“Motivated” Numerical Science Problems: for GPT4 + Wolfram Alpha and GPT4 + Code Interpreter Problems and Solutions

Scott Aaronson
University of Texas at Austin
aaronson@cs.utexas.edu

Ernest Davis
New York University
davise@cs.nyu.edu

August 15, 2023

This document includes 21 college-level numerical math and physics problems with problems, answers, and outputs from GPT4 with the Wolfram Alpha plug-in (henceforth GPT4+WA) and with the Code Interpreter plug-in (henceforeth GPT4+CI). The experiments on the two systems were carried out between July 27 and August ?? 2023.

1. **Question:** If the earth collapsed to a black hole, how big would the black hole be? Please include all calculations.

   Answer: 8.87 millimeters.

2. If you fell into the black hole at the center of the Milky Way, how long would you have before hitting the singularity? Please include all calculations.

   Answer: 66.7 seconds.

3. Consider the first binary black hole system discovered by LIGO. Roughly how close to that system would a person have to have been, before they were killed by the gravitational waves? Please include all calculations.

   Answer: So close that you would be killed by the tidal forces rather than by the gravitational waves.

4. How far off from 10! is Stirling's approximation? Please include all calculations.

   Answer: 30,104.

5. How high would an airplane have to be, before you could notice 10 degrees of the earth's curvature when looking out the window? Please include all calculations.

   Answer: 98.4 km.

   Matlab

   ```
   r = 6378 % Radius of earth in kilometers
   d = r/cos(10*pi/180)  % distance from center of earth to airplane
   answer = d-r %height above earth's surface
   ```

6. Approximately how much time would a commercial airliner save in going from New York to Tel Aviv, if it could go in a straight line, through a tunnel in the earth, at the same speed as usual? Please include all calculations. Please include all calculations.

   Answer: About 52 minutes, assuming a flight speed of 500 mph.

   Matlab:

```
nyclat = (40+42/60+46/3600)*pi/180
nyclong = -(74+22/3600)*pi/180
talat = (32+8/60)*(pi/180)
talong = (34+78/60)*(pi/180)
earthRadius = 3959 % miles
nycDirection = [cos(nyclat)*sin(nyclong), cos(nyclat)*cos(nyclong), sin(nyclat)]  % unit vector
taDirection = [cos(talat)*sin(talong), cos(talat)*cos(talong), sin(talat)]
tunnel = earthRadius*norm(taDirection-nycDirection)
flight = earthRadius*acos(dot(nycDirection,taDirection))
timeSaved = (flight-tunnel)/550 % assuming 550 mph.
```

7. For what fraction of the lifetime of the universe has there been life in it? Give upper and lower bounds. Please include all calculations.

   Answer: Between 25% and 99.93%.

   Explanation: The age of the universe is taken to be 13.8 billion years. It is generally accepted that life emerged on earth at least 3.5 billion years ago, and it is speculated that it might have been possible for life to appear in the universe about 10 years after the Big Bang.

8. Approximately how long would it take to transmit an entire human genome over a standard WiFi connection? Please include all calculations.

   Answer: 4 minutes.

   Explanation: The human genome has about 3 billion base pairs = 6 billion bits. Assuming a 25Mbs Wifi connection, that gives 240 seconds = 4 minutes.

9. Approximately how large would an asteroid have to be, in diameter (assume it's approximately spherical), before an Olympic high jumper could no longer reach escape velocity by jumping off it? Please include all calculations.

   Answer About 10 km.

   Explanation: The kinetic energy needed for escape velocity from the surface of a planet is inversely proportional to the radius and proportional to the mass of the planet, which in turn is proportional to the density and radius cubed. So over planets of constant density energy is proportional to the radius squared and density. So the velocity is proportional to the radius and square root of the density. So the radius as a proportional to the velocity and inversely proportional to the square root of the density.

   The world record for high jump is 2.45 meters, set by Javier Sotomayor in 1993. The jumper's center of mass starts about half way up his body, around .9 m. (Sotomayor is 193 cm tall.) At the apex of the jump, it is perhaps .3m above the bar. So the jump raises the center of mass by about 1.8 meters. The initial velocity is therefore $v = \sqrt{2gh} \approx 6$ m/s.

   Escape velocity from earth is 11 km/s and the diameter of earth is 12,750 km. The densities of asteroids is not at all well known (see the Wikipedia article, "Standard asteroid physical characteristics" but a reasonable estimate might be about 4/9 the density of earth.

   So the diameter of the asteroid would be about $12750 \cdot (6/11000) \cdot (3/2) \approx 10$ km.

10. What is the length of the $y = x^2$ parabola in the region $-1 \leq x \leq 1$? Please include all calculations.

    Answer:
$$\int_{-1}^{1} \sqrt{1 + 4x^2}\, dx = \sqrt{5} + \frac{1}{2} \sinh^{-1}(2) = 2.95789$$

11. Approximately how large a supply of antimatter would be needed, in order to propel a spacecraft with the mass of the International Space Station into orbit around Proxima Centauri, in one year as experienced by its crew? Please include all calculations.

    Answer: 706,000 kg

    Explanation:
    We'll assume that the Space Station is accelerated rapidly to its cruising speed, and we'll ignore the need to decelerate when it reaches Proxima Centauri.

    The first step is to compute the velocity. We'll use the speed of light as the unit of speed and a year as the unit of time. Proxima Centauri is 4.25 light years away. If the spaceship travels at speed v, then as measured from earth, the time required $\tau = 4.25/v$. Because of time dilation, this appears to the crew as $\tau\sqrt{1 - v^2}$. So we have the equation $1 = (4.25/v)\sqrt{1 - v^2}$, so $1 = (4.25^2/v^2)(1 - v^2)$, so $v^2 - 18.0625(1 - v^2)$, so $v^2 = 18.0625/19.062 = 0.9475$ and $v = 0.9734$.

    If the Space Station has mass M, then its total energy at speed v is $M/\sqrt{1 - v^2}$, so the additional energy is $M/\sqrt{1 - v^2} - M$ (c is still 1). If we annihilate mass m of antimatter with mass m of matter, then that liberates $2m$ of energy We have $2m = M \cdot ((1/\sqrt{1 - v^2}) - 1) = 3.3661M$ so $m = 1.683M$. Taking m to be 420,000 kg, that gives a value of 706,000 kg of antimatter.

12. Approximately how many errors will a standard laptop suffer over its lifetime, due to cosmic rays hitting the microchip? Please include all calculations.

    Answer: Estimates vary widely, but one commonly cited figure is about 1 error per 256 megabyes of RAM per month. Assuming an 8 GByte laptop and a 5 year lifespan, that will total 1920 errors.

13. What is the approximate probability that a randomly-chosen 100-digit integer is prime? Please include all calculations.

    Answer: Approximately $1/\log_e(5.5 \cdot 10^{99}) = 1/(\log_e(5.5) + 99 \cdot \log_e(10)) = 0.00435$. That should be accurate to within 1%.

14. What is the probability that a randomly-chosen 100*100 matrix, over the finite field F_2, is invertible? Please include all calculations.

    Answer: 0.289

    Explanation: Calculate the probability that the 2nd row is linearly independent from the first, then the probability that the 3rd row is outside the span of the first two, the probability that the 4th row is outside the span of the first three, etc. and multiplying them all together. In backward order this is

    (1/2) * (3/4) * (7/8) * (15/16) * ... $\approx$ 0.289.

15. How does the total weight of all the uranium that humans mined, compare to the total weight of all the gold that they've mined? Please include all calculations.

    Answer: It is estimated that humans have mined about 209,000 tones of gold and about 2.8 million tons of uranium, a difference of a factor of about 13.

Explanation: The figure for gold comes from the World Gold Council. The figure for uranium was estimated by the authors based on the information on this web page of the World Nucler Association

16. What is the Shannon entropy of a positive integer n that's chosen with probability Pr[n] = $6/(\pi^2 * n^2)$? Please include all calculations.

    Answer: 2.362

    Matlab:

```
ent=0;
for i=1:1000000
  p = 6/((pi^2)*(i^2));
  ent = ent - p*log(p);
  sum = sum+p;
end
ent/log(2)

% It is easily shown that the sum of the remaining terms is less than 0.001.
```

    Revised version of problem: On consideration, after running the above, we thought that perhaps the wording of the problem was unfair to GPT. After all, the answer probably has no elegant closed-form expression, and it is not easy to get a highly precise numerical answer (the convergence rate of the naïve summation is $O(\log(n)/n)$). So we tried the two systems with the following revised question:

    Compute the Shannon entropy of a positive integer n that's chosen with probability Pr[n] = $6/(\pi^2 * n^2)$ to 3 digit accuracy. Please include all calculations.

17. Assume that IQs are normally distributed, with a mean of 100 and a standard deviation of 15. For which n does the number of people with IQ n exceed the number of people with IQ n+1 by the maximum amount? Please include all calculations.

    Answer: Approximately $n = 14.5$

    Explanation: Let $f(x)$ be the probability density function for the normal distribution with mean 100 and standard deviation 15.

    $f(x) = \frac{1}{\sqrt{2\pi}\cdot 15}\exp(-(x-100)^2/2\cdot 15^2)$

    At the infinitesimal scale, the change between people with IQ x and people with IQ $x+\epsilon$ is maximized when the derivative of the probability distribution is negative and its magnitude is maximal; thus, at a value x greater than the mean where the second derivative is 0.

    The first derivative of f, $f'(x) = -((x-100)/15^2)f(x)$.

    Setting the second derivative of f to be zero, we have, $f'' = ((x-100)^2/15^2)^2 - (1/15^2)f(x)$, so $(x-100)^2 = 15^2$, which has two roots: $x = 85$, where the first derivative reaches a maximum positive value and $x = 105$ where it reaches a maximum magnitude negative value.

    Now, that does not directly give us the value of x that maximizes $f(x) - f(x+1)$, and finding the true maximum would involve solving an inelegant transcendental equation. However, since the third derivative of f is fairly small and the second derivative is 0, it is safe to assume that picking x and $x+1$ evenly across $x = 15$ gives values close to the maximum value. Using binary search, one can determine that the true maximum is 114.5028.

18. Suppose a randomized algorithm outputs the correct answer (yes or no) with probability 0.9. If we take the majority answer out of 100 independent runs, with what probability will the answer be wrong? Please include all calculations.

Answer: $3.232 \cdot 10^{-24}$.

Explanation: Assume that if the results are exactly 50/50, then you flip a fair coin. Then the exact value is

$$(1/2) \cdot C(100.50) \cdot 0.9^{50} \cdot 0.1^{50} + \sum_{i=1}^{50} C(100, 50+i) \cdot 0.9^{50-i} \cdot 0.1^{50+i}$$

However, since this is (approximately) a geometrically decreasing series with ratio of 1/9, it suffices to take the first five terms to get three digit accuracy.

Matlab: (Matlab warns that its value may have a relative error of 10^{-14}).

```
p=nchoosek(100,50)*0.9^50*0.1^50
sum=p/2
for i=1:5
   p=nchoosek(100,50+i)*0.9^(50-i)*0.1^(50+i)
   sum = sum+p
end
sum
```

19. In a Manhattan-like two-dimensional grid, how many distinct shortest paths are there that go from a fixed starting point to a fixed endpoint 5 blocks to the north and 5 blocks to the east? Please include all calculations.

    Answer: 252.

    Explanation: You have to take 10 steps: 5 of them north and 5 of them east. They can be in any order. So the numbr is C(10,5) = 252.

20. When 3-dimensional spheres are packed in their densest possible configuration, what percentage of the space is empty? Please include all calculations.

    Answer: $1 - \pi/3\sqrt{2} = 0.2595$. I looked it up in the Wikipedia article Kepler conjecture