“Arbitrary Numerical” Science Problems for GPT4 + Wolfram Alpha and GPT4 + Code Interpreter: Problems and Solutions

Ernest Davis
New York University
davise@cs.nyu.edu

Scott Aaronson
University of Texas at Austin
aaronson@cs.utexas.edu

September 13, 2024

This document includes 32 problems in high school and college-level physics and mathematics with numerical or vector answers. It includes the questions and answers, with an explanation and in some cases Matlab code.

Unless otherwise noted, all questions should be marked correct if the relative error is less than 1%. If the answer is a vector, then the question will be marked correct if $|\vec{u} - \vec{v}|/|\vec{v}| < 0r., 1$ where $\vec{v}$ is the correct value and $\vec{v}$ is the proposed answer.

1. Object A is a spherical shell of gold, 1 meter in radius, 10 cm thick. Object B is a spherical shell of copper, 10 cm in radius, 5 cm thick. They are placed so that there is an 8 cm gap between them. What is the net gravitational force they exert on each other?

   Answer: 3.4486 dyne

   Matlab

```
% By a theorem of Newton’s, the gravitational force exerted by spherical shell
% is the same as if all the mass were concentrated at the center.
GoldDensity = 19.3; % gm/cm^3 --  Wikipedia.
GoldVolume = (4/3) * pi * (100^3-90^3); %cm^3
GoldMass = GoldDensity * GoldVolume; % gm
CuDensity = 8.96    % gm/cm^3 --- Wikipedia
CuVolume = (4/3) * pi  * (10^3-5^3);  % cm^3
CuMass = CuDensity * CuVolume;
Distance = 100+10+8; %cm
GravConst = 6.674 * 10^(-8); % dyne cm^2/gm^2 -- Wikipedia
Force = GravConst * GoldMass * CuMass / Distance^2
```

2. Let b, c, d be points at the center of Buenos Aires, Cincinatti, and Delhi, respectively. Consider the plane P that contains b, c, and d (it cuts through the earth). What is the area in square kilometers of the triangle b,c,d lying in P?

   Answer: 41,111,600 km^2.

   Matlab

```
% Convert latitude and longitude to a unit 3D vector
function v=LatAndLongToUnit(LatDeg, LatMin, LatS, LongDeg, LongMin, LongS)
```

```
Theta = LatS*(LatDeg+(LatMin/60))*(pi/180);
Phi = LongS*(LongDeg+(LongMin/60))*(pi/180);
v = [cos(Theta)*cos(Phi), cos(Theta)*sin(Phi), sin(Theta)];
end

b=LatAndLongToUnit(34, 36, -1, 58, 22, -1)
c=LatAndLongToUnit(39, 6, 1, 84, 30, -1)
d=LatAndLongToUnit(28, 37, 1, 77, 14, 1)
bc=c-b
bd=d-b
cd=d-c
EarthRadius=6371;                              % The magnitude of the cross product
answer = norm(cross(bc,bd))*EarthRadius^2/2  % is the area of the parallelogram
x=norm(bc)
y=norm(bd)
z=norm(cd)
s=(x+y+z)/2
check = sqrt(s*(s-x)*(s-y)*(s-z))*EarthRadius^2  % Heron's formula
```

3. X is a regular tetrahedron of zinc with a side length of 2 moving at 100 m/sec. Y is a molecule of NaCl that is moving with speed v. X and Y have equal kinetic energy. What is $1 - (v/c)$, where c is the speed of light?

   Answer: $3.364 \cdot 10^{-20}$

   Matlab:

```
ZincVolume=2^3/(6*sqrt(2)) % Wolfram Alpha
ZincDensity = 7.14 % gm/cm^3. Wikipedia
ZincMass = ZincVolume*ZincDensity
VZinc = 100/(3*10^8) % Speed of Zinc in units where C=1
ZincKE = ZincMass*VZinc^2/2
NaClMass = 9.7*10^(-23) % gm. Wolfram Alpha
LorenzFactor = ZincKE/NaClMass
Q = 1/LorenzFactor^2 % 1-(v/c)^2
Answer = Q/2
```

4. Viewed from Vega, what is the angle between Sirius and the Sun?

   Answer: 0.0972 radians

   Matlab:

```
AngleSEV = 2.755 % radians. Wolfram Alpha
DistES = 8.597 % light years. Wolfram Alpha
DistSV = 33.41 % light years. Wolfram Alpha
AngleEVS = asin(sin(AngleSEV)*DistES/DistSV)
```

5. You have an empty cylindrical open container whose inside has a diameter of 8 centimeters and a height of 20 centimeters. and a pitcher with 200 ccs of water. You first empty the pitcher into the cylinder, then put a solid rock cube, 4 cm on a side, into the container so that it is sitting flush against the bottom of the container. What is the height of the water in the container?

   Answer: 5.2521 cm

   Matlab:

```
AreaOfCyl = pi*4^2
AreaAtBottom = AreaOfCyl-4^2
VolumeAtBottom = 4*AreaAtBottom
VolumeAtTop = 200 - VolumeAtBottom
HeightAtTop = VolumeAtTop/AreaOfCyl
Height = 4+HeightAtTop
```

6. There are six small spherical masses placed at various positions in space. Measured in a coordinate system with a 1 meter unit length:
A has mass 2 kg and is at $\langle 2,0,0\rangle$;
B has mass 3 kg and is at $\langle 0,2,0\rangle$;
C has mass 1 kg and is at $\langle 0,0,2\rangle$;
D has mass 4 kg and is at $\langle 1,1,1\rangle$;
E has mass 5 kg and is at $\langle 1,0,1\rangle$;
F has mass 2 kg and is at $\langle 0,1,1\rangle$.
What is the instantaneous acceleration of F under the gravitational attraction of A,B,C,D,E? Express your answer as a vector.

   Answer: $\langle 0.403, -0.080, -0.056\rangle * 10^{-9}$ m/sec^2

   Matlab:

```
% Gravitational force exerted on mass ma at point pa by mass
% mb at point pb, taking the gravitational constant to be 1.

function f = GravForce(ma,pa,mb,pb)
    v = pb-pa;
    r = norm(v);
    u = v/norm(v);
    f = ma*mb*u/(r^2)
end

g1 = GravForce(2,[0,1,1],2,[2,0,0]) + ...
GravForce(2,[0,1,1],3,[0,2,0]) + ...
GravForce(2,[0,1,1],1,[0,0,2]) + ...
GravForce(2,[0,1,1],4,[1,1,1]) + ...
GravForce(2,[0,1,1],5,[1,0,1])
GravConst = 6.674*10^(-11) % N*kg^2/m^2 -- Wikipedia
acc = g1*GravConst/2
```

7. How many total eclipses of the moon were there between Jules Verne's death and Neil Armstrong's moon landing? An exact integer value is required.

   Answer: 52.

   Jules Verne died March 24, 1905.
Neil Armstrong landed on the moon July 20, 1969
There were 52 total lunar eclipses in between:
02-09-06, 08-04-06, 06-04-09, 11-27-09, 05-24-10, 11-17-10,
03-22-13, 09-15-13, 01-08-17, 07-04-17, 12-28-17, 05-03-20,
10-27-20, 04-22-21, 02-20-24, 08-14-24, 06-15-27, 12-08-27,
06-03-28, 11-27-28, 04-02-31, 09-26-31, 01-19-35, 07-16-35,
01-08-36, 05-14-38, 11-07-38, 05-03-39, 03-03-42, 08-26-42,
12-19-45, 06-14-46, 12-08-46, 04-13-49, 10-07-49, 04-02-50
09-26-50, 01-29-53, 07-25-53, 01-19-54, 11-18-56, 05-13-57,

11-07-57, 03-13-60, 09-05-60, 12-30-63, 06-25-64, 12-19-64,
04-24-67, 10-18-67, 04-13-68, 10-06-68
Source: Wikipedia article, "List of lunar eclipses in the twentieth century."

8. A quantity of chlorine gas is in a right prism whose base is a triangle with sides 5 cm, 7 cm, and 4 cm and whose altitude is 8 cm. The temperature is the freezing point of mercury, and the pressure is 2 atmospheres. What is the mass of the chlorine?

   Answer: 0.5781 gm

   Matlab:

```
s=(5+7+4)/2
Area = sqrt(s*(s-5)*(s-7)*(s-4))  % Heron's formula
VolumeCC = Area * 8
Liters = VolumeCC / 1000
Temp = 273.15 -38.83  % Wikipedia
IdealGasConst = 0.082057 % L*atm*K^{-1}*mol^{-1}. Wikipedia
Moles = Liters*2/(Temp*IdealGasConst)
MassPerMoleCl2 = 70.9 % gm. Wolfram Alpha.
Mass = Moles * MassPerMoleCl2
```

9. A train has two whistles, one at middle C and one at the F above middle C. The train is driving past a station without stopping or changing speed. On approaching the station, it blows the low whistle; when it passes the station, it switches to the high whistle. However, to the people standing at the station, it sounds like the whistle dropped by a whole tone. How fast is the train moving?

   Answer: 68.8 m/s

   Explanation: A half-tone in music is a factor of $2^{1/12}$. The interval from C to F is 5 half-tones. So the effect is that a note as the train approaches is dropped by 7 half tones or a factor of $2^{7/12}$. Let s be the speed of sound, then $(s+v)/(s-v) = 2^{7/12}$. Solving for v we get

$$v = s \cdot \frac{2^{7/12} - 1}{2^{7/12} + 1}$$

   The speed of sound is 343 m/s so the train is moving at 68.41 m/s. (One can work this out by hand to an accuracy of 1% using the fact that a fifth in music – 7 half tones – is 1.5 in the natural musical scale.)

10. A physical process generates photons whose energies follow a random distribution of the following form: For positive energy e, the probability density at e is proportional to the value of e in a Gaussian distribution with mean 2 Ev and standard deviation 0.01 Ev. The probability of a negative value is zero. What is the expected value of the wavelength of a photon produced by this process? (Give the mathematical answer, assuming that the above description is exact, and assuming the standard relation between energy and wavelength in a photo. The answer is not physically plausible.)

    Answer: Infinite.

    Explanation: The wavelength of a photon with energy e is hc/e. The mean wavelength of a photon in the beam is therefore

$$\int_{e=0}^{\infty} N_{2,0.01}(e) \cdot \frac{hc}{e} \, de$$

where $N_{2,0.01}$ is the Gaussian distribution with mean 2.0 and standard deviation 0.1. In the limit as $e \rightarrow 0^+$, the quantity $N_{2,0.01}(e)$, though extremely tiny, is still a positive quantity, so the integral diverges at 0.

11. The wavelengths of the photons in a beam of light are uniformly distributed across the range of visible (to humans) light. What is the mean energy of a photon in electron volts?

    Answer: 2.3688 eV.

    Explanation: The range of visible light is around 3800 to 7000 Angstroms. A photon with wavelength λ has energy hc/λ. Thus, if λ is uniformly distributed between 3800 and 7000, the mean energy is

    $$\int_{\lambda=3800}^{7000} (1/3200)(hc/\lambda)\, d\lambda = (hc/3200) \log_e \lambda|_{3800}^{7000}$$

    Matlab:

```
Planck = 4.136 * 10^(-15) % eV*sec
SpeedOfLight = 3*10^8 % m/sec
Meter2Ang = 10^(10)
SpeedOfLight * Meter2Ang * Planck / 3200 * log(7000/3800)
```

12. A pendulum is hanging on a 2 meter cord attached to the ceiling 3 meters above the floor. It is brought to a position 25 degrees from the vertical and released. It swings past the bottom and the cord is cut when it is 10 degrees from the vertical on the far side. Then the bob flies through the air and hits the ground. What is the distance from the point where the bob is released to the point where it hit the ground?

    Answer: 2.3606 meters

```
% Take the point of attachmemt of the pendulum as the origin
g = 9.8  % Acceleration of terrestrial gravity in m/sec^2
d2r = pi/180 % Degrees to radians
s25 = sin(25*d2r) % Handy abbreviations
c25 = cos(25*d2r)
s10 = sin(10*d2r)
c10 = cos(10*d2r)
pr = [-2 * s25, -2*c25]  % release point
pc = [2 * s10,  -2*c10]   % cut point
hd = 2 * (c10 - c25)  % height difference between the cutpont and the release point.
s = sqrt(2*g*hd) % speed of the bob at release time
vc = s * [c10,s10]; % velocity of bob when cut
tca = vc(2)/g      % time between cut and apex of flight
hca = (g*tca^2)/2  % height difference from cut point to apex
ha = -2*c10 + hca % height of apex of flight
haf = 3 + ha      % height difference between apex and floor
taf = sqrt(2*haf/g) % time from apex to floor
pf = [pc(1) + (tca+taf)*vc(1),-3]
answer = norm(pf-pr)
```

13. An spherical asteroid 500 km in diameter travels in an essentially perfect circular orbit of radius 2.4 astronomical units. On January 4, 2023, the earth was at perihelion, and as it

happens, the asteroid was in perfect opposition. What was the solid angle of the asteroid in the sky as seen by an earth observer?

Answer: $4.35 \cdot 10^{-12}$

Matlab:

```
perihelion = 1.471 * 10^8 % km. Wikpedia
au = 1.498 * 10^8 % km. Wikipedia
asteroid = 2.4*au
distance = asteroid - perihelion
radiusAngle = 250/distance
solidAngle = pi*radiusAngle^2
```

14. Two twin stars, one of 5 solar masses, the other of 10 solar masses, orbit each other in circular orbits in a period of 4 earth years. How far apart are they, in astronomical units?

Answer: 6.21 Au.

Explanation:
Let L be the distance between them in AUs. They each rotate around in a circle centered at their center of gravity, which is $2L/3$ from the small star and $L/3$ from the large star.
Let $M = 5$ be the mass of the smaller star in solar masses.
Let Q be the ratio of the centripetal accelleration of the large star to the centripetal acceleration of the earth in its orbit.
Comparing the gravitational acceleration, $Q = M/L^2 = 5/L^2$ Comparing the centripetal acceleration, the star is moving on a path of radius $L/3$ Au at an angular velocity 1/4 of earth. Hence $Q = L/48$. So $L^3 = 5 \cdot 48$, so L=6.21 Au.

Matlab: Alternative solution using a generalized form of Kepler's third law:
$a^3/T^2 = G(m_1 + m_2)/4\pi^2$.

```
Year = 3.154*10^7 %sec
Grav = 6.6743*10^(-11) % N*m^2/kg^2
SolarMass = 1.99*10^(30) % kg
M1 = 10*SolarMass
M2 = 5*SolarMass
Period = 4*Year
DistInMeter = (Grav*(M1+M2)*Period^2/(4*pi^2))^(1/3)
DistInAU = DistInMeter/(1.496*10^11)
```

15. Draw a circle, on the earth's surface, going through Cairo, Peking, and Moscow. Let S be the area of the part of the earth's surface inside the circle and let P be the area of the circle in the plane of the circle. What is S/P?.

Answer: 1.108.

Matlab:

```
% Since we are computing a ratio, we take the earth's radius to be 1
c = LatAndLongToUnit(30,2,1,31,14,1) % Unit vector corresponding to Cairo
p = LatAndLongToUnit(39,54,1, 116,24,1) % Peking
m = LatAndLongToUnit(55,45,1,37,37,1) % Moscow
a = zeros(3);  % coefficient matrix
q = zeros(3,1); % constant terms
% Equation of the perpendicular bisector plane of line cp is dot(x,p-c) = 0
% (Note that the origin is always in the plane.)
```

```
a(1,:) = p-c
% Likewise perpendicular bisector of cm
a(2,:) = m-c
a(3,:) = cross(p-c,m-c) % normal to plane of three cities
q(3,1) = dot(c,a(3,:))
f = a\q  % circumcenter is intersection of these 3 planes
f=f';
r = norm(f-c) % radius of circumcenter
norm(m-f)  % check for correctness
norm(p-f)
PlanarArea = pi*r^2  % area of planar circle
SphereArea = 2*pi*(1-norm(f)) % Formula from Wikipedia "Spherical sector"
Answer = SphereArea/PlanarArea
```

16. Assume that the probability of having any particular isotope of a chemical follows their frequency on earth. What is the probability that a randomly constructed molecule of glucose will have 4 atoms of ^{12}C, 2 atoms of ^{13}C, 11 atoms of ^{1}H, 1 atom of ^{2}H, 3 atoms of ^{18}O, 3 atoms of ^{16}O.

    Answer: $4.797 \cdot 10^{-13}$

    Matlab:

```
nchoosek(6,4) * .0106^2 *0.989^4 * ...
nchoosek(12,1) * .99985^11 *0.000145 * ...
nchoosek(6,3) * 0.998^3 * 0.00205^3
Answer: 4.797 * 10^-13
```

17. Two ^{31}P phosphorus nuclei, with no electrons, are isolated in space, with coordinates ⟨0,0,0⟩ and ⟨10,10,10⟩ in a coordinate system whose unit length is 1 Angstrom. What is the instantanous acceleration (a vector with unit length of Angstrom/sec^2) of the nucleus at the origin due to the electrostatic force?

    Answer: $\langle -1.9420 \cdot 10^{27}, -1.9420 \cdot 10^{27}, -1.9420 \cdot 10^{27} \rangle$

    Matlab

```
Mass = 5.1433*10^(-26) % kg
Charge = 15*1.602*10^(-19) % coulomb
CoulombConst = 8.988 * 10^9 % Newton * m^{2}/C^{2}
Angstrom2Meter = 10^(-10)
Distance = 10*sqrt(3)*Angstrom2Meter
Force = CoulombConst*Charge^(2)/Distance^(2) % Newtons
Acceleration = -Force/(Mass*Angstrom2Meter) * [1,1,1]/sqrt(3)
```

18. An irregular (house-shaped) pentagon has vertices numbered 1 through 5 in order. The pentagon has right angles at vertices 1, 3, and 4, and 135-degree angles at 2 and 5. Side 2-3 and 4-5 have length 1 and side 3-4 has length 2. The pentagon is placed on a planar coordinate system so that the numbering of the vertices is in clockwise order, vertex 3 is at the origin, and vertex 5 is on the positive y-axis. What are the coordinates of vertex 1?

    Answer: $\langle 1.5269, 2.3446 \rangle$

    Matlab:

```
% Problem 18
% Consider the pentagon in a standard orientation where vertex 3 is at the origin and
% vertex 4 is on the negative x-axis. Compute the rotation necessary to place 5 above
% 3, and then apply that rotation to 1.

theta = atan2(1,-2)-(pi/2)
standard1 = [-sqrt(2);1+sqrt(2)]
answer = [cos(theta), sin(theta); -sin(theta), cos(theta)]*standard1
standard5 = [-2;1]
check = [cos(theta), sin(theta); -sin(theta), cos(theta)]*standard5 % check
```

19. Consider a cube with unit length sides, where the vertices of one face are numbered A..D in counterclockwise order, as viewed from the center of the cube; the vertices of the opposite face are named E to H; and there are edges AE, BF, CG, and DH. Rotate the cube so that vertex A is at the origin, vertex G is on the positive z axis, and vertex B is in the x-z plane, with positive x coordinate. What are the coordinates of vertex E?

    Answer: $\langle -0.4082, -0.7071, .5774\rangle = \langle -\sqrt{1/6}, -\sqrt{1/2}, \sqrt{1/3}\rangle$

    Explanation: Consider the cube in a standard position where $\vec{a} = \vec{0}$, $\vec{b} = \hat{\imath}$, $\vec{d} = \hat{\jmath}$. $\vec{e} = \hat{k}$; thus $\vec{g} = \langle 1, , 1, 1\rangle$. Since $\vec{a}$ is at the origin after the rotation, this is a pure rotation around the origin; it is therefore equivalent to multiplication by an orthogonal matrix M with determinant 1. The dot product $M(2,:)$ with any vector v is the y-coordinate of the rotated position of v. In particular since after the rotation both $\vec{b}$ and $\vec{g}$ have y-coordinate zero, $M(2,:)$ must be orthogonal to both; hence, it is their cross-product, normalized. Since the x coordinate of the rotated place of g is 0, and since M is orthogonal, $M(1,;)$ is orthogonal to both $\vec{g}$ and $M(2,:)$; hence, it is the normalized form of their cross product. Since M is orthogonal, $M(3,:)$ is the cross product of $M(1,:)$ and $M(2,:)$. All that remains is to make sure the signs are OK, to check, and to compute $M \cdot \vec{e}$.

    Matlab

```
% Problem 19

b = [1,0,0]
g = [1,1,1]
e = [0,0,1]
m = zeros(3);
u = cross(b,g) % both b and g get mapped to a vector with y component = 0
m(2,:) = u / norm(u)
v = cross(m(2,:),g)
m(1,:) = sign(dot(v,b))*v/norm(v)
m(3,:) = cross(m(1,:),m(2,:))
if (dot(m(3,:),g) < 0)
   m(2,:) = -m(2,:)
   m(3,:) = -m(3,:)
end
m*m' %Check: rotation matrix?
det(m)
m*b'  %Check: b mapped into xz plane with positive x?
m*g'  %Check: g mapped into positive z axis?
answer = m*e'
```

20. Joe and Jim each have a bank account which they started on January 1, 2000. Joe started his account with with \$1000; Jim started his with \$900. Joe's account pays 5% every December

31, and he adds an additional $500 every January 1. Jim's account pays 10% every December 31. When will Jim's account have more money in it than Joe's?

Answer: 2053. (More precisely, December 31, 2052).

Matlab:

```
joe = 1000
jim = 900
count = 0
while (jim <= joe)
  count = count+1
  jim = jim*1.10
  joe = joe*1.05 + 500
end
2000+count
```

21. Two points **p** and **q** are independently randomly chosen following a uniform distribution in a three-dimensinal sphere of radius 10^7. With a relative error of less than 1%, give an estimate with an error no greater than 1% that the Euclidean distance from **p** to **q** is less than 1?

    Answer: 10^{-21}.

    Explanation: For any point **p** not close to the boundary of the large sphere, **q** will be within distance 1 of **p** if and only if it is inside the sphere radius 1 around **p**. That breaks down if **p** is within 1 of the boundary of the large sphere, but the probability of that is tiny (around $4\pi \cdot 10^{-7}$). So the probability that `q` is inside the small sphere given that it is in the large sphere is the ratio of the volumes of the spheres $= 10^{-21}$.

22. A point **p** is chosen at random within the 100-dimensional box $\mathbf{B} = [0, 100]^{100}$ following a uniform distribution. What is the probability that the Euclidean distance from **p** to the boundary of **B** is less than 1?

    Answer: $1 - 0.98^{100} \approx 1 - 1/e^2 = 0.8647$

    Explanation: The distance from **p** to the boundary is $\min_{i=1..k} \min(p_k, 100 - p_k)$ That is, the distance is more than 1 if and only if $\mathbf{p} \in [1, 99]^{100}$. So the probability that the distance is less than one is $(100^{100} - 98^{100})/100^{100}$.

23. A point **p** is chosen at random within the 100-dimensional box $\mathbf{B} = [0, 1]^{100}$ following a uniform distribution. What is the mean value of the Euclidean distance from **p** to the boundary of **B**?

    Answer: 1/202.

    Explanation: Pick one particular face of the hypercube; for convenience the lower face in the x-dimension $\mathbf{F} = \{0\} \times [0, 100]^{99}$. Let $\mathbf{c} = \langle 0.5, 0.5, \ldots 0.5 \rangle$ be the center of **B**. Let **P** be the hyperpyramid consisting of the convex hull of $\mathbf{F} \cup \{ \mathbf{c} \}$. It is obvious, and easy to show, that the points in **B** whose closest boundary point is **F** are exactly those in **P** and that the distance of a point in **P** to **F** is just its x-coordinate. Moreover, since **B** can be partitions into such pyramids, one for each face of **B** and they all have identical distributions of distances to the boundary, the mean distance to the boundary is the same in **P**. Therefore the mean distance for a point in **G** to the boundary of **B** is the same as mean distance for the distance from a point in **P** to `F` which is just the mean x-coordinate of a point in **P**.

    The rest is just calculus. The cross-section of **P** at x-coordinate x is the box $[x, 1 - x]^{99}$ of 99-d volume $(1 - 2x)^{99}$. We could integrate this to compute the volume of **P** but it's simpler to note that there is one such pyramid for each face of **B**; there are 200 such faces; and their union of the pyramids is all of **B** which has volume 1. So the volume of **P** is 1/200. So the probability density of a uniform choice in **P** as a function of `x` is $200(1 - 2x)^{99}$.

The expected value of x in $\mathbf{P}$ is therefore

$$\int_{x=0}^{0.5} 200(1-2x)^{99} x dx = 200 \int_{u=0}^{1} u^{99} \cdot ((1-u)/2) \cdot (du/2) = (200/4) \int_{u=0}^{1} u^{99} - u^{100} du =$$

$$(200/4) \cdot (u^{100}/100 - u^{101}/101)|_0^1 = 1/202.$$

24. A pound of graphite is inside a closed cylindrical container with a radius of 50 cm and a height of 200 cm, filled with air at room temperature (22° C) and atmospheric pressure. The graphite reacts with the oxygen, producing equal masses of carbon monoxide and carbon dioxide until one of the reactant chemicals is exhausted. How many moles of each chemical is in the container at the end?

    Answer: No oxygen, 11.86 moles of carbon monoxide, 7.61 moles of carbon dioxide, 18.2 moles of carbon.

    Matlab:

```
CAtomMass = 12.011
OAtomMass = 16
COMolecMass = CAtomMass+OAtomMass
CO2MolecMass = CAtomMass+2*OAtomMass
MolesOfCarbon = 453.6/CAtomMass % Wikipedia
Temp = 273.15 + 22  % Wikipedia
Liters = pi*200*50^2/1000 % Volume in liters
IdealGasConst = 0.082057 % l*atm*K^{-1}*mol^{-1}. Wikipedia
MolesOfAir = Liters/(Temp*IdealGasConst)
MolesOfO2 = MolesOfAir * .2095
MoleFracOfCO = CO2MolecMass/(COMolecMass+CO2MolecMass)% Molar fraction of CO in product
A = 2*MolesOfO2/(2-MoleFracOfCO)
MolesOfCO = MoleFracOfCO * A
MolesOfCO2 = (1-MoleFracOfCO) * A
MolesOfCO*COMolecMass - MolesOfCO2*CO2MolecMass %check
MolesOfCO + 2*MolesOfCO2 - 2*MolesOfO2  %check
EndMolesOfCarbon = MolesOfCarbon - (MolesOfCO+MolesOfCO2)
```

25. Suppose that you have a gram of pure radium 233. What is the mass of the helium that that will generate via fission in a week?

    Answer: 0.0248 gm

    Explanation: Radium 223 undergoes a series of nuclear reactions:
    Radium 233 → Radon 219 + alpha
    Radon 219 → Polonium + alpha
    Polonium 215 → Lead 211 + alpha
    Lead 211 → Bismuth 211 + beta
    Bismuth 211 → Thallium 207 + alpha
    Thallium 207 → Lead 207 + beta
    Thus each atom of radium generates 4 alpha particles.

    The first reaction has a half-life of 11.43 days; the others have half-lifes ranging from fractions of a second to a few minutes.

    Matlab:

```
radiumMass = 223
heliumMass = 4
halfLife = 11.43 % Half life of radium 223 in days
f = (1/2)^(7/halfLife)
answer = (1-f)*4*heliumMass/radiumMass
```

26. A small school has 100 students and three extramural activities: the basketball team, the chess club, and the drama society. There are 60 students in the band, 48 in the chess club, and 55 in the drama club. There are 18 students who do both band and chess, 28 who do both band and drama, 31 who do both chess and drama, and 8 students who are in all three groups. What is the probability that a student is not in the band given that they are either in the drama club or the chess club but not both?

27. You draw 20 cards from a deck of cards labelled 1..60. To your suprise, the cards can be arranged as an arithmetic sequence. What is the probability that the successive difference is 3? Give an exact answer as a rational number.

    Answer: $3/66 = 1/22$.

    Explanation: There are 41 sequences of difference 1 (starting with every value from 1 to 41); 22 sequences with difference 2 (starting from 1 to 22); and 3 with difference 3 (starting points 1, 2, and 3).

28. You draw 5 cards from a deck of cards labelled 1 ... 200. To your surprise the cards can be arranged as a geometric series. What is the probability that the smallest number is less than or equal to 7? (Note that the ratio need not be an integer.)

    Answer: 9/16

    Explanation: The possible increasing ratios are 2, 3, and 3/2.

    If the ratio is 2, the sequence has the form a, 2a, 4a, 8a, 16a. a can be any integer from 1 to 12. In 7 of these, $a \leq 7$.

    If the ratio is 3, the sequence has the form a, 3a, 9a, 27a, 81a. a is either 1 or 2. In both of these, $a \leq 7$.

    If the ratio is 3/2, the sequence has the form 16a, 24a, 36a, 54a, 81a, and a can be either 1 or 2. In both of them the smallest number is greater than 7.

    So the probability is 9/16.

29. In this question and the following three, assume that the satellite is moving in a closed orbit around the Earth and that the only influence on the satellite's motion is the Earth's gravity. Assume that the Earth is a perfect sphere. Ignore the revolution of the Earth around the sun, but do not ignore the rotation of the Earth around its axis.

    A satellite in a circular geosynchronous orbit remains directly above the point on the earth's equator 0° N, 50° E. What is the straight line distance from the satellite to Houston?

    Answer: 46,934 km.

    Matlab:

```
% Method 1: Compute the positions of Houston and the satellite
% as 3 dimensional vectors
r = 6378.1 % km. Earth's radius
q = 35786 % height of geosynchronous orbit over earth's surface
rs = r+q % distance from center of earth to satellite
```

```
deg2rad = pi/180;  % degrees to radians
hlat = 29.75*deg2rad    % Houston latitude and longitude
hlong = -95.38*deg2rad
uh = [cos(hlat)*cos(hlong), cos(hlat)*sin(hlong),sin(hlat)] % unit vector Houston
h = r*uh % Houston location as vector
slong = 50*deg2rad
us = [cos(slong),sin(slong),0]
s = rs*us % Satellite location as vector
answer1 = norm(s-h)

% Method 2 Apply law of cosines to triangle Houston - center of earth- satellite
cosTheta = dot(us,uh) % cosine of angle at center of earth
sqrt(r^2 + rs^2-2*cosTheta*r*rs)
```

30. A satellite in a circular geosynchronous orbit passes directly above the North and South Poles. When it crosses the North Pole, its velocity is in the plane of the 0° circle of longitude. At what longitude does it pass directly above a point in the Tropic of Cancer? (An answer will be marked correct if it is within 3 degrees of the correct answer.)

    Answer: 66.56° W.

    Explanation: The period of a geosychronous orbit is equal to a sidereal day. In that time, the satellite executes a 360° revolution and the earth executes a 360° rotation, so their angular velocities are equal. So the longitude is equal to the difference in latitude between the North Pole and the Tropic of Cancer. Since the Tropic of Cancer is at 23.44 degrees, the difference is 66.56 degrees.

31. A satellite orbits the earth in a circular orbit. It completes an orbit every 14 hours 40 minutes. How high is it above the Earth's surface?

    Answer: 24,042 km

    Matlab:

```
%  By Kepler's 3rd law, R^3 is proportional to T^{2}, so R is proportional to T^(2/3)
r = 6378.1 % km. Earth's radius
q = r+35786 % km. distance of geosynchronous orbit from center of earth
e = 23+(56/60) % hours. Period of geosynchronous orbit.
z = q*((14+(40/60))/e)^(2/3) % Distance from satellite to center of earth
answer = z-r
```

32. A satellite orbits the earth in a circular orbit. It passes directly over the North and South poles and completes an orbit every 14 hours 40 minutes. On one orbit going southward it was directly above the earth location 40° N, 10° W at 1:00 PM EST. At what time will it next cross the plane that contains the circle of latitude 40° N, and what will be its longitude? (An answer will be marked correct if the time is within 5 minutes of the correct answer, and the longitude is within 3 degrees.)

    Answer: Time: 2:19 PM. Longitude: 29.77 degrees W ($= 20^\circ \circ 40'$ W)

    Matlab:

```
% Let c be the center of the earth.
% Let P be the plane containing the circle of 40 degrees latitude.
% Let s be the position of the satellite when it crosses P
% Let a be the point where the line from c to the North Pole crosses P.
```

```
% Let r be the radius of the earth and let h be the height of the satellite

% cas is a right triangle. |ca| = r*sin(40 degrees). |cs| = r+h
% So the angle acs = arcsin(|ca|/|cs|).
% The angle traversed since 1 PM is 40-(90-acs)
% The rest of the calculation is as in problem 29

r = 6378.1 % km. Earth's radius
h = 24042 % height of satellite over Earth's surface. Problem 30
w = 14+(40/60) % period of satellite
d = 23+(56/60) % sidereal day
deg2rad = pi/180
ca = r*sin(40*deg2rad)
cs = r+24042 % Problem 30
acs = asin(ca/cs)
at = 40-(acs/deg2rad) % angle traversed in degrees
lat = 10 + (w/d)*at % latitude
t = 1 + w*(at/360) % time
```