the answer to the question as stated is "no." (We do not assert this for other areas of pure or applied mathematics.)

Our reasoning is by historical parallel. Let us consider the contribution of the method of Descartes to geometry. It is no misreading of the history of geometry to state that the method of Descartes, for all its power, and all its ability to handle problems in classical geometry in a deterministic way, did not contribute substantially to the stock of geometrical theorems of the classical type. The method of Descartes began to take wing when the method, i.e., the algebra, was freed from the visual, spatial, or deductive goals of classical geometry. To put it in other terms: the means of analytic geometry replaced the ends of classical geometry. (This is a frequent historical process in mathematics and one of the authors (Davis) hopes shortly to present some descriptions and speculations on it. The current cliché "the medium is the message" is an aspect of this process.) Only in this way were the immense and beautiful structures of algebraic geometry raised.

By the same token, it is unlikely that the computer will contribute substantially either to classical geometry or to algebraic geometry in terms of the original goals of these subjects. However, if the characteristic means of the computer can supersede these goals—whether the means are to be found in computer languages or in combinatorial power or in heuristic power is not clear—then a genuinely new subject of historical significance can emerge. The Descartes of computer geometry must point an identifying finger firmly at these means.

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# ON NEWTON'S INEQUALITY FOR REAL POLYNOMIALS

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**1. Introduction.** The result of Newton on the coefficients of real polynomials with all roots real, namely

$$\frac{A_r^2}{(n-r+1)} \geqq \frac{A_{r-1}A_{r+1}}{(n-r)},$$

where the polynomial is given in the form

$$P_n(z) = \sum_{r=0}^{n} \frac{A_r}{r!} z^{n-r}$$

is well known. There is an elegant and short proof of this, depending on Rolle's Theorem, in [1]. What is not generally known however, and not as easy to prove (although the ideas involved are still completely elementary), is that there exists a dual to Newton's Inequality. In it the zeros of the real polynomial must