On embedding expanders into ℓ_p spaces

Jiří Matoušek*
Department of Applied Mathematics
Charles University
Malostranské nám. 25, 118 00 Praha 1
Czech Republic

Abstract

In this note we show that the minimum distortion required to embed all n-point metric spaces into the Banach space ℓ_p is between $(c_1/p)\log n$ and $(c_2/p)\log n$, where $c_2 > c_1 > 0$ are absolute constants and $1 \le p < \log n$. The lower bound is obtained by a generalization of a method of Linial et al. [LLR95], by showing that constant-degree expanders (considered as metric spaces) cannot be embedded any better.

1 Introduction

Let M be a metric space with a metric ρ, let X be a normed space (whose norm will be denoted by $\|.\|$), and let $f : M \to X$ be a mapping. We say that f is a *D-embedding* (or a mapping with *distortion* at most D), $D \ge 1$ a real number, if we have

$$\frac{1}{D}\,\rho(x,y) \le \|f(x) - f(y)\| \le \rho(x,y)$$

for any two points $x, y \in M$. We say that M D-embeds into X if there exists a D-embedding[1] $f : M \to X$.

The D-embeddability of finite metric spaces into various normed spaces was investigated in the context of the local Banach space theory ([Enf69a], [Enf69b], [JL84], [Bou85], [BMW86], [JLS87], [AR92], [Ma95]), and it seems that it can be of considerable interest also in more applied areas (see [LLR95]).

Let the symbol ℓ_p^n denote the n-dimensional real vector space equipped with the L_p-norm, given by $\|(x_1, x_2, \ldots, x_n)\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ (for $1 \le p < \infty$). Similarly ℓ_p denotes the space of countable sequences of real numbers with a finite L_p-norm. For a metric space M, let $D_p(M)$ be the minimum D such that M can be D-embedded into ℓ_p, and let $D_p(n)$ be the

*Research supported by Czech Republic Grant GAČR 201/94/2167 and Charles University grants No. 351 and 361.

[1] A number of various terms besides the mentioned ones are used in the literature in this context; e.g., a D-embedding is also called a *D-isomorphism*, a *D-lipeomorphism*, etc.

supremum of $D_p(M)$ over all n-point metric spaces M. Since any n-point subset of ℓ_p can be isometrically embedded into $\ell_p^{n(n-1)/2}$ (see e.g., [Fic88]), we can restrict our attention to embeddings into finite-dimensional ℓ_p spaces.

The most well-studied case is that of $p = 2$, where we are dealing with embeddings into the usual Euclidean space. It is easy to find a 4-point metric space which cannot be isometrically embedded into any Euclidean space, but it is not so easy to prove that $D_2(n) \to \infty$ for $n \to \infty$; this was probably first done by Enflo [Enf69b], whose proof yields $D_2(n) = \Omega(\sqrt{\log n})$ (his example is the cube $\{0,1\}^k$ with the L_1 metric, $n = 2^k$; see also [Enf69a]). Bourgain [Bou85] proved an upper bound $D_2(n) = O(\log n)$ and a lower bound $D_2(n) = \Omega(\log n/\log\log n)$. The lower bound is non-constructive, using random graphs and a counting argument. Linial et al. [LLR95] discovered another lower bound technique, which allowed them to show the asymptotically tight lower bound $D_2(n) = \Omega(\log n)$; their proof, unlike Bourgain's, yields an explicit metric space exhibiting the lower bound (one can say that their method slightly resembles Enflo's, with expander graphs replacing the L_1-cube).

The situation for other values of p has been understood less satisfactorily. Concerning upper bounds, Bourgain's embedding technique in fact proves $D_p(n) = O(\log n)$ for any p (with the constant of proportionality independent of p). Concerning lower bounds, the argument of Linial et al. [LLR95] shows $D_p(n) = \Omega(\log n)$ for any $p \in [1,2]$. For $p > 2$, however, the best known lower bound was apparently one following from the results of Bourgain et al. [BMW86], which is $c_\varepsilon(\log n)^{1/2-\varepsilon}$, with $\varepsilon > 0$ an arbitrarily small number and $c_\varepsilon > 0$ depending on ε. Here we obtain asymptotically tight bounds:

Theorem 1

(i) *There exist constants $c_1 > 0$ and n_0 such that for any $p \geq 1$ and any $n \geq n_0$ there exists an n-point metric space which D-embeds into ℓ_p only for $D \geq (c_1/p)\log n$.*

(ii) *Any n-point metric space can be embedded into ℓ_p with distortion at most $(c_2/p)\log n$, where c_2 is a constant and $1 \leq p < \log n$.*

Part (i) is proved in section 2 by generalizing the method of Linial et al. [LLR95]. Part (ii) is proved in section 4 by modifying Bourgain's embedding method very slightly.

2 Expanders

Let $G = (V,E)$ be a (simple, unoriented) graph on the vertex set $V = \{1,2,\ldots,n\}$. We assume that G is d-regular (every vertex has exactly d neighbors) with d a constant (while n is a variable attaining arbitrarily large values). The graph G is called an *expander* if there exists a constant $\Phi > 0$ (independent of n; Φ is called the *conductance* of G) such that for any subset $A \subseteq V$ with $|A| \leq n/2$ we have

$$|\{\{i,j\} \in E;\ i \in A, j \in V \setminus A\}| \geq \Phi|A|. \qquad (1)$$

It can be shown that a random d-regular graph is an expander (with a suitable $\Phi = \Phi(d)$) with a positive probability. Sophisticated explicit constructions of expanders are also known — see e.g., [AS92] for background information and references.

If $x_1, x_2, \ldots, x_n$ are real numbers, their *median* is defined as a real number m such that $|\{i;\, x_i \le m\}| \ge \lfloor n/2 \rfloor$ and $|\{i;\, x_i \ge m\}| \ge \lfloor n/2 \rfloor$ (thus, for n even, the median need not be determined uniquely). A basic property of expanders we use is the following (apparently due to Sinclair and Jerrum [JS88]; see e.g., Lovász [Lov93], Ex. 11.30):

Lemma 2 *Let G be an expander with conductance $\Phi > 0$, let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers, and let m be their median. Then*

$$\sum_{\{i,j\} \in E} |x_i - x_j| \ge \Phi \sum_{i \in V} |x_i - m| \,. \tag{2}$$

The proof of (2) has few lines. Inequality (1) can be viewed as a special case of (2), where the x_i's only attain values 0 and 1 (set $x_i = 1$ for $i \in A$ and $x_i = 0$ otherwise).

The key to the non-embeddability result of Linial et al. [LLR95] is a suitable generalization of (2) for the x_i's being elements of a Banach space instead of real numbers. Let $K = \binom{V}{2}$ denote the set of edges of the complete graph with vertex set V. A simple consequence of (2) is the following:

$$\sum_{\{i,j\} \in E} |x_i - x_j| \ge \frac{\Phi}{n-1} \sum_{\{i,j\} \in K} |x_i - x_j| \,. \tag{3}$$

Indeed, since (3) is clearly translation-invariant (it is not changed by adding the same real number to each x_i), we may assume that 0 is a median of the x_i's. Then we have $\sum_{\{i,j\} \in K} |x_i - x_j| \le \sum_{\{i,j\} \in K} (|x_i| + |x_j|) = (n-1) \sum_{i \in V} |x_i|$, so (3) follows from (2).

Linial et al. [LLR95] essentially observe (although they formulate it in a somewhat different manner) that (3) holds also for any $x_1, x_2, \ldots, x_n \in \ell_1$:

$$\sum_{\{i,j\} \in E} \|x_i - x_j\|_1 \ge \frac{\Phi}{n-1} \sum_{\{i,j\} \in K} \|x_i - x_j\|_1 \,. \tag{4}$$

This is an immediate consequence of (3): write each $x_i \in \ell_1$ in coordinates, use (3) for each coordinate separately, and sum the resulting inequalities. Since any finite subset of any ℓ_p with $1 \le p \le 2$ can be isometrically embedded into ℓ_1 [BDK66], an analog of (4) holds also in these ℓ_p's, in particular in a Euclidean space.

For $p > 2$, we need an "L_p-analog" of (3):

Proposition 3 *Let G be a d-regular expander with conductance Φ, let $x_1, x_2, \ldots, x_n$ be real numbers with median m, and let $p > 1$. Then we have*

$$\begin{aligned} \sum_{\{i,j\} \in E} |x_i - x_j|^p &\ge \frac{(\Phi/2p)^p}{d^{p-1}} \sum_{i \in V} |x_i - m|^p \\ &\ge \frac{(\Phi/4p)^p}{d^{p-1}} \frac{1}{n-1} \sum_{\{i,j\} \in K} |x_i - x_j|^p \,. \end{aligned} \tag{5}$$

The version of this result for $p = 2$ was essentially proved by Sinclair and Jerrum [JS88] (the method goes back to Alon [Alo86]); the result for a general p doesn't seem to be known.

We prove it in section 3 below by generalizing the known proof of the $p = 2$ case (following the presentation of Lovász [Lov93], Ex. 11.32).

As a consequence we get that for any $x_1, x_2, \ldots, x_n \in \ell_p$,

$$\sum_{\{i,j\}\in E} \|x_i - x_j\|_p^p \geq \left(\frac{c_0}{p}\right)^p \frac{1}{n-1} \sum_{\{i,j\}\in K} \|x_i - x_j\|_p^p \,, \tag{6}$$

with $c_0 = \Phi/4d^{1-1/p}$ a positive constant (this again follows by applying (5) to each coordinate and summing up).

To derive Theorem 1(i) from (6), we consider the expander G as an n-point metric space, with the metric ρ given by the usual graph-theoretic distance of vertices. Let us consider the ratio

$$R_\rho = \frac{\left(\frac{1}{|E|}\sum_{\{i,j\}\in E}\rho(i,j)^p\right)^{1/p}}{\left(\frac{1}{|K|}\sum_{\{i,j\}\in K}\rho(i,j)^p\right)^{1/p}} \,.$$

The numerator is the pth degree average of the edge length, which is 1 by definition. The denominator is the pth degree average of the distance of two vertices of G. Since G is d-regular, at most $1 + d + \cdots + d^k \leq 2d^k$ vertices of G have distance at most k from a given vertex, and from this one can see that at least a fixed fraction of the pairs $\{i, j\} \in K$ satisfies $\rho(i,j) \geq \log_d(n/4)$ (say). Therefore $R_\rho = O(1/\log n)$.

Next, suppose that $f : V \to \ell_p$ is a D-embedding, and let σ be the metric on V given by $\sigma(i,j) = \|f(i) - f(j)\|_p$. Define the ratio R_σ analogously to R_ρ. Then inequality (6) shows that $R_\sigma \geq c'/p$, with $c' = \Phi/4d$ a positive constant (we use $|E| = dn/2$ and $|K| = n(n-1)/2$). On the other hand, if f is a D-embedding, we should have $R_\sigma \leq DR_\rho$, and hence $D \geq (c_1/p)\log n$ as claimed in Theorem 1. □

3 A p-inequality for expanders

For a real number $x < 0$, let x^p stand for $-(-x)^p$. First we note the following estimate:

Lemma 4 *For any real numbers a, b and any $p \geq 1$, we have*

$$|a^p - b^p| \leq p|a-b|\left(|a|^{p-1} + |b|^{p-1}\right) .$$

Proof. By symmetry, we may assume $a \geq |b| > 0$, and by re-scaling we may suppose $a = 1$, so it is enough to show $1 - b^p \leq p(1-b)$ for any $b \in (-1, 1)$. Finally by writing $x = 1 - b$ we pass to $(1-x)^p \geq 1 - px$ which is a well-known (Bernoulli's) inequality. □

Proof of Proposition 3. Let p, $x_1, x_2, \ldots, x_n$, and m be as in the Proposition; we may assume $m = 0$. For the sake of brevity, put

$$S = \sum_{\{i,j\}\in E} |x_i - x_j|^p, \qquad T = \sum_{i\in V} |x_i|^p \,.$$

By (2) applied to the numbers $x_1^p, \ldots, x_n^p$ we have

$$\Phi T \leq \sum_{\{i,j\}\in E} |x_i^p - x_j^p| \,.$$

Using Lemma 4, we further get that the right-hand side is at most $p\sum_{\{i,j\}\in E} u_{ij}v_{ij}$, where $u_{ij} = |x_i - x_j|$, $v_{ij} = |x_i|^{p-1} + |x_j|^{p-1}$. By Hölder's inequality, we have

$$\sum_E u_{ij}v_{ij} \le \left(\sum_E u_{ij}^p\right)^{1/p} \left(\sum_E v_{ij}^q\right)^{1/q}$$

where $q = p/(p-1)$. We note that $\sum_E u_{ij}^p = S$, and we estimate $v_{ij}^q = (|x_i|^{p-1} + |x_j|^{p-1})^q \le 2^q \left(|x_i|^{(p-1)q} + |x_j|^{(p-1)q}\right) = 2^q\left(|x_i|^p + |x_j|^p\right)$; hence, using the d-regularity of G, we get

$$\left(\sum_E v_{ij}^q\right)^{1/q} \le 2\left(\sum_E \left(|x_i|^p + |x_j|^p\right)\right)^{1/q} = 2d^{1/q}\left(\sum_{i\in V} |x_i|^p\right)^{1/q} = 2d^{1/q}T^{1/q}\,.$$

Combining the whole chain of inequalities yields $\Phi T \le p\,S^{1/p}2d^{1/q}T^{1/q}$, hence $S \ge (\Phi/2p)^p d^{-(p-1)}T$, which is the first inequality in Proposition 3. The second inequality follows by estimating $|x_i - x_j|^p \le 2^p\left(|x_i|^p + |x_j|^p\right)$. $\Box$

4 An upper bound

Here we prove part (ii) of Theorem 1. We use the method invented by Bourgain [Bou85]. This method has been used, with small modifications, in a number of other papers ([JLS87], [Ma91], [LLR95], [Ma95]), and the author of the present note finds it already somewhat embarrassing to repeat essentially the same thing here again; so the presentation is somewhat sketchy. On the other hand, it is interesting that one can get a tight upper bound also in terms of p.

The proof is based on the following lemma:

Lemma 5 *Let M be an n-point metric space with a metric ρ. Let x, y be two distinct points of M, and let $s \ge 2$ be a parameter. Then there exist real numbers $\Delta_1, \Delta_2, \ldots, \Delta_t \ge 0$ with $\Delta_1 + \cdots + \Delta_t = \rho(x,y)/4$, where $t = \lfloor \log_s n\rfloor + 1$, and such that the following holds for each $i = 1, 2, \ldots, t$: if $A_i \subseteq M$ is a randomly chosen subset of X, with each point of X included in A_i independently with probability $1/s^i$, then the probability P_i of the event "$|\rho(x, A_i) - \rho(y, A_i)| \ge \Delta_i$" satisfies $P_i \ge 1/8s$.*

Proof sketch. As shown in [LLR95] (or [Bou85] with a slightly different formulation)[2], the numbers Δ_i can be chosen in such a way that P_i is at least the probability that $A_i \cap S_1 = \emptyset$ and at the same time $A_i \cap S_2 \ne \emptyset$, where S_1 is a certain subset of X of size $< s^i$ and S_2 is another subset of X, disjoint from S_1, of cardinality $\ge s^{i-1}$. A detailed calculation showing that the latter probability is at least $1/8s$ is given in [Ma95, Lemma 4.1]. $\Box$

Proof of Theorem 1(ii). Let (M,ρ) be a given n-point metric space. Fix $s = 2^p$, $t = \lfloor \log_s n\rfloor + 1$, and for each $i = 1, 2, \ldots, t$ choose r independent random subsets $A_{i1}, \ldots, A_{ir}$, each $A_{ij} \subseteq X$ being chosen as the A_i in Lemma 5 (i.e. each point included with probability

[2] The proofs in [LLR95] and [Bou85] argue for the $s = 2$ case, but the generalization to an arbitrary s is entirely straightforward.

$1/s^i$). If r is chosen sufficiently large, by Lemma 5 we may assume that *for each* $x, y \in X$ and each $i = 1, 2, \ldots, t$, the inequality $|\rho(x, A_{ij}) - \rho(y, A_{ij})| \geq \Delta_i$ holds for at least $r/16s$ indices j, where Δ_i depends on x, y and is as in the Lemma (one can take $r = const.s \log n$, as can be shown using a suitable version of the Chernoff inequality). We fix such a collection of the A_{ij} and define a mapping $f : M \to \ell_p^{tr}$: if the coordinates in ℓ_p^{tr} are indexed the same way as the sets A_{ij}, we define f componentwise by $f(x)_{ij} = \rho(x, A_{ij})$.

Since $|\rho(x, A) - \rho(y, A)| \leq \rho(x, y)$ holds for any set A, we obtain $\|f(x) - f(y)\|_p \leq t^{1/p} r^{1/p} \rho(x, y)$. On the other hand, we have

$$\|f(x) - f(y)\|_p = \left(\sum_{i=1}^{t} \sum_{j=1}^{r} |\rho(x, A_{ij}) - \rho(y, A_{ij})|^p \right)^{1/p} \geq \left(\sum_{i=1}^{t} \frac{r}{16s} \Delta_i^p \right)^{1/p} ;$$

using Hölder's inequality, we get $\sum_{i=1}^{t} \Delta_i^p \geq \left(\sum_i \Delta_i\right)^p / t^{p-1} = (\rho(x, y)/4)^p / t^{p-1}$, hence $\|f(x) - f(y)\|_p \geq \frac{1}{64} (r/s)^{1/p} t^{-(p-1)/p} \rho(x, y)$. Thus after an appropriate scaling, f is a D-embedding with $D = O(t) = O(\log n/p)$. $\square$

References

[Alo86] N. Alon: Eigenvalues and expanders, *Combinatorica* 6(1986) 83–96.

[AS92] N. Alon and J. Spencer: *The probabilistic method*, J. Wiley & Sons, 1992.

[AR92] J. Arias-de-Reyna and L. Rodríguez-Piazza: Finite metric spaces needing high dimension for Lipschitz embeddings in Banach spaces, *Israel J. Math.* 79(1992) 103–111.

[Bou85] J. Bourgain: On Lipschitz embedding of finite metric spaces in Hilbert space, *Israel J. Math.* 52(1985) 46–52.

[BDK66] J. Bretagnolle, D. Dacunha-Castelle, and J. L. Krivine: Lois stables et espaces L^p, *Ann. Inst. H.Poincaré*, Sect. B2 (1966) 231–259.

[BMW86] J. Bourgain, V. Milman, H. Wolfson: On type of metric spaces, *Trans. Amer. Math. Soc.* 294(1986), 295–317.

[Enf69a] P. Enflo: On a problem of Smirnov, Ark. Mat. 8(1969), 107–109.

[Enf69b] P. Enflo: On the nonexistence of uniform homeomorphisms between L_p-spaces, Ark. Mat. 8(1969), 103–105.

[Fic88] B. Fichet: L_p-spaces in data analysis, in: *Classification and related methods of data analysis*, H. H. Bock ed., North Holland 1988, pp. 439–444.

[JL84] W. Johnson and J. Lindenstrauss: Extensions of Lipschitz maps into a Hilbert space, *Contemporary Math.* 26 (Conference in Modern Analysis and Probability) pages 189–206, Amer. Math. Soc. 1984.

[JLS87] W. Johnson, J. Lindenstrauss, G. Schechtman: On Lipschitz embedding of finite metric spaces in low dimensional normed spaces, in: *Geometrical aspects of functional analysis* (J. Lindenstrauss, V. D. Milman eds.), Lecture Notes in Mathematics 1267, Springer-Verlag, Berlin-Heidelberg 1987, pages 177–184.

[JS88] M. Jerrum and A. Sinclair: Conductance and the rapid mixing property for Markov chains: The approximation of the permanent resolved, in: *Proc. 20th ACM Symposium on Theory of Computing*, 1988, pages 235–244.

[LLR95] N. Linial, E. London and Yu. Rabinovich: The geometry of graphs and some its algorithmic applications, *Combinatorica* 15(1995) 215–245.

[Lov93] L. Lovász: *Combinatorial Problems and Exercises (2nd ed.)*, Akadémiai Kiadó, Budapest 1993.

[Ma91] J. Matoušek: Note on bi-Lipschitz embeddings into normed spaces, *Comment. Math. Univ. Carolinae* 33,1(1992) 51–55.

[Ma95] J. Matoušek: On the distortion required for embedding finite metric spaces into normed spaces, *Israel J. Math.* 93(1996) 333–344.