A QUALITATIVE CALCULUS FOR THREE-DIMENSIONAL ROTATIONS

THESIS

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Vitae

The author was born on November 30, 1984, in Tehran, Iran. She has graduated from Sharif University of Technology in Computer Engineering and expects to be graduated in the Master of Science in Computer Science from Polytechnic Institute of New York University by the end of December 2011. The thesis was started on June 2011 and finished on December 2011. The research has been conducted in Department of Computer Science in Courant Institute of Mathematical Science, New York University, under supervision of Professor Ernest Davis.

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Azam A. Asl, December 2011.

AN ABSTRACT

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by

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This thesis presents a qualitative calculus for three-dimensional directions and rotations. A direction is characterized in terms of the signs of its components relative to a fixed coordinate system. A rotation is characterized in terms of the signs of the components of the associated 3×3 rotation matrix. A system has been implemented that can solve the following problems:

1. Given the signs of direction $\vec{V}$ and rotation P, find the possible signs of the image of $\vec{V}$ under P, $\vec{V} \cdot P$. Moreover, for each possible sign vector $\vec{S}$ for $\vec{V} \cdot P$, generate an exact instantiation of $\vec{V}$ and P for which the sign of $\vec{V} \cdot P$ is $\vec{S}$.
2. Given the signs of rotations P and Q, find the possible signs of the composition $P \cdot Q$. Moreover for each possible sign S of $P \cdot Q$, generate an exact instantiation of P and Q for which the sign of $P \cdot Q$ is S.

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Introduction

The field of Qualitative Spatial Reasoning (QSR) develops methods for carrying out geometric computations using qualitative information about spatial properties and relations, rather than numerically precise information [8]. The majority of the QSR literature has addressed reasoning about topological constraints between regions; the best known theory is the RCC-8 system of relations [10]. However, other work in the area has addressed other geometric properties such as convexity, relative position, and relative size. A qualitative calculus is a theory that describes how an inference engine can use a constraint network of qualitative geometric relations to draw conclusions that are implicit but not explicit in the network.

The research described in this thesis develops the first qualitative calculus for three-dimensional directions and rotations. Our calculus is based on the well-known sign calculus over the three values $+, -$, and 0. A standard reference rectangular coordinate system is fixed. A direction $\vec{v}$ is then characterized in terms of the signs of the components of $\vec{v}$ in the x, y and z direction.

There are thus 27 possible combinations of signs; however, since $< 0, 0, 0 >$ is not a direction, there are 26 possible sign vectors for directions. A rotation Q is characterized in terms of the signs of the elements of the rotation matrix. As we will discuss below, we have determined that there are 336 sign matrices that correspond to possible rotation matrices, which we will call base rotations. A system has been implemented that can solve the following problems:

1. Given the signs of direction V and rotation P, what are the possible signs of the image

of V under P?

2. Given the signs of two rotations P and Q, what are the possible signs of the composition of P and Q?

Moreover, for each of these problems, the program outputs an exact instantiation demonstrating the feasibility of the solution. That is, for each sign equation $\vec{v} \cdot P = \vec{u}$ or $P \cdot Q = R$, where $\vec{v}$ and $\vec{u}$ are sign vectors and P, Q, and R are sign matrices, the program generates values $\overrightarrow{v'}$, P', $\overrightarrow{u'}$ or P', Q', R' that satisfy the equations and have the specified sign.

A number of qualitative calculi, including the *Star Calculus* and $\mathcal{OPRA}_m$ [2, 4] have been developed for two-dimensional directions and rotations. However, that is a much simpler theory, for three reasons:

1. The space of two-dimensional rotations is isomorphic to the space of two-dimensional directions.
2. Two-dimensional rotations commute.
3. In two dimensions, both applying a rotation to a direction and composing two rotations correspond to the simple operation of adding angles mod 2π. In three dimensions, (1) and (2) are false and no simple formula analogous to (3) exists.

As implied above, a qualitative calculus is basically a constraint satisfaction problem; Arc-consistency methods were first applied to qualitative reasoning in Allen's [1] temporal interval calculus, and thereafter it became a common approach in temporal and spatial reasoning. In our project there are two different categories of constraint networks:

1. Each node is a set of sign directions and each arc is a rotation.
2. Each node is a set of sign rotations and each arc is a rotation as well.

Each one of these categories was divided into independent constraint networks. To determine the consistency of each one of these networks we used path-consistency to rule out the inconsistent cases as much as possible and then used Waltz propagation [7] to instantiate the remaining with reals.

By taking advantage of symmetries among the base rotations, we were able to categorize them into 14 distinct categories. Likewise the 26 sign directions can be divided into 3 categories. In this way, the number of the CSPs we need to solve for the vector rotation problem is reduced to 3×14 from 27×336 and the number of rotation composition problems is reduces to 14×14 from 336×336.

Chapter 1

Background

In this chapter we first provide a review of the mathematics notations used and then present a definition of the qualitative calculus for three-dimensional rotations. In continue we give a review of the relevant background on two-dimensional calculi.

1.1 ROTATION IN THREE-DIMENSIONAL SPACE

Three-dimensional rotations Γ can be characterized in terms of systems of three angles such as the Euler angles or Yaw-pitch-roll. However, none of these angular systems are at all convenient to use for computing compositions of rotations. Instead, we use a rotation matrix M as defined; Let $\hat{x} = [1 \quad 0 \quad 0]$, $\hat{y} = [0 \quad 1 \quad 0]$ and $\hat{z} = [0 \quad 0 \quad 1]$ be the unit coordinate vectors in a fixed coordinate system and let Γ be a rotation. If $\hat{X} = \Gamma(\hat{x})$, $\hat{Y} = \Gamma(\hat{y})$ and $\hat{Z} = \Gamma(\hat{z})$, then the corresponding rotation matrix M is the 3×3 matrix:

$$M = \begin{bmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{bmatrix}.$$

In general, for arbitrary row vectors $\vec{v}$ and $\vec{u}$, if the equation $\vec{u} = \Gamma(\vec{v})$ holds, one can rewrite it as $\vec{u} = \vec{v} \cdot M$. Some useful properties of M are as follows:

1. The domain of rotation matrices is closed under composition and inverse.
2. M is an orthogonal matrix. That is $M^T \cdot M = I$, which implies that $M^{-1} = M^T$.

3. $M[i,:] \cdot M[i,:] = 1$. Using (1) and (2) we have $M[:,i] \cdot M[:,i] = 1$.
4. If $i \neq j$, $M[i,:] \cdot M[j,:] = 0$. Using (1) and (2) we have $M[:,i] \cdot M[:,j] = 0$
5. For any two rows (columns) of M; $M[i,:] = [x_i \quad y_i \quad z_i]$ and $M[j,:] = [x_j \quad y_j \quad z_j]$, the third row is the plus or minus the cross-product: $M[k,:] = [y_i z_j - z_i y_j \quad z_i x_j - x_i z_j \quad x_i y_j - y_i x_j]$.
6. The determinant of a rotation matrix without reflection is 1. The determinant of a rotation matrix with reflection is -1. In this project, we have excluded reflections.
7. The number of zeros in a rotation matrix is zero, one, four or six.

1.2 SIGN CALCULATION

A sign is the arithmetic sign: –, 0 or +. The table of sign negation, addition and multiplication in sign calculus are shown below. Subtraction and division are symmetric to addition and multiplication respectively. Within the tables, *I* means indefinite.

~	–	0	+	I
	+	0	–	I

Table 1- Negation

+	–	0	+	I
–	–	–	I	I
+	I	+	+	I
0	–	0	+	I
I	I	I	I	I

Table 2-Addition

×	–	0	+	I
–	+	0	–	I
+	–	0	+	I
0	0	0	0	0
I	I	0	I	I

Table 3- Multiplication

1.3 SIGN VECTOR

The $\hat{x}, \hat{y}$ and $\hat{z}$ coordinate axes divide three-dimensional space into 27 distinct parts. These parts can be perfectly represented by a triple of signs; $< 0, 0, 0 >$, represents the single-point origin in the coordinate system. The positive x-*axis* could be represented by $< +, 0, 0 >$, the negative x-*axis* could be represented by $< -, 0, 0 >$. Similarly $< 0, +, 0 >, < 0, -, 0 >$, $< 0, 0, + >$ and $< 0, 0, - >$ represent the positive y-*axis,* negative y-*axis,* positive z-*axis and* negative z-*axis* respectively. Any of the 12 one-zero sign triples represent the corresponding plane in the coordinate system and finally any of the 8 non-zero sign triples corresponding three-dimensional region of space. The same representation of the partition can be applied for any sign vector which lies within the partition.

1.4 PROBLEM DEFINITION

Our formulation of qualitative inference for three-dimensional rotations, which we call the *signed matrix rotation problem* can be stated as follows: Given

- A collection of variables over three-dimensional vectors $\vec{v}_i, i = 0 \dots n$.
- A collection of variables over three-dimensional rotation matrices $M_p, p = 1 \dots m$. (i.e. orthonormal matrices with determinant 1).
- For some subset (possibly null) of the vector variables, a specification of the signs.
- A collection of equations of the form $\vec{v}_i = \vec{v}_j \cdot M_p$
- For each matrix M_p, a specification of the signs of its elements.

Determine whether the specification is consistent; that is, whether there exist vectors and matrices over the reals that satisfy both the equations and the sign constraints.

For example, given the equations

$$\vec{v}_0 = [1, 0, 0] \qquad \vec{v}_2 = \vec{v}_1 \cdot M_1 \qquad \vec{v}_0 = \vec{v}_2 \cdot M_1 \qquad \vec{v}_0 = \vec{v}_1 \cdot M_2$$

and the constraints

$$sg(M_1) = \begin{bmatrix} + & + & + \\ + & - & - \\ - & + & - \end{bmatrix}$$

$$sg(M_2) = \begin{bmatrix} + & + & + \\ + & - & + \\ + & - & - \end{bmatrix}$$

One solution is:

$$\vec{v}_1 = \left[\frac{7}{9}, \frac{4}{9}, \frac{4}{9}\right] \qquad \vec{v}_2 = \left[\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right]$$

$$M_1 = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \frac{7}{9} & \frac{28}{45} & \frac{4}{45} \\ \frac{4}{9} & -\frac{29}{45} & \frac{28}{45} \\ \frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \end{bmatrix}$$

1.5 A CALCULUS FOR THREE-DIMENSIONAL ROTATIONS

A common AI technique for solving systems of constraints, particularly in qualitative reasoning, is to use a combinations of label propagation (also known as Waltz propagation) and arc propagation [11]. A constraint network is a directed graph, where each node is labeled with a set of possible qualitative values and each arc is labeled with a set of possible qualitative relations. To implement label propagation, a constraint satisfaction engine uses a module that can solve the following problem:

"Given that variable x has qualitative value V and x and y are related by qualitative relation R, what are the possible values of y?"

To implement arc propagation, a constraint satisfaction engine uses a module that solve this problem:

"Given that x and y are related by qualitative relation P and that y and z are related by R, what are the possible qualitative relations between x and z?"

On our particular application, the nodes are the directions, and the qualitative labels are the sign vectors. The relations between nodes are rotations, and the qualitative relations are the sign matrices. We have developed modules that solve these two problems for this domain. Having implemented these subroutines, the incorporation of these in a full constraint-propagation engine that does label and arc propagation is a standard programming exercise; we have not implemented these.

1.6 OTHER DIRECTION CALCULI

The two-dimensional qualitative directional calculi divide the plane into different parts in respect to a reference point. The semantic of this partitioning is based on human sense of direction; left- right, front- back and up-down. Another common terminology for this is in geographical usage; there they usually use North, West, East and South.

Among the recent direction calculi there are Star Calculus ($\mathcal{STAR}_m$) [2], $\mathcal{OPRA}_m$ [4]. Both of them are calculi with arbitrary granularity. $\mathcal{STAR}_m$ is a generalization of a number of earlier calculi, including those introduced by Frank [5]. Given a global reference point, using m lines $\mathcal{STAR}_m$ divides the plan into $4m + 1$ disjoint zones with respect to reference direction: $2m$ half lines resulting from m lines, $2m$ two-dimensional sectors and the reference point. If you want a Star calculus with different sector angles you need to mention the desired angle each line forms with reference direction in order. (Figure 1).

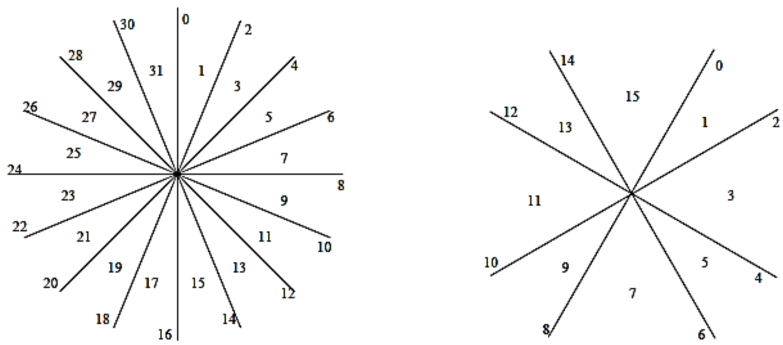

Figure 1- Left: $\mathcal{STAR}_8(0)$, right : $\mathcal{STAR}_4(30,\ 60, 120\ , 150)$

Oriented Point Relation Algebra $\mathcal{OPRA}_m$ is a calculus consists of directional relations on the domain of oriented points, o-points, which can be considered as $\mathcal{OPRA}_1$ (Figure 2). $\mathcal{OPRA}_m$ is the generalization of $\mathcal{OPRA}_1$; fore the given level of granularity m, it divides the space into $2m$ equal sectors. Figure 3 shows the existing relation between two o-points A and B in $\mathcal{OPRA}_2$.

In figure 3, m=2 results in relation A 2 $<_{7}^{1}$ B. That is B lies in segment 7 regarding A and A lies in segment 1 relative to B.

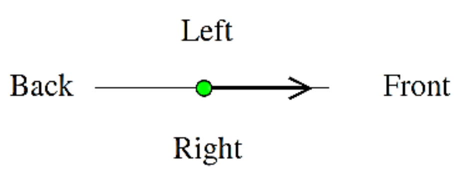

Figure 2- An o-point and its relative directions

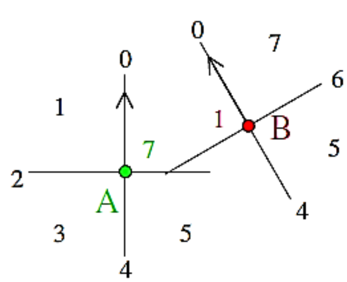

Figure 3- Two o-points in relation $A\ 2 <_{7}^{1}\ B$.

1.7 IMPLEMENTATION DETAILS

The program is written in Java and is about 2000 line of code. The size of the pre-computed real instances is about 2000. The source code can be downloaded from: http://cs.nyu.edu/QualitativeRotations/

The interface is as follows: enter 1 to run the vector rotation calculus or enter 2 to run the rotation composition calculus. To enter a sign vector simply enter a sequence of $-, 0$ or $+$. To enter a sign rotation matrix, first enter the first and the second rows respectively. The program will then display all possible third rows; choose one of them. If there is no matching rotation matrix, the system would exit.

Chapter 2

Three-Dimensional Sign Rotations

In order to define the qualitative calculus for rotations we need to find the set of all possible three-dimensional sign rotation matrices. This chapter explains the method we used to identify 336 possible rotation matrices. We will show that all of these 336 rotation matrices are distinct in their geometrical aspect and cover the entire space of three-dimensional rotations. That is, the set of these 336 base rotation matrices $\mathcal{A}$, is a set of Jointly Exhaustive and Pairwise Disjoint (JEPD) binary relations.

We came up with a categorization of rotation matrices. Each member of a category can be converted to the other member in the same category, by applying some appropriate operation. In this way, we can greatly reduce the size of the problem. The idea was to take advantages of the symmetries of an octahedron.

2.1 SIGN ROTATION MATRICES

We can map any sign triple (vector) to one of the faces/edges/vertices of a regular octahedron centered at the origin of the coordinate system with vertices located on unit distance of the origin. For example $< +, +, + >$, $< +, -, 0 >$ and $< -, 0, 0 >$ can be mapped to a face, edge and vertex in octahedron respectively. In this way, a sign rotation matrix could be defined by three features of the octahedron.

There are 24 rotations that map the coordinate directions to the positive or negative coordinate directions. To show this, one can map $\hat{x}$ to any of the 6 coordinate directions: $\pm\hat{x}, \pm\hat{y}$ or $\pm\hat{z}$. Having chosen $\Gamma(\hat{x})$ you can map $\hat{y}$ to any of the 4 orthogonal directions. Once $\Gamma(\hat{x})$ and $\Gamma(\hat{y})$ are both determined, so is $\Gamma(\hat{z})$. The other proof is that the triangle of the octahedron formed by the $\hat{x}, \hat{y}$ and $\hat{z}$ axes is mapped to one of the 8 faces of the octahedron; and for each choice of target face, it can be aligned in any of 3 ways.

These 24 coordinate rotations map the octahedron to itself, mapping vertices to vertices, edges to edges, faces to faces and preserving structure. We will call these "octahedral" rotations. Clearly they form a group; the composition of two octahedral rotations is an octahedral rotation and so is the inverse. Table 4 represents the list of 24 octahedral rotations. If $\hat{u}$ and $\hat{v}$ are directions with the same sign triple, they are located on the same piece of the octahedron, and therefore they will still be on the same piece after an octahedral rotation. Therefore, if A and B are two rotational matrices with the same signs, and P is an octahedral rotation then $A \cdot P$ has the same sign as $B \cdot P$. Therefore, we can define an equivalence relation over the sign rotation matrices: A is equivalent to B if $B = A \cdot P$ for some octahedral rotation P.

Since every equivalence class of sign rotation has 24 elements; it is not necessary to enumerate all valid sign rotation matrices; it suffices to identify a representative from each equivalence class. Once we can instantiate a representative *Q,* we can then instantiate the other 23 elements by rotating the instance with the octahedral rotations. This greatly reduces the complexity of analyzing the sign rotation matrices.

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} \quad \begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix} \quad \begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\quad\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\quad\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&-1&0\\0&0&1\\-1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\quad\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&-1\\-1&0&0\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\-1&0&0\\0&-1&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}$$

Table 4- 24 octahedral rotations

Since the octahedral rotations preserve the geometry of the octahedron, we can be sure that two sign rotations *A* and *B* are from different equivalence classes if the two triples $\langle \hat{x}\cdot A,\ \ \hat{y}\cdot A,\ \ \hat{z}\cdot A\rangle$ and $\langle \hat{x}\cdot B,\ \ \hat{y}\cdot B,\ \ \hat{z}\cdot B\rangle$ are geometrically different, in terms of features of the octahedron. Note that these are the rows of *A* and *B* respectively.

2.2 REDUCED LIST OF SIGN ROTATIONS

Below is a list of 14 sign rotation representatives, with geometric features that guarantee that they are distinct. These fall into 6 categories:

- Category 1 includes three representatives. In these *Q [1, :], Q [2, :] and Q [3, :]* (each one of the three rows in matrix) are all in the faces that share the vertex $\langle 1, 0, 0\rangle$. See figure 4 bellow.
    1. *Q[1, :]* shares an edge with both *Q [2, :], Q [3, :],* $Q\colon\begin{bmatrix}+&+&+\\+&-&+\\+&+&-\end{bmatrix}$.
    2. *Q [2, :]* shares an edge with both *Q [1, :], Q [3, :],* $Q\colon\begin{bmatrix}+&+&+\\+&-&+\\+&-&-\end{bmatrix}$.
    3. *Q [3, :]* shares an edge with both *Q [1, :], Q [2, :],* $Q\colon\begin{bmatrix}+&+&+\\+&-&-\\+&+&-\end{bmatrix}$.

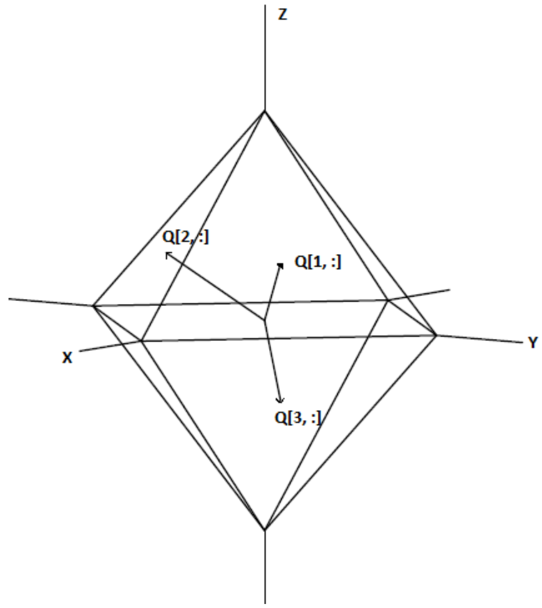

Figure 4- ***Q[1, :]*** **shares an edge with both** ***Q [2, :], Q [3, :].***

- Category 2 includes a single representative. In this *Q [1, :], Q [2, :],* and *Q [3, :]* are all inside faces. Each of the faces connects to both of the other at a vertex. See the figure 5 bellow.

4. $Q\colon \begin{bmatrix} + & + & + \\ + & - & - \\ - & + & - \end{bmatrix}$

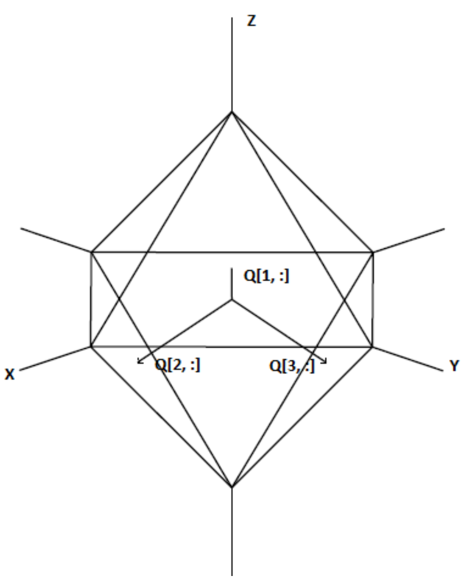

Figure 5

- Category 3 includes 3 representatives. In these, two of the row vectors lie in a face and one lies in an edge. The two faces have a common vertex; the edge connects two of the other vertices of the edge of the third vector in a vertex. See figure 6 bellow.

5. *Q [1, :]* is in the edge, Q: $\begin{bmatrix} + & + & 0 \\ - & + & + \\ + & - & + \end{bmatrix}$.
6. *Q [2, :]* is in the edge, Q: $\begin{bmatrix} + & + & + \\ + & 0 & - \\ - & + & - \end{bmatrix}$.
7. *Q [3, :]* is in the edge, Q: $\begin{bmatrix} + & + & + \\ - & - & + \\ + & - & 0 \end{bmatrix}$.

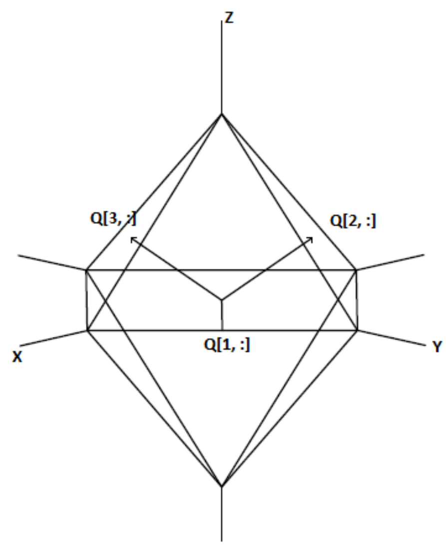

Figure 6- *Q [1, :]* is in the edge.

- Category 4 includes 3 representatives. In these, two of the row vectors lie in a face and one lies in an edge. The two faces have a common edge; one vertex of the edge meets one of the shared vertices of the faces.

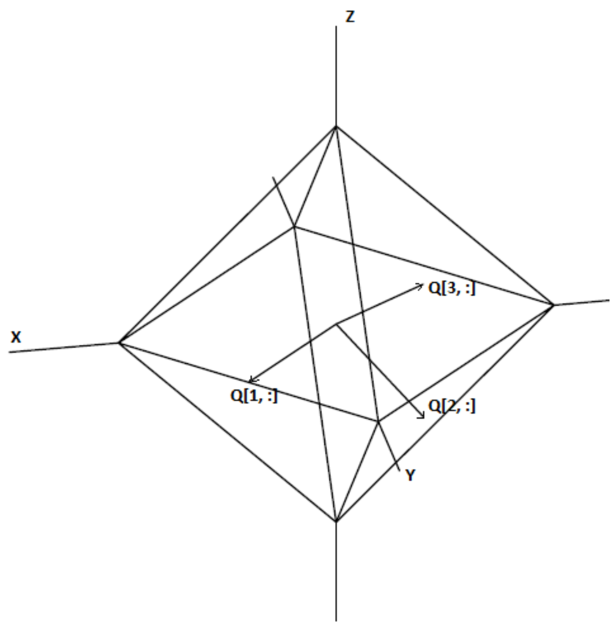

Figure 7- *Q [1, :]* is in the edge.

8. *Q [3, :]* is in the edge, Q: $\begin{bmatrix} + & + & + \\ + & + & - \\ - & + & 0 \end{bmatrix}$.
9. *Q [2, :]* is in the edge, Q: $\begin{bmatrix} + & + & + \\ - & 0 & + \\ + & - & + \end{bmatrix}$.
10. *Q [1, :]* is in the edge, Q: $\begin{bmatrix} + & + & 0 \\ - & + & - \\ - & + & + \end{bmatrix}$.

- Category 5 includes 3 representatives. In these, one vector is mapped to a vertex and the other two are mapped to edges.

11. *Q [1, :]* is mapped to the vertex, Q: $\begin{bmatrix} + & 0 & 0 \\ 0 & + & - \\ 0 & + & + \end{bmatrix}$.
12. *Q [2, :]* is mapped to the vertex, Q: $\begin{bmatrix} + & 0 & - \\ 0 & + & 0 \\ + & 0 & + \end{bmatrix}$.
13. *Q [3, :]* is mapped to the vertex, Q: $\begin{bmatrix} + & - & 0 \\ + & + & 0 \\ 0 & 0 & + \end{bmatrix}$.

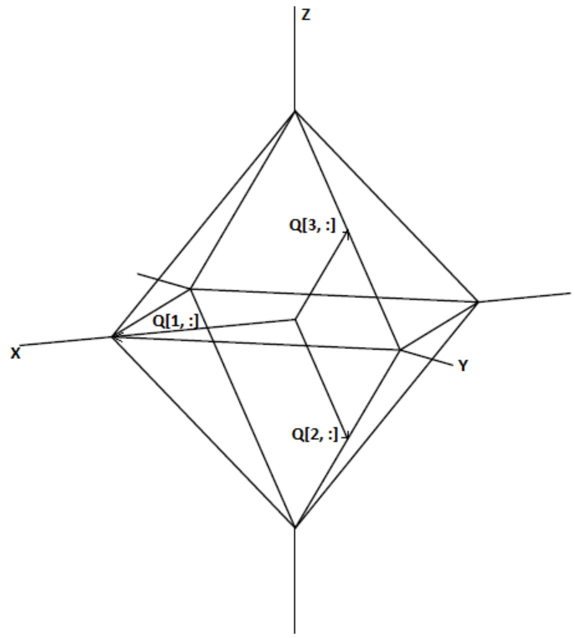

Figure 8- ***Q [1, :]*** **is mapped to the vertex.**

- Category 6 includes 1 representative. In this all the vectors are mapped to vertices.

  14. $Q: \begin{bmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{bmatrix}$.

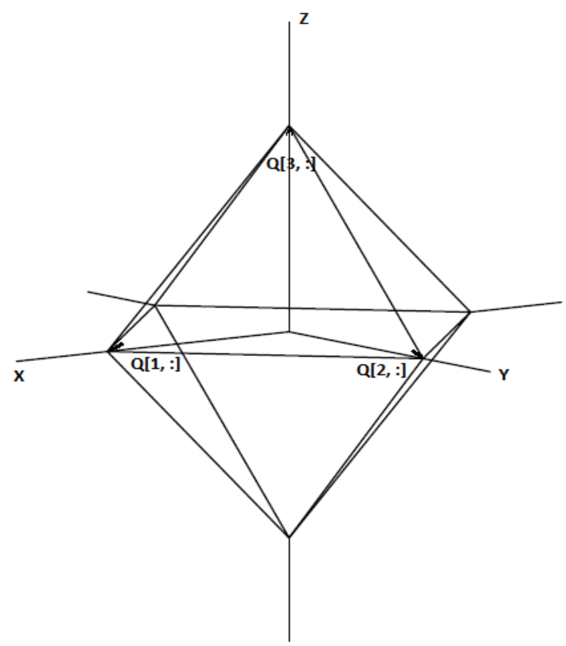

Figure 9

2.3 INSTANTIATION OF SIGN ROTATIONS

To randomly instantiate the sign rotations, we first instantiate any of the representatives and then rotate the instance with each one of the octahedral rotations. The necessary constraints to instantiate a rotation matrix can be derived from 1.1 cases 3-6. Note that all of these constrains are definable in sign calculus as well. In implementing the instantiation of the representatives we look at the zeros in the matrix and after randomly initializing enough number of the non-zero entries in the matrix, we get the values for the rest of none-zero entries based on the existing constraints in 1.1.

Chapter 3

Three-Dimensional Sign Vector Rotation

In this chapter we explain our algorithm to answer the first question mentioned in 1.6; that is, for any given sign vector $\vec{v}$, and any signed rotation matrix R, what is the maximal set of signed vectors $\vec{u}$, such that: $\vec{u} = \vec{v} \cdot R$? The possible number of these kind of questions are 26×336, as for 26 non-zero sign vectors and 336 base rotations.

3.1 VECTOR ROTATION

To compute $\vec{v} \cdot R$, our calculus converts the operation to a basic operation, which is pre-computed and saved. Once we look up the solution for the basic case, then we convert the solution to the solution of the current problem by applying an appropriate matrix multiplication.

Let us define the three sign directions $\overrightarrow{U_1}, \overrightarrow{U_2}$ and $\overrightarrow{U_3}$ as $\overrightarrow{U_1} = \langle +, 0, 0 \rangle$, $\overrightarrow{U_2} = \langle +, +, 0 \rangle$ and $\overrightarrow{U_3} = \langle +, +, + \rangle$. Any of the sign directions $\overrightarrow{U_1}, \overrightarrow{U_2}$ and $\overrightarrow{U_3}$ is a representative of the class of vertices, edges and faces in the octahedron respectively. Any arbitrary sign vector $\vec{v}$ maps to one of the vertex, edge or face in the octahedron. Using octahedral rotations it is possible to map $\vec{v}$ to its class representative. This factorization is not unique. In fact based on the class of $\vec{v}$, there are a number of octahedral rotations that map $\vec{v}$ to its class representative. If $\vec{v}$ has exactly two zeros, it belongs to class of vertex in octahedron. Since an octahedron has six vertices and there are 24 distinct octahedral rotations and since octahedral rotations maintain the structure of the octahedron, this means there are 24/6 or 4 octahedral rotations that map $\vec{v}$ to $\overrightarrow{U_1}$. Similarly if $\vec{v}$ belongs to class of edge in octahedron and since an octahedron has 12 edges, there are two different octahedral rotations that map $\vec{v}$ to $\overrightarrow{U_2}$. Lastly, if $\vec{v}$ belongs to class of face in octahedron and since there are eight faces in octahedron, there are three different octahedron rotations that map $\vec{v}$ to $\overrightarrow{U_3}$.

After we factor $\vec{v}$ as $\vec{v} = \overrightarrow{U_\iota} \cdot P$, we can compute $P \cdot R$. Recall that P is a real number matrix. $P \cdot R$ permutes the columns of R, which is still a sign rotation. Then we can factor this rotation to its equivalence class representative: $P \cdot R = Q_j . P'$. Since $\vec{v} \cdot R = \overrightarrow{U_\iota} \cdot Q_j \cdot P'$, once we was able to compute $\overrightarrow{U_\iota} \cdot Q_j$ we can simply output the solution to $\vec{v} \cdot R$ after we carry out a octahedral rotation P' on the result. Following is the summary of the algorithm:

1. Finds $\vec{U}$ and P such that $\vec{v} = \vec{U} \cdot P$;
2. Compute $P \cdot R$;
3. Find a representative Q_j and an octahedral rotation P' such that $(P \cdot R) = Q_j \cdot P'$;
4. Look up the pre-computed value of $\vec{U} \cdot Q_j$;
5. Compute $\left(\vec{U} \cdot Q_j\right) \cdot P'$;
6. Returns $(\left(\vec{U} \cdot Q_j\right) \cdot P')$;

To optimize the calculation we have pre-computed $\overrightarrow{U_\iota} \cdot Q_j$, where $i = 1,2,3$ and $j = 1, \ldots, 14$, which is the topic of the next part.

3.2 INSTANTIATION OF THE BASE CASES

The problem of computing the possible values of $\overrightarrow{U_\iota} \cdot Q_j$, can be decomposed into a set of individual CSPs. For example let $\vec{U} = < +, +, 0 >$ and

$$Q = \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix}.$$

Using the sign calculus directly, one can compute:

$$\vec{U} \cdot Q = [+ \quad + \quad 0] \cdot \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix} = [+ \quad I \quad I].$$

Instantiating the I's in all possible ways give the following:

$\{[+ \quad - \quad -],\ [+ \quad - \quad 0],\ [+ \quad - \quad +], [+ \quad 0 \quad -],\ [+ \quad 0 \quad 0],\ [+ \quad 0 \quad +],$

$[+ \quad + \quad -],\ [+ \quad + \quad 0],\ [+ \quad + \quad +]\}.$

However not all of these are possible. We must check each of these suggested values independently for consistency:

$$[+ \quad + \quad 0] \cdot \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix} = [+ \quad - \quad -]?$$

$$[+ \quad + \quad 0] \cdot \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix} = [+ \quad - \quad 0]?$$

$\vdots$

$$[+ \quad + \quad 0] \cdot \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix} = [+ \quad + \quad +]?$$

The consistency of each one of these decision problems can be computed using backtracking; That is, for each problem we try to find an instantiation with real values. The basic backtracking algorithm [8] takes as input a set of constraints θ over the set of relations $\mathcal{S} \subseteq 2^{\mathcal{A}}$. ($\mathcal{A}$ is the set of base relations) It selects an unprocessed constraint $x\{R\}y$ of θ, splits R into its base relations $B_1, \ldots, B_k$, replaces $x\{R\}y$ with $x\{B_i\}y$ and repeats this process until all constraints are refined. If the resulting constraints is consistent, which can be shown using the local consistency algorithm, then θ is consistent. Otherwise the algorithm backtracks and replaces the last constraint with the next possible base relation. For example let $\vec{U} = [U_x , U_y , 0]$,

$$Q = \begin{bmatrix} Q_{x1} & Q_{y1} & Q_{z1} \\ Q_{x1} & Q_{y2} & Q_{z2} \\ Q_{x3} & Q_{y3} & Q_{z3} \end{bmatrix}$$

and for the second possible resultant sign vector $[+ \quad - \quad 0]$, let :

$U_x , U_y, Q_{x1}, Q_{y1}, Q_{z1}, Q_{x2}, Q_{x3}$, and $Q_{y3} \in \mathbb{R}^+$ and Q_{y2}, Q_{z2} and $Q_{z3} \in \mathbb{R}^-$.

The set of constraints θ, is consist of 4 constraints as follows:

- Q must be an orthogonal matrix.
- $U_x Q_{x1} + U_y Q_{x2} > 0$.
- $U_x Q_{y1} + U_y Q_{y2} < 0$.
- $U_x Q_{z1} + U_y Q_{z2} = 0$.

We can refine the general constraint Q must be an orthogonal matrix in 3 different ways:

B_1: The dot product of first and second row in Q is zero: $Q_{x1}Q_{x2} + Q_{y1}Q_{y2} + Q_{z1}Q_{z2} = 0$.

B_2: The dot product of second and third row in Q is zero: $Q_{x2}Q_{x3} + Q_{y2}Q_{y3} + Q_{z2}Q_{z3} = 0$.

B_3: The dot product of third and first row in Q is zero: $Q_{x3}Q_{x1} + Q_{y3}Q_{y1} + Q_{z3}Q_{z1} = 0$.

For example if the backtracking algorithm (1.6) selects B_1, by applying constraint propagation the following three constraints would be added to the set of constraints:

- $Q_{y1}Q_{z2} > Q_{z1}Q_{y2}$.
- $Q_{z1}Q_{x2} > Q_{x1}Q_{z2}$.
- $Q_{x1}Q_{y2} < Q_{y1}Q_{x2}$.

The inconsistency of some of the CSPs is detectable before applying backtracking. For example in the given example above, we proved that four out of the nine cases are inconsistent. This is expanded in the next part.

3.3 COMPUTING $\overrightarrow{U}.Q$

In this section for any $\overrightarrow{U_i} \cdot Q_j$ operation, we identify the achievable (consistent) cases within the hypothesis space of the results. Since there are 3 direction representatives and 14 rotation representatives; there are 42 cases to study:

- $Q_1 = \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix}$

Case 1.1. $\overrightarrow{U_1} \cdot Q_1 = \langle +,+,+ \rangle$: This is a unique case. It is always achievable.

Case 1.2. $\overrightarrow{U_2} \cdot Q_1 = \langle +, \mathrm{I}, + \rangle$: All three possible cases are achievable.

Case 1.3. $\overrightarrow{U_3} \cdot Q_1 = \langle +, \mathrm{I}, \mathrm{I} \rangle$: All nine possible cases are achievable.

- $Q_2 = \begin{bmatrix} + & + & + \\ + & - & + \\ + & - & - \end{bmatrix}$

Case 2.1. $\overrightarrow{U_1} \cdot Q_2 = \langle +,+,+ \rangle$: This is a unique case. It is always achievable.

Case 2.2. $\overrightarrow{U_2} \cdot Q_2 = \langle +, \mathrm{I}, + \rangle$: All three possible cases are achievable.

Case 2.3. $\overrightarrow{U_3} \cdot Q_2 = \langle +, \mathrm{I}, \mathrm{I} \rangle$: All nine possible cases are achievable.

- $Q_3 = \begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix}$

Case 3.1. $\overrightarrow{U_1} \cdot Q_3 = \langle +,+,+ \rangle$: This is a unique case. It is always achievable.

Case 3.2. $\overrightarrow{U_2} \cdot Q_3 = \langle +, \mathrm{I}, \mathrm{I} \rangle$: Five out of nine possible cases are achievable.

Proof. Assume $x_i, y_i \; and \; z_i \in \mathbb{R}^+$,

$$Q = \begin{bmatrix} +x_1 & +y_1 & +z_1 \\ +x_2 & -y_2 & -z_2 \\ +x_3 & +y_3 & -z_3 \end{bmatrix}, \; Q \in Q_3$$

Since Q is a rotation matrix and the dot product of any two row vectors of it are zero:

$$x_1x_2 - y_1y_2 - z_1z_2 = 0 \;\rightarrow\; \begin{cases} x_1x_2 > y_1y_2 & (3.1) \\ x_1x_2 > z_1z_2 & (3.2) \end{cases}$$

$$x_2x_3 - y_2y_3 + z_2z_3 = 0 \;\rightarrow\; \begin{cases} y_2y_3 > x_2x_3 & (3.3) \\ y_2y_3 > z_2z_3 & (3.4) \end{cases}$$

$$x_3x_1 + y_3y_1 - z_3z_1 = 0 \;\rightarrow\; \begin{cases} z_3z_1 > x_3x_1 & (3.5) \\ z_3z_1 > y_3y_1 & (3.6) \end{cases}$$

Let also assume u_1 and $u_2 \in \mathbb{R}^+$, $\vec{u} = \langle u_1, u_2, 0\rangle$, so $\vec{u} \in \overrightarrow{U_2}$. Let's have $\vec{u} \cdot Q = \vec{a}$ and $\vec{a} = \langle a_1, a_2, a_3\rangle$, where $a_1 \in \mathbb{R}^+, a_2$ and $a_3 \in \mathbb{R}$. Therefore, $a_2 = u_1y_1 - u_2y_2$ and $a_3 = u_1z_1 - u_2z_2$.

$$if\ a_2 \geq 0 \;\rightarrow\; u_1y_1 \geq u_2y_2 \;\rightarrow\; \frac{u_1}{u_2} \geq \frac{y_2}{y_1} \qquad (3.7)$$

$$\xrightarrow{(3.4),(3.6)} y_2y_3z_3z_1 > z_2z_3y_3y_1 \;\rightarrow\; y_2z_1 > z_2y_1 \;\rightarrow \frac{y_2}{y_1} > \frac{z_2}{z_1} \xrightarrow{(3.7)} \frac{u_1}{u_2} > \frac{z_2}{z_1} \rightarrow$$

$$u_1z_1 > u_2z_2 \;\rightarrow\; a_3 > 0.$$

$$if\ a_3 \leq 0 \;\rightarrow\; u_1z_1 \leq u_2z_2 \;\rightarrow\; \frac{u_1}{u_2} \leq \frac{z_2}{z_1} \qquad (3.8)$$

$$\xrightarrow{(3.4),(3.6)} \frac{y_2}{y_1} > \frac{z_2}{z_1} \xrightarrow{(3.8)} \frac{u_1}{u_2} < \frac{y_2}{y_1} \rightarrow u_1y_1 < u_2y_2 \;\rightarrow\; a_2 < 0.$$

Case 3.3. $\overrightarrow{u_3} \cdot Q_3 = \langle +, \mathrm{I}, \mathrm{I}\rangle$: All nine possible cases are achievable.

- $Q_4 = \begin{bmatrix} + & + & + \\ + & - & - \\ - & + & - \end{bmatrix}$

Case 4.1. $\overrightarrow{U_1} \cdot Q_4 = \langle +, +, +\rangle$, This is a unique case. It is always achievable.

Case 4.2. $\overrightarrow{U_2} \cdot Q_4 = \langle +, \mathrm{I}, \mathrm{I}\rangle$: Five out of nine possible cases are achievable

Proof. Assume $x_i, y_i\ and\ z_i \in \mathbb{R}^+$,

$$Q = \begin{bmatrix} +x_1 & +y_1 & +z_1 \\ +x_2 & -y_2 & -z_2 \\ -x_3 & +y_3 & -z_3 \end{bmatrix}, Q \in Q_4.$$

$$x_1x_2 - y_1y_2 - z_1z_2 = 0 \ \rightarrow \begin{cases} x_1x_2 > y_1y_2 & (4.1) \\ x_1x_2 > z_1z_2 & (4.2) \end{cases}$$

$$-x_2x_3 - y_2y_3 + z_2z_3 = 0 \ \rightarrow \begin{cases} z_2z_3 > x_2x_3 & (4.3) \\ z_2z_3 > y_2y_3 & (4.4) \end{cases}$$

$$-x_3x_1 + y_3y_1 - z_3z_1 = 0 \ \rightarrow \begin{cases} y_3y_1 > x_3x_1 & (4.5) \\ y_3y_1 > z_3z_1 & (4.6) \end{cases}$$

Assume u_1 and $u_2 \in \mathbb{R}^+$, $\vec{u} = \langle u_1, u_2, 0\rangle$, so $\vec{u} \in \overrightarrow{U_2}$. Let's have $\vec{u} \cdot Q = \vec{a}$ and $\vec{a} = \langle a_1, a_2, a_3\rangle$, where $a_1 \in \mathbb{R}^+, a_2$ and $a_3 \in \mathbb{R}$.

$a_1 > 0$, $a_2 = u_1y_1 - u_2y_2$ and $a_3 = u_1z_1 - u_2z_2$.

$$if\ a_2 \leq 0 \ \rightarrow \ u_1y_1 \leq u_2y_2 \ \rightarrow \frac{u_1}{u_2} \leq \frac{y_2}{y_1} \qquad (4.7)$$

$$\xrightarrow{(4.4),(4.6)} z_2z_3y_3y_1 > y_2y_3z_3z_1 \ \rightarrow \ z_2y_1 > y_2z_1 \ \rightarrow \frac{z_2}{z_1} > \frac{y_2}{y_1} \xrightarrow{(4.7)} \frac{z_2}{z_1} > \frac{u_1}{u_2} \rightarrow$$

$$u_1z_1 < u_2z_2 \ \rightarrow \ a_3 < 0.$$

$$if\ \ a_3 \geq 0 \ \rightarrow \ u_1z_1 \geq u_2z_2 \ \rightarrow \frac{u_1}{u_2} \geq \frac{z_2}{z_1} \qquad (4.8)$$

$$\xrightarrow{(4.4),(4.6)} \frac{z_2}{z_1} > \frac{y_2}{y_1} \xrightarrow{(4.8)} \frac{u_1}{u_2} > \frac{y_2}{y_1} \rightarrow u_1y_1 > u_2y_2 \rightarrow \ a_2 > 0.$$

Case 4.3. $\overrightarrow{U_3} \cdot Q_4 = \langle \mathrm{I}, \mathrm{I}, \mathrm{I}\rangle$: Seven cases out of twenty seven hypotheses are achievable.

Proof. Assume u_1, u_2 and $u_3 \in \mathbb{R}^+$, $\vec{u} = \langle u_1, u_2, u_3\rangle$, so $\vec{u} \in \overrightarrow{U_3}$. Let's have $\vec{u} \cdot Q = \vec{a}$ and $\vec{a} = \langle a_1, a_2, a_3\rangle$, where $a_1, a_2$and $a_3 \in \mathbb{R}$.

$a_1 = u_1x_1 + u_2x_2 - u_3x_3$, $a_2 = u_1y_1 - u_2y_2 + u_3y_3$ and $a_3 = u_1z_1 - u_2z_2 - u_3z_3$.

$$if\ a_1 \leq 0 \begin{cases} u_3x_3 \geq u_1x_1 \ \rightarrow \ \dfrac{x_3}{x_1} \geq \dfrac{u_1}{u_3} & (4.9) \\ u_3x_3 \geq u_2x_2 \ \rightarrow \ \dfrac{x_3}{x_2} \geq \dfrac{u_2}{u_3} & (4.10) \end{cases}$$

$$\xrightarrow{(4.1),(4.3)} x_1x_2y_1y_3 > y_1y_2x_1x_3 \ \rightarrow \ x_2y_3 > y_2x_3 \ \rightarrow \frac{y_3}{y_2} > \frac{x_3}{x_2} \xrightarrow{(4.10)} \frac{y_3}{y_2} > \frac{u_2}{u_3} \rightarrow$$

$$y_3u_3 > y_2u_2 \ \rightarrow \ a_2 > 0.$$

$$\xrightarrow{(4.2),(4.5)} x_1x_2z_2z_3 > z_1z_2x_2x_3 \rightarrow x_1z_3 > z_1x_3 \rightarrow \frac{z_3}{z_1} > \frac{x_3}{x_1} \xrightarrow{(4.9)} \frac{z_3}{z_1} > \frac{u_1}{u_3} \rightarrow$$

$$z_3u_3 > z_1u_1 \rightarrow a_3 < 0.$$

$$if\ a_2 \le 0 \rightarrow \begin{cases} y_2u_2 \ge u_1y_1 \rightarrow \frac{u_2}{u_1} \ge \frac{y_1}{y_2} & (4.11) \\ y_2u_2 \ge u_3y_3 \rightarrow \frac{u_2}{u_3} \ge \frac{y_3}{y_2} & (4.12) \end{cases}$$

$$\xrightarrow{(4.1),(4.3)} x_1x_2y_1y_3 > y_1y_2x_1x_3 \rightarrow x_2y_3 > y_2x_3 \rightarrow \frac{y_3}{y_2} > \frac{x_3}{x_2} \xrightarrow{(4.12)} \frac{u_2}{u_3} > \frac{x_3}{x_2} \rightarrow$$

$$x_2u_2 > x_3u_3 \rightarrow a_1 > 0.$$

$$\xrightarrow{(4.4),(4.6)} z_2z_3y_3y_1 > y_2y_3z_3z_1 \rightarrow z_2y_1 > y_2z_1 \rightarrow \frac{y_1}{y_2} > \frac{z_1}{z_2} \xrightarrow{(4.11)} \frac{u_2}{u_1} > \frac{z_1}{z_2} \rightarrow$$

$$z_2u_2 > z_1u_1 \rightarrow a_3 < 0.$$

$$if\ a_3 \ge 0 \rightarrow \begin{cases} z_1u_1 \ge z_2u_2 \rightarrow \frac{z_1}{z_2} \ge \frac{u_2}{u_1} & (4.13) \\ z_1u_1 \ge z_3u_3 \rightarrow \frac{u_1}{u_3} \ge \frac{z_3}{z_1} & (4.14) \end{cases}$$

$$\xrightarrow{(4.4),(4.6)} z_2z_3y_3y_1 > y_2y_3z_3z_1 \rightarrow z_2y_1 > y_2z_1 \rightarrow \frac{y_1}{y_2} > \frac{z_1}{z_2} \xrightarrow{(4.13)} \frac{y_1}{y_2} > \frac{u_2}{u_1}$$

$$\rightarrow y_1u_1 > y_2u_2 \rightarrow a_2 > 0.$$

$$\xrightarrow{(4.2),(4.5)} x_1x_2z_2z_3 > z_1z_2x_2x_3 \rightarrow x_1z_3 > z_1x_3 \rightarrow \frac{z_3}{z_1} > \frac{x_3}{x_1} \xrightarrow{(4.14)} \frac{u_1}{u_3} > \frac{x_3}{x_1}$$

$$\rightarrow x_1u_1 > x_3u_3 \rightarrow a_1 > 0.$$

- $Q_5 = \begin{bmatrix} + & + & 0 \\ - & + & + \\ + & - & + \end{bmatrix}$

Case 5.1. $\overrightarrow{U_1} \cdot Q_5 = \langle +, +, + \rangle$: This is a unique case. It is always achievable.

Case 5.2. $\overrightarrow{U_2} \cdot Q_5 = \langle \text{I}, +, + \rangle$: All three possible cases are achievable.

Case 5.3. $\overrightarrow{u_3} \cdot Q_5 = \langle \text{I}, \text{I}, + \rangle$: Five out of nine possible cases are achievable.

Proof. Assume x_i, y_i and $z_i \in \mathbb{R}^+$,

$$Q = \begin{bmatrix} +x_1 & +y_1 & 0 \\ -x_2 & +y_2 & +z_2 \\ +x_3 & -y_3 & +z_3 \end{bmatrix}, Q \in Q_5.$$

$$\left.\begin{array}{r} x_1x_2 = y_1y_2 \\ y_3y_1 = x_3x_1 \end{array}\right\} \rightarrow\ x_2y_3 =\ x_3y_2\ \rightarrow\ \frac{x_2}{x_3} = \frac{y_2}{y_3} \qquad (5.1)$$

Assume u_1, u_2 and $u_3 \in \mathbb{R}^+$, $\vec{u} = \langle u_1, u_2, u_3 \rangle$, so we have $\vec{u} \in \overrightarrow{U_3}$. Let have $\vec{u}.Q = \vec{a}$ and $\vec{a} = \langle a_1, a_2, a_3 \rangle$, where $a_1, a_2 \in \mathbb{R}$ and $a_3 \in \mathbb{R}^+$. $a_1 = u_1x_1 - u_2x_2 + u_3x_3$, $a_2 = u_1y_1 + u_2y_2 - u_3y_3$.

$$if\ a_1 \le 0\ \rightarrow\ u_3x_3 < u_2x_2\ \rightarrow \frac{u_3}{u_2} < \frac{x_2}{x_3} \xrightarrow{(5.1)} \frac{u_3}{u_2} < \frac{y_2}{y_3} \rightarrow\ u_3y_3 < u_2y_2\ \rightarrow\ a_2 > 0.$$

$$if\ a_2 \le 0\ \rightarrow\ u_3y_3 > u_2y_2\ \rightarrow \frac{u_3}{u_2} > \frac{y_2}{y_3} \xrightarrow{(5.1)} \frac{u_3}{u_2} > \frac{x_2}{x_3} \rightarrow\ u_3x_3 > u_2x_2\ \rightarrow\ a_1 > 0.$$

- $Q_6 = \begin{bmatrix} + & + & + \\ + & 0 & - \\ - & + & - \end{bmatrix}$

Case 6.1. $\overrightarrow{U_1} \cdot Q_6 = \langle +, +, + \rangle$:This is a unique case. It is always achievable.

Case 6.2. $\overrightarrow{U_2} \cdot Q_6 = \langle +, +, \mathrm{I} \rangle$: All three possible cases are achievable.

Case 6.3. $\overrightarrow{U_3} \cdot Q_6 = \langle \mathrm{I}, +, \mathrm{I} \rangle$: Five cases out of nine possible cases are achievable.

Proof. Assume x_i, y_i and $z_i \in \mathbb{R}^+$,

$$Q = \begin{bmatrix} +x_1 & +y_1 & +z_1 \\ +x_2 & 0 & -z_2 \\ -x_3 & +y_3 & -z_3 \end{bmatrix}, Q\ \in\ Q_6\ .$$

$$\left.\begin{array}{r} x_1x_2 = z_1z_2 \\ z_2z_3 = x_2x_3 \end{array}\right\} \rightarrow\ x_1z_3 =\ x_3z_1\ \rightarrow\ \frac{x_1}{x_3} = \frac{z_1}{z_3} \qquad (6.1)$$

Assume u_1, u_2 and $u_3 \in \mathbb{R}^+$, $\vec{u} = \langle u_1, u_2, u_3 \rangle$, so we have $\vec{u} \in \overrightarrow{U_3}$. Let have $\vec{u} \cdot Q = \vec{a}$ and $\vec{a} = \langle a_1, a_2, a_3 \rangle$ where $a_1, a_3 \in \mathbb{R}$ and $a_2 \in \mathbb{R}^+$.

$a_1 = u_1x_1 + u_2x_2 - u_3x_3$, $a_3 = u_1z_1 - u_2z_2 - u_3z_3$.

$$if\ a_1 \le 0\ \rightarrow\ u_1x_1 < u_3x_3\ \rightarrow \frac{x_1}{x_3} < \frac{u_3}{u_1} \xrightarrow{(6.1)} \frac{z_1}{z_3} < \frac{u_3}{u_1} \rightarrow\ u_1z_1 < u_3z_3\ \rightarrow\ a_3 < 0.$$

$$if\ a_3 \geq 0 \rightarrow u_1 z_1 > u_3 z_3 \rightarrow \frac{z_1}{z_3} > \frac{u_3}{u_1} \xrightarrow{(6.1)} \frac{x_1}{x_3} > \frac{u_3}{u_1} \rightarrow u_1 x_1 > u_3 x_3 \rightarrow a_1 > 0.$$

- $Q_7 = \begin{bmatrix} + & + & + \\ - & - & + \\ + & - & 0 \end{bmatrix}$

Case 7.1. $\overrightarrow{U_1} \cdot Q_7 = \langle +,+,+ \rangle$: This is a unique case. It is always achievable.

Case 7.2. $\overrightarrow{U_2} \cdot Q_7 = \langle \mathrm{I}, \mathrm{I}, + \rangle$: Three out of nine possible cases are achievable.

Proof. Assume x_i, y_i and $z_i \in \mathbb{R}^+$,

$$Q = \begin{bmatrix} +x_1 & +y_1 & +z_1 \\ -x_2 & -y_2 & +z_2 \\ +x_3 & -y_3 & 0 \end{bmatrix}, Q \in Q_7 .$$

$$\left.\begin{matrix} x_1 x_3 = y_1 y_3 \\ y_2 y_3 = x_2 x_3 \end{matrix}\right\} \rightarrow x_1 y_2 = x_2 y_1 \rightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2} \qquad (7.1)$$

Assume u_1 and $u_2 \in \mathbb{R}^+$, $\vec{u} = \langle u_1, u_2, 0 \rangle$, so $\vec{u} \in \overrightarrow{U_2}$. Let's have $\vec{u}.Q = \vec{a}$ and $\vec{a} = \langle a_1, a_2, a_3 \rangle$, where $a_1, a_2 \in \mathbb{R}$ and $a_3 \in \mathbb{R}^+$.

$$\left.\begin{matrix} a_1 = u_1 x_1 - u_2 x_2 \\ a_2 = u_1 y_1 - u_2 y_2 \end{matrix}\right\} \xrightarrow{(7.1)} a_1 \text{ and } a_2 \text{ have the same sign.}$$

Case 7.3. $\overrightarrow{U_3}. Q_7 = \langle \mathrm{I}, \mathrm{I}, + \rangle$: Five out of nine possible cases are achievable.

Proof. Assume u_1, u_2 and $u_3 \in \mathbb{R}^+$, $\vec{u} = \langle u_1, u_2, u_3 \rangle$, so $\vec{u} \in \overrightarrow{U_3}$.

$a_1 = u_1 x_1 - u_2 x_2 + u_3 x_3$, $a_2 = u_1 y_1 - u_2 y_2 - u_3 y_3$.

$$if\ a_1 \leq 0 \rightarrow u_2 x_2 > u_1 x_1 \rightarrow \frac{u_2}{u_1} > \frac{x_1}{x_2} \xrightarrow{(7.1)} \frac{u_2}{u_1} > \frac{y_1}{y_2} \rightarrow u_2 y_2 > u_1 y_1 \rightarrow a_2 < 0.$$

$$if\ a_2 \leq 0 \rightarrow u_1 y_1 > u_2 y_2 \rightarrow \frac{u_1}{u_2} > \frac{y_2}{y_1} \xrightarrow{(7.1)} \frac{u_1}{u_2} > \frac{x_2}{x_1} \rightarrow u_1 x_1 > u_2 x_2 \rightarrow a_1 < 0.$$

For the remaining representative matrices, all the possible cases of $\overrightarrow{U_i} \cdot Q_j$ are achievable. Table 5 represents these achievable sign products for different $\overrightarrow{U_i}$ and Q_j.

3.4 $\vec{U}.Q = \vec{a}$ INSTANTIATION

The algorithm to instantiate $\vec{U} \cdot Q = \vec{a}$, is dependent on the equivalence classes of both $\vec{U}$ and Q. The general idea is to instantiate Q randomly and try to assign value to $\vec{U}$. For example let's assume we want to instantiate $\overrightarrow{U_3} \cdot Q_1 = \vec{a}$, where $\vec{a} = \langle +, a_2, a_3 \rangle, a_2$ and $a_3 \in \mathbb{R}$. From the last section we know that the general form of $\overrightarrow{U_3} \cdot Q_1$ is as <+, I, I>. The general idea is first to try to instantiate the special case of <+, 0, 0>. To solve the problem of $\overrightarrow{U_3} \cdot Q_1 =< +, 0, 0 >$, after assigning a random positive real value to u_1 and generating a random instance of Q_1;

$$Q = \begin{bmatrix} +x_1 & +y_1 & +z_1 \\ +x_2 & -y_2 & +z_2 \\ +x_3 & +y_3 & -z_3 \end{bmatrix},$$

where x_i, y_i and $z_i \in \mathbb{R}^+$ the values of u_2 and u_3 are simply derived from solving the two existing equations:

$$u_1 y_1 - u_2 y_2 + u_3 y_3 = 0, \qquad u_1 z_1 + u_2 z_2 - u_3 z_3 = 0.$$

Once we have the values of u_2 and u_3 for the state of $< +, 0, 0 >$ in hand, since $a_2 = signOf(u_1 y_1 - u_2 y_2 + u_3 y_3)$ and $a_3 = signOf(u_1 z_1 + u_2 z_2 - u_3 z_3)$, to reach the desired product of $\langle +, a_2, a_3 \rangle$ the rest of algorithm works as follows:

$if\ a_2\ is +$

$\quad Increase\ the\ value\ of\ u_1\ proportional\ to\ the\ value\ of\ z_2;$

$\quad Decrease\ the\ value\ of\ u_2\ proportional\ to\ the\ value\ of\ z_1;$

$else\ if\ a_2\ is -$

$\quad Decrease\ the\ value\ of\ u_1\ proportional\ to\ the\ value\ of\ z_2;$

$\quad Increase\ the\ value\ of\ u_2\ proportional\ to\ the\ value\ of\ z_1;$

$if\ a_3\ is +$

$\quad Increase\ the\ value\ of\ u_1\ proportional\ to\ the\ value\ of\ y_3;$

$Decrease\ the\ value\ of\ u_3\ proportional\ to\ the\ value\ of\ y_1;$

$else\ if\ a_3\ is\ -$

$Decrease\ the\ value\ of\ u_1\ proportional\ to\ the\ value\ of\ y_3;$

$Increase\ the\ value\ of\ u_3\ proportional\ to\ the\ value\ of\ y_1;$

Note that the change of the value of u_1 and u_2 in order to make the value of a_2 positive/negative does not affect the sign of a_3 and vice versa.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Q	+ + + + – + + + –	+ + + + – + + – –	+ + + + – – + + –	+ + + + – – – + –	+ + 0 – + + + – +	+ + + + 0 – – + –	+ + + – – + + – 0	+ + + + + – – + 0	+ + + – 0 + + – +	+ + 0 – + – – + +	+ 0 0 0 + – 0 + +	+ 0 – 0 + 0 + 0 +	+ – 0 + + 0 0 0 +	+ 0 0 0 + 0 0 0 +
$\overrightarrow{U_1}.Q$	+ + +	+ + +	+ + +	+ + +	+ + 0	+ + +	+ + +	+ + +	+ + +	+ + 0	+ 0 0	+ 0 –	+ – 0	+ 0 0
$\overrightarrow{U_2}.Q$	+ – + + 0 + + + +	+ – + + 0 + + + +	+ – – + – 0 + – + + 0 + + + +	+ – – + 0 – + + – + + 0 + + +	– + + 0 + + + + +	+ + – + + 0 + + +	+ + + – – + 0 0 +	+ + + + + 0 + + –	+ + + 0 + + – + +	+ + – 0 + – – + –	+ + –	+ + –	+ – 0 + 0 0 + + 0	+ + 0
$\overrightarrow{U_3}.Q$	+ – – + – 0 + – + + 0 – + 0 0 + 0 + + + – + + 0 + + +	+ – – + – 0 + – + + 0 – + 0 0 + 0 + + + – + + 0 + + +	+ – – + – 0 + – + + 0 – + 0 0 + 0 + + + – + + 0 + + +	– + – 0 + – + – – + 0 – + + – + + 0 + + +	– + + 0 + + + – + + 0 + + + +	– + – 0 + – + + – + + 0 + + +	– – + 0 – + + – + + 0 + + + +	– + – – + 0 – + + 0 + – 0 + 0 0 + + + + – + + 0 + + +	– – + – 0 + – + + 0 – + 0 0 + 0 + + + – + + 0 + + + +	– + – – + 0 – + + 0 + – 0 + 0 0 + + + + – + + 0 + + +	+ + – + + 0 + + +	+ + – + + 0 + + +	+ – + + 0 + + + +	+ + +

Table 5- Representative sign vector rotation achievable products

Chapter 4

Three-Dimensional Sign Rotation Composition

In this chapter we explain our algorithm to answer the second question mentioned in 1.6; that is, for any given signed rotation matrices S and R, what is the maximal set of signed rotations M, such that: $M = R \cdot S$?

4.1 ROTATION COMPOSITION

As we have seen in the last chapter, we can factor any sign rotation as the product of its class representative and the octahedral rotation. Once we do that we can reduce the problem of composition of any two sign rotations R and S, $R \cdot S$, to the problem of composition of two representative rotations Q_i and Q_j. Note that the size of the problem in the former case is 336×336, while it is 14×14 in the latter one. If we have pre-computed the solutions for representatives composition computed, we can look up the solution whenever it is needed. The following summarizes the above process:

1. Factor R as $R = Q_i \cdot P_i$;
2. Compute $P_i \cdot S$;
3. Factor $(P_i \cdot S)$ as $(P_i.S) = Q_j \cdot P_j$;
4. Look up $Q_i \cdot Q_j$;
5. Compute $(Q_i.Q_j).P_j$;
6. Return $\left((Q_i.Q_j).P_j\right)$;

4.2 CALCULATE $\boldsymbol{Q_i . Q_j}$

The main purpose here is to compute the set of possible values of the product of $Q_i \cdot Q_j$ for any given representative sign rotations Q_i and Q_j. To do this, it is easier to think of Q_i as a collection of three row vectors. Then we can divide the problem of $Q_i \cdot Q_j$ to three separate vector rotation problems: $Q_i[1,:] \cdot Q_j$, $Q_i[2,:] \cdot Q_j$ and $Q_i[3,:] \cdot Q_j$, which we already have seen the solution in the last chapter.

Once we found the set of solution for each one of these problems, we can roughly estimate the set of solutions for $Q_i \cdot Q_j$. This estimation is a set of matrices whose first row is a member of the first vector rotation solution set, second row is a member of the second vector rotation solution set and finally the third row is from the third vector rotation solution set. Since the product of composition of any two rotations is a rotation as well, we can eliminate any combination that is not a sign rotation. The resultant is the list of possible products for the $Q_i \cdot Q_j$. In order to recognize the achievable cases out of these possible products, for each possible case such as *R,* we can try to find some random instances of Q_i and Q_j such that their composition is an instance of *R.* Next section elaborates the algorithm to do this.

4.3 INSTANTIATION OF $\boldsymbol{Q_i \cdot Q_j = R}$

We want to find a random real instance of representative sign rotations Q_i and Q_j such that their composition is an instance of sign rotation *R*. The general idea is that we will use the results of section 3.4 to instantiate the sign equation $\vec{U}.Q = \vec{a}$ to find an exact instantiation of $Q_i[1,:] \cdot Q_j[1,:] = R[1,:]$ and $Q_i[2,:] \cdot Q_j[2,:] = R[2,:]$. Two extra conditions must be met. First the instance of Q_j must be the same in both instantiations. Second the two instances of row vectors in Q_i must be perpendicular to each other. Once we found such desired row instances of Q_i we can simply get the third row for Q_i as the cross product $Q_i[1,:] \times Q_i[2,:]$ of the resultant matrix is an instance of Q_i and the product of it with the above instance of Q_j is an instance of *R,* we are done. In some cases, we do not start with the first and second row of Q_i but a different pair of rows, as described below.

We can elaborate the above solution as follows. Before that we need to mention that this solution works for those cases of $Q_i \cdot Q_j = R$, such that Q_i has at least one non-zero row. Otherwise Q_i has at least four zeros in its body, which is a trivial case.

- Between the three rows of Q_i choose a non-zero row and mark it.
- Choose an unmarked row from Q_i; $Q_i[p,:]$, mark it and take the corresponding row from *R;* $R[p,:]$. Instantiate the problem of $Q_i[p,:] \cdot Q_j = R[p,:]$.
- Take the unmarked row from Q_i; $Q_i[q,:]$, and its corresponding row from *R;* $R[q,:]$. Repeat the last step. Use the same instance of Q_j for instantiation. The solution for this instantiation should satisfy one more condition: having dot-product of zero with the row instance of Q_i from the last step. In this way the resulting row instances in R; $R[p,:]$ and $R[q,:]$, have dot-product of zero to each other as well.
- Normalize both instances of $\overrightarrow{Q_\iota[p,:]}$ and $\overrightarrow{Q_\iota[q,:]}$.
- Once we have two normalized instance vectors of Q_i , we can output the third row vector of Q_i as the cross product of $Q_i[p,:]$ and $Q_i[q,:]$, let's assume; $\overrightarrow{Q_\iota[p,:]} = [p_x,\ p_y,\ p_z]$ and $\overrightarrow{Q_\iota[q,:]} = [q_x,\ q_y,\ q_z]$, we can get the instance of third row of Q_i as: $\overrightarrow{Q_\iota[p,:]} \times \overrightarrow{Q_\iota[q,:]} = [p_y q_z - p_z q_y\ ,\ p_z q_x - p_x q_z,\ p_x q_y - p_y q_x]$.
- If any permutation of row vectors $\overrightarrow{Q_\iota[p,:]}, \overrightarrow{Q_\iota[q,:]}$ and $\overrightarrow{Q_\iota[p,:]} \times \overrightarrow{Q_\iota[q,:]}$ is an instance of Q_i, calculate the product of composition of this and the existing instance of Q_j. If the product is an instance of *R,* return the instances. Otherwise repeat the algorithm.

- Terminate after a certain number of trials and omit R from the set of solutions for $Q_i \cdot Q_j$.

The implementation for those cases of $Q_i \cdot Q_j = R$, in which Q_i has at least four zeros in its body is trivial. We can always pick two row vectors of Q_i that are perpendicular to each other. Once we solve the problem of direction rotation for these two vectors with the same instance of Q_j, we can produce the instance of third row of Q_i as before and see whether or not the overall instantiation satisfies $Q_i \cdot Q_j = R$.

4.4 AN EXAMPLE OF CALCULATION OF $Q_i \cdot Q_j$

Let assume we want to compute $Q_8 \cdot Q_7 = \begin{bmatrix} + & + & + \\ + & + & - \\ - & + & 0 \end{bmatrix} \times \begin{bmatrix} + & + & + \\ - & - & + \\ + & - & 0 \end{bmatrix}$.

(1) Identify the plausible products:

From table 5 compute the solutions for each case of $Q_8[1,:] \cdot Q_7$, $Q_8[2,:] \cdot Q_7$ and $Q_8[3,:] \cdot Q_7$ and construct all possible sign matrices as described above. After eliminating all those non-rotation matrices, we end up with following matrices:

$$\begin{bmatrix} - & - & + \\ - & + & + \\ - & - & - \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ + & + & + \\ - & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & - & + \\ - & - & - \end{bmatrix} \quad \begin{bmatrix} + & + & + \\ - & + & + \\ - & - & + \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & 0 & + \\ - & - & - \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & + \\ - & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & + \\ - & - & - \end{bmatrix} \quad \begin{bmatrix} 0 & - & + \\ - & + & + \\ - & - & - \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & + \\ - & - & 0 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ 0 & + & + \\ - & - & + \end{bmatrix} \quad \begin{bmatrix} + & 0 & + \\ - & + & + \\ - & - & + \end{bmatrix}$$

(2) For each one of the plausible solutions R, try to instantiate $Q_8 \cdot Q_7 = R$.
For instance, let assume R is the second matrix in the set:

$$\begin{bmatrix} + & + & + \\ + & + & - \\ - & + & 0 \end{bmatrix} \times \begin{bmatrix} + & + & + \\ - & - & + \\ + & - & 0 \end{bmatrix} = \begin{bmatrix} + & - & + \\ + & + & + \\ - & - & + \end{bmatrix}$$

Get a random instance for Q_7 ; $Qins_7 = \begin{bmatrix} 0.123 & 0.988 & 0.084 \\ -0.01 & -0.084 & 0.996 \\ 0.992 & -0.123 & 0 \end{bmatrix}$;

Find an instance of $Qins_8[1,:]$ such that the product of $Qins_8[1,:] \times Qins_7$ is an instance of $R[1,:]$:

$$[0.061 \quad 0.003 \quad 0.998] \times \begin{bmatrix} 0.123 & 0.988 & 0.084 \\ -0.01 & -0.084 & 0.996 \\ 0.992 & -0.123 & 0 \end{bmatrix} = [0.997 \quad -0.063 \quad 0.008]$$

Find an instance of $Qins_8[2,:]$ such that the product of $Qins_8[2,:] \times Qins_7$ is an instance of $R[2,:]$ and $Qins_8[1,:] \cdot Qins_8[2,:] = 0$:

$$[0.996 \quad 0.058 \quad -0.061] \times \begin{bmatrix} 0.123 & 0.988 & 0.084 \\ -0.01 & -0.084 & 0.996 \\ 0.992 & -0.123 & 0 \end{bmatrix} = [0.061 \quad 0.987 \quad 0.143].$$

Compute $Qins_8[1,:] \times Qins_8[2,:]$:

$$[0.061 \quad 0.003 \quad 0.998] \times [0.996 \quad 0.058 \quad -0.061] = [-0.059 \quad 0.998 \quad 0].$$

Since the cross product is an instance of third row in Q_8, Compute $Qins_8[2,:] \times Qins_7$:

$$[-0.059 \quad 0.998 \quad 0] \times \begin{bmatrix} 0.123 & 0.988 & 0.084 \\ -0.010 & -0.084 & 0.996 \\ 0.992 & -0.123 & 0 \end{bmatrix} = [-0.017 \quad -0.142 \quad 0.989].$$

Since the cross product is an instance of $R[3,:]$; return $Qins_8$ and $Qins_7$ as solution.

$$\begin{bmatrix} 0.061 & 0.003 & 0.998 \\ 0.996 & 0.058 & -0.061 \\ -0.059 & 0.998 & 0 \end{bmatrix} \times \begin{bmatrix} 0.123 & 0.988 & 0.084 \\ -0.010 & -0.084 & 0.996 \\ 0.992 & -0.123 & 0 \end{bmatrix} = \begin{bmatrix} 0.997 & -0.063 & 0.008 \\ 0.061 & 0.987 & 0.143 \\ -0.017 & -0.142 & 0.989 \end{bmatrix}$$

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}	Q_{13}	Q_{14}
Q_1	$\frac{48}{51}$	$\frac{29}{31}$	$\frac{29}{31}$	$\frac{30}{31}$	$\frac{29}{31}$	$\frac{20}{21}$	$\frac{21}{21}$	$\frac{29}{31}$	$\frac{29}{31}$	$\frac{21}{21}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{1}{1}$
Q_2	$\frac{29}{31}$	$\frac{29}{31}$	$\frac{44}{51}$	$\frac{29}{31}$	$\frac{21}{21}$	$\frac{20}{21}$	$\frac{29}{29}$	$\frac{21}{21}$	$\frac{29}{29}$	$\frac{29}{31}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{1}{1}$
Q_3	$\frac{29}{31}$	$\frac{47}{51}$	$\frac{29}{31}$	$\frac{29}{31}$	$\frac{21}{21}$	$\frac{29}{29}$	$\frac{20}{21}$	$\frac{27}{29}$	$\frac{20}{21}$	$\frac{28}{29}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{1}{1}$
Q_4	$\frac{29}{31}$	$\frac{30}{31}$	$\frac{46}{51}$	$\frac{30}{31}$	$\frac{21}{21}$	$\frac{20}{21}$	$\frac{27}{29}$	$\frac{20}{21}$	$\frac{29}{29}$	$\frac{29}{31}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{1}{1}$
Q_5	$\frac{21}{21}$	$\frac{20}{21}$	$\frac{29}{29}$	$\frac{19}{29}$	$\frac{18}{19}$	$\frac{14}{19}$	$\frac{15}{16}$	$\frac{9}{11}$	$\frac{17}{19}$	$\frac{14}{19}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{1}{1}$
Q_6	$\frac{29}{29}$	$\frac{20}{21}$	$\frac{29}{29}$	$\frac{20}{21}$	$\frac{14}{19}$	$\frac{9}{11}$	$\frac{18}{19}$	$\frac{17}{19}$	$\frac{15}{16}$	$\frac{17}{19}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{5}{5}$	$\frac{1}{1}$
Q_7	$\frac{29}{31}$	$\frac{27}{29}$	$\frac{21}{21}$	$\frac{20}{21}$	$\frac{13}{19}$	$\frac{17}{19}$	$\frac{9}{11}$	$\frac{14}{16}$	$\frac{14}{19}$	$\frac{18}{19}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{1}{1}$
Q_8	$\frac{29}{31}$	$\frac{27}{29}$	$\frac{21}{21}$	$\frac{20}{21}$	$\frac{17}{19}$	$\frac{17}{19}$	$\frac{8}{11}$	$\frac{12}{19}$	$\frac{14}{19}$	$\frac{18}{19}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{1}{1}$
Q_9	$\frac{29}{31}$	$\frac{20}{21}$	$\frac{29}{29}$	$\frac{20}{21}$	$\frac{14}{19}$	$\frac{8}{11}$	$\frac{18}{19}$	$\frac{17}{19}$	$\frac{13}{19}$	$\frac{17}{19}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{5}{5}$	$\frac{1}{1}$
Q_{10}	$\frac{21}{21}$	$\frac{20}{21}$	$\frac{29}{29}$	$\frac{29}{29}$	$\frac{18}{19}$	$\frac{16}{19}$	$\frac{16}{16}$	$\frac{9}{11}$	$\frac{18}{19}$	$\frac{16}{19}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{1}{1}$
Q_{11}	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{3}{3}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{2}{3}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$
Q_{12}	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{5}{5}$	$\frac{1}{1}$	$\frac{2}{3}$	$\frac{1}{1}$	$\frac{1}{1}$
Q_{13}	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{3}{3}$	$\frac{3}{3}$	$\frac{5}{5}$	$\frac{5}{5}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{2}{3}$	$\frac{1}{1}$
Q_{14}	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$

Table 6- Number of achievable cases out of the possible cases, in the transitivity table of representative rotations.

Conclusion

In this project we introduced and implemented a qualitative calculus for three-dimensional rotation. We defined a triple sign vector as a qualitative vector and identified 336 possible 3×3 base sign rotations matrices. The calculus consists of two parts. First it computes and instantiates the set of all possible products for the given vector and rotation matrix. It also computes and instantiates the maximal set of the composition for the given two basic sign rotations. Since the size of problem was large, in terms of the number of CSPs that we need to implement (27×336 in the first part and 336×336 in the second part) by mapping the 26 non-zero possible sign row vectors to unit length octahedral centered in origin we categorized 26 vectors to 3 categories (face/edge/vertex) and 336 rotation matrices to 14 distinct categories. This way the former size became 3×14 and the latter became 14×14.

As future work, the current calculus can be extended to a qualitative calculus and other constraint propagation systems with more expressivity and functionality, which is the end product of most QSR projects. That would be also worthwhile to establish reliably that the set of achievable products for each case in composition part is not missing any possible product. That could be more desirable to work out this in linear algebra as we did in chapter 3. But it would be also possible to try to refine the current code and may be to increase the number of trials to see if it gets better results.

A more challenging would be to create a qualitative calculus of rotations based on one of the other regular polyhedra like icosahedron or cube rather than the octahedral. Any of the regular polyhedral gives a qualitative calculus though none of the others can be analyzed in terms of the sign calculus.

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