BOOK REVIEW

*By Ernest Davis*

**Numerical Notation: A Comparative History.** *By Stephen Chrisomalis, Cambridge University Press, New York, 2010, 496 pages, $114.99.*

The construction of written notation systems for numbers, closely following the development of words for numbers, is one of the most fundamental and most widespread initial steps in the development of mathematics. Numerical notation can be found on artifacts from Egypt and Mesopotamia dating back to about 3200 BC, in Egypt concurrent with the earliest written records of language, and in Mesopotamia predating the records by centuries. But it is possible that numerical notation is even older: Marks on Neolithic pottery and tortoise shells found at Jiahu in Henan province dating from 6600–6200 BC may well be numeral signs.

Among cultural artifacts, numerical notation systems are unusual in two respects. First, they are extremely well defined and limited semantically; they represent some or all of the natural numbers, and sometimes certain fractions. Second, they are not very numerous; records document only about 100 distinct systems that have ever been in any kind of general use. (The exact count depends on how you individuate.) It is therefore possible to present an account of numerical notational systems throughout history that is essentially complete and definitive, up to the limits in the historical record.

Stephen Chrisomalis's *Numerical Notations: A Comparative History* is such an account. He gives complete descriptions of all known numerical notation systems: how they work and how they have been used. Each system is illustrated with a clear hand-drawn table of symbols; most are also accompanied by photographs showing their use on historical or archaeological artifacts.* Chrisomalis also traces the evolutionary history of these systems. When, as often, this history is obscure, he surveys the scholarly literature and makes his own judgment of the probable historical relations between notations, taking into account such considerations as similarity of structure, similarity of symbols, known contacts between cultures, and proximity in time and space.† Considering the complete survey, he analyzes a substantial number of regularities and near regularities that govern individual systems, and a much smaller number of regularities that govern how one numerical system can evolve from another. Finally, he discusses how these regularities relate to characteristics of human cognition and human society.

Chrisomalis identifies five main structures for notational systems. All numerical notation systems are built around powers of a fixed base, generally but not always 10.CONTRADICTS THE FIRST UNIVERSAL PRINCIPLE LATER? A *cumulative-additive* system, such as Roman numerals, has a symbol for each power of the base (I, X, C, M); these are repeated and the values are then added. (The Roman numerals also have a *subbase* of 5 (V, L, D), common in additive systems, and a *subtractive* feature (IX for 9), which is extremely rare.) In a *ciphered-additive* system, such as the Greek or Hebrew numerals, each multiple of a power of 10 has its own symbol, and the values of these are added together. For instance, in the Greek alphabetic system, $\nu$ represents 400, $\lambda$ 30, and $\delta$ 4; $\nu\lambda\delta$ thus represents 434.

In a *multiplicative-additive system*, signs for the digits 1 through 9 alternate with signs for the power of 10. Traditional Chinese numerical notation works this way; so (to some extent) does the English language, e.g., "two *thousand* three *hundred* for*ty*-seven." In *ciphered-positional* notation, such as the Western numerals, there are symbols for the numbers from either 0 or 1 up to the base minus 1; the power of the base is then indicated by the position of the symbol in the numeral. To be unambiguous (not all such systems were), a system must have either a symbol for zero or some other way of indicating powers with a zero coefficient. Finally, *cumulative-positional* systems represent powers of ten positionally, as in the Western numbers, but the coefficients cumulatively; the famous base-60 ancient Babylonian system followed this principle. (A ciphered base-60 system would of course need 60 distinct symbols for the digits.)

About 30% of the systems that Chrisomalis discusses are *hybrids* that combine different principles for different ranges of numbers—often, a cumulative or ciphered-additive system for lower powers of the base and a multiplicative-additive system for higher powers. However, no naturally arising systems of pure numbers use any other principles. One can imagine a system that represents numbers by their prime factorization; or that uses division (as in "a half-dozen"); or that uses the factorials as a base (e.g., representing 301 as [2,2,2,0,1] since 301 = 2 • 5! + 2 • 4! + 2 • 3! +1 • 1) etc.; but these do not actually arise.

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Chrisomalis's historical accounts are always impeccably clear, but unavoidably somewhat dry; after 100 numerical notational systems, one's eyes begin to glaze. However, he provides all kinds of fascinating historical and cultural tidbits along the way. Large numbers, and their use in wild exaggerations, go back to the very earliest days of numerical notation; an Egyptian macehead from 3100 BC records the supposed capture of 120,000 prisoners, 1,422,000 goats, and 400,000 cattle. Some numerical systems were used only for counts of quite specific categories; in fact, in ancient Uruk in Sumeria, there were 15 different numerical systems for different kinds of quantities, including "the regular Š system [for] barley, the Š' system for germinated barley for brewing beer, and the Š* system for barley groats." In modern China, six numerical systems are to some degree active, depending on the region and the particular use. One of these, the "accountants' system," uses deliberately complex symbols in order to avoid falsification.

The historical understanding of the origins of the Western numerals, which originated in India and were transmitted via the Islamic world to Europe, deteriorated over the centuries. In the earliest European sources, as well as in Arabic sources, they are called, correctly, "Indian" numerals. By the 16th century, they were often called, less correctly, "Arabic" numerals. In the early 20th century, a number of scholars proposed, on the basis of no evidence other than pure Eurocentrism, that they must actually have originated in Greece.

Chrisomalis emphasizes strongly that the use of numerical notation varies significantly from one culture and time to another—we should not make the mistake of supposing that our own uses of numbers apply universally. In particular, in most times and places, written numbers were not used for calculation; calculations were done by some method of finger calculation or with tools, such as an abacus or counting sticks. It is therefore a mistake to suppose that the inefficiency of a notational system for calculation was any kind of drawback.

By way of analogy (mine, not Chrisomalis's), consider the numerical notation for dates, e.g., 2/24/2015 (American style) for February 24, 2015. What is it good for? Well, it makes it easy to approximate the time between dates, particularly if they don't cross a boundary: 7/15/2015 is about 5 months after 2/24/2015; 9/26/1898 was about 117 years earlier. It is also easy to judge the relation of dates to yearly events: 12/25/1898 was Christmas, was about 4 days after the winter solstice, and was the beginning of winter in New York and of summer in Sydney. That's about it. Calculating the day of the week for 12/25/1898 or the exact number of days between 12/25/1898 and 2/24/2015 is laborious by hand, and requires several lines in a computer program (and you have to be very careful to avoid off-by-one errors).

The irregularity of the calendar is a constant source of inefficiency and trouble for the construction of calendars, either printed or automated. Why do we put up with this? First, we rarely have to compute the number of days that have elapsed since some date in the past (though we do often have to determine the day of the week of a future date). Second, the costs of changing it would be prohibitive. Third, because an earth year happens to be 365.2425 earth days, no calendar that incorporates both years and days can possibly be very elegant.‡ If our descendants living on seasonless space stations make fun of us for measuring time in such an obviously awkward way, they will simply be missing the point.

Even within the basic Western numerical notation are suboptimalities that we tend to overlook because we are so used to them. Who knows how many man-hours and dollar-equivalents have been lost over the last five centuries because the handwritten digits 4 and 9, and 1 and 7, are easily confused. The Roman numerals are much clearer in that respect.

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To me, the most interesting part of Chrisomalis's book is his analysis of the regularities that govern numerical systems. He adduces 14 principles that hold in all the systems he has studied, 8 that hold in nearly all. Among the universals: "Every base is a multiple of 10"; "Any system that can represent $N$ + 1 can also represent $N$." For the near-universals: "No numerical notation explicitly represents arithmetic operations such as addition and multiplication" (in contrast are linguistic forms, such as "a thousand *and* fourteen," "vingt-*et*-un"). The single exception occurs in the Shang Chinese numerals: "All numerical notation systems are ordered and read from the highest to the lowest power of the base." This is, of course, necessarily true for positional systems, but would not have to be true for additive systems. One could imagine, in Roman numerals, that you could write IICVCX to mean 217; but in fact this is not allowed. There are a few exceptions in some alphabetic systems, where the notation follows the word order for the lexical number. Chrisomalis's explanations of these in terms of human cognitive capacity, such as the limited size of working memory, and the relation of numerical notations to language, are thought-provoking and no doubt true in part. I don't think that they are a sufficient explanation of all the regularities found, but probably no such explanation can be found.

All in all, Chrisomalis's book is an impressive accomplishment and a valuable contribution to our understanding of the fundamentals of mathematics as a cultural activity.

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*In the online copy that I was reading, these photographs were not always very clear; but the reproduction quality may well be better in printed copies.NEED TO CHECK THAT OUT; IF YOU SUGGEST A GOOD ILLUSTRATION, WE'LL ASK CUP FOR A FILE

†Some of the scholarly literature in the area suggests wildly conjectured relations between notational systems that are similar in some respects, but separated by millennia.

‡It seems to me, by the way, that this is a clear counter-example to the common theory that we find mathematical regularities in nature because we impose them as a conceptual framework. If we could impose preferred mathematical regularities on nature, we wouldn't be dealing with this.

*Ernest Davis is a professor of computer science at the Courant Institute of Mathematical Sciences, NYU.*