Problem: Given $x_1 \dots x_n$ find $\max_{i<j}(x_i - x_j)^2/(j-i)$.

Approximation Algorithm:

```
% alpha is a parameter between 2/3 and 1.
% large alpha: accurate

function F(float[n] x; alpha)   % 1-based indexing
   define beta = 3*alpha-2
   define c = n^alpha  % number of chunks
   define q = n/c % size of chunks
   define h = n^beta
% Assume w.l.o.g.  that n, c, q, h are integers

   float ChunkMax[q], ChunkMin[q];
   for a=1:c
      ChunkMax(a) = max(x[q*(a-1)+1:q*a])
      ChunkMin(a) = min(x[q*(a-1)+1:q*a])
   return max(CloseF(), FarF())

   function CloseF()
        m = 0
        for i=1:n-1
           for j=i+1:min(i+h*q,n)
              m=max(m,(x[i]-x[j])^2/(j-i))
        return m
   end CloseF

   function FarF()
      m = 0
      for a=1:(c-h)
        for b=(a+h):c
           m=max(m,FarChunksApprox(a,b))
      return m
   end FarF

   function ChunksApprox(a,b)
        z = max(ChunkMax(b)-ChunkMin(a), ChunkMax(b)-ChunkMin(a))
        return z^2/((b-a)*q)
   end ChunksApprox
end F
```

Running time: CloseF takes time $\mathrm{O}(nhq) = \mathrm{O}(n^{1+\beta+(1-\alpha}) = \mathrm{O}(n^{2\alpha})$ FarF takes time $\mathrm{O}(c^2) = \mathrm{O}(2\alpha)$.
Overall: Time $\mathrm{O}(n^{2\alpha})$

Lemma: Let $a; b$ be between 1 and c such that $b - a \geq h$.
Let z=`ChunksApprox`(a, b).
Let $y = \max_{u \in (a-1)q+1:qa,\, b \in (b-1)q+1:bq}(x_v - x_u)^2/(v-u)$.
Then $|z - y|/y < 4/h + O(1/h)$

Proof: Let $i \in (a-1)q + 1 : qa$. Let $j \in (b-1)q + 1, bq$.

then $(b-a-1)q < j-i \leq (b+1-a)q$
so $(x_j - x_i)^2/(b+1-a)q \leq (x_j - x_i)^2/(j-i) \leq (x_j - x_i)^2/(b-a-1)q$,
and of course $(x_j - x_i)^2/(b+1-a)q \leq (x_j - x_i)^2/(b-a)q \leq (x_j - x_i)^2/(b-a-1)q$.

So $|(x_j - x_i)^2/(j-i) - (x_j - x_i)^2/(b-a)q| \leq (x_j - x_i)^2 \cdot (1/(b-a-1)q - 1/(b+1-a)q) \leq (x_j - x_i)^2/(1(/hq-q) - (1/hq+q)) \leq 2(x_j - x_i)^2/(h^2 q)$.

Let $s \in (a-1)q+1 : qa$, $t \in (b-1)q + 1r : bq$ be the indices that maximize $(x_t - x_s)^2/(t-s)$.
Let $u \in (a-1)q+1 : qa, v \in (b-1)q+1, bq$ be the indices that maximize $(x_u - x_v)^2$. Thus ChunksApprox(a,b) = $(x_u - x_v)^2/(b-a)q$

So we have: $(x_v - x_u)^2/(b-a)q \geq (x_t - x_s)^2/(b-a)q$.
$(x_v - x_u)^2/(v-u) \leq (x_t - x_s)^2/(t-s)$.

So $(x_t - x_s)^2 \leq (x_v - x_u)^2 \leq (x_t - x_s)^2(v-u)/(t-s)$.

However $(v-u)/(t-s) \leq (hq+q)/(hq-q)$.

So $|(x_v - x_u)^2 - (x_t - x_s)^2| \leq ((h+1)/(h-1)) - 1)(x_t - x_s)^2 = 2/(h-1)(x_t - x_s)^2$.

So $|z-y| =$
$|(x_v - x_u)^2/(b-a)q - (x_t - x_s)^2/(t-s)| \leq$
$|(x_v - x_u)^2/(b-a)q - (x_t - x_s)^2/(b-a)q| + |(x_t - x_s)^2/(b-a)q - (x_t - x_s)^2/(t-s)| \leq$
$(x_t - x_s)^2 \cdot [2/(h-1)^2) + 2/h^2 q)$.

QED.

Theorem: Let $z = max_{i<j}\,(x_j - x_i)^2/(j-i)$. Let $\beta = 3\alpha - 2$ Then $|F(x,\alpha) - z|/z \leq 2/n^\beta$ + l.o.t.,

Proof: Immediate from the above lemma.

Note: The only property of $(x_j - x_i)$ used in the algorithm is that
$\max_{i,j}(x_i, y_j)^2 = \max((\max_i(x_i) - \min_j(y_j))^2, (\max_j(y_j) - \min_i(x_i))^2)$
so it will work to find the approximate maximum of $g(x_j, x_i)/(j-i)$ for any g with that property.

I think with a tighter analysis, you can probably shave off a factor of 2.

Thanks to Daniel Kane for pointing out an error in an earlier draft.