Advanced Approximation Algorithms **(CMU 15-854B, Spring 2008)**

Lecture 21: $1 - \delta$ vs. $\frac{1}{2} + \epsilon$ Hardness for Max-E3Lin

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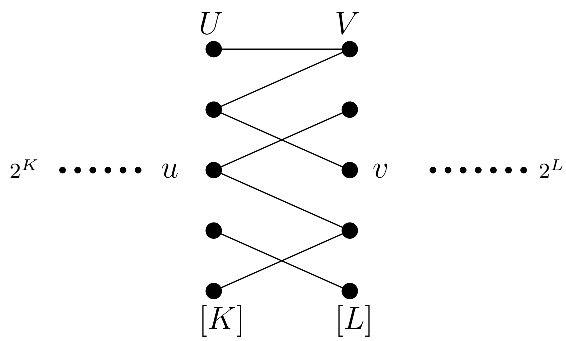
1 Recap

In this lecture, we finish the hardness reduction for Max-E3Lin[1]. Recall we began with a Label-Cover(K,L) instance, with left vertices U and right vertices V. We associate 2^K variables $u_1, \dots,$ u_{2^K} with each $u \in U$ and 2^L variables $v_1, \dots, v_{2^L}$ for each $v \in V$. Furthermore, we identify these variables with functions $f_u : \{-1,1\}^K \to \{-1,1\}$ and $g_v : \{-1,1\}^L \to \{-1,1\}$ respectively.



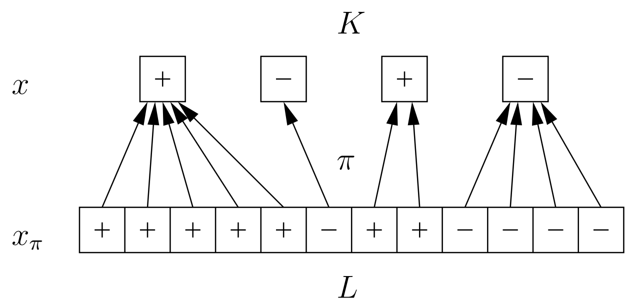
Also recall that for the completeness direction of our proof, we will "encode" the assignment that $u \in U$ gets key $a \in [K]$ by taking f_u to be the a^{th} dictator function, $f(x) = x_a$ (and similarly for V, L). Thus, for soundness, our test should also have the property that if f_u passes with sufficiently high probability, we can "decode" f_u into a small set of suggested keys. Last time, we saw how to come up with such a test.

However, it's not enough to merely test that the functions suggest a labelling—we need them to suggest a *good* labelling. That is, we could have all of the f_u and g_v be perfectly decodable dictator functions, and they would pass all the tests proposed last time with high probability, even if that assignment doesn't actually satisfy any of the constraints. Thus, our test needs to somehow also simultaneously check that the suggested key/label pairs satsify the edge constraints.

2 The New Test

We alter the test from last time to take the constraints into account.

- Choose $b \in \{-1,1\}, x \in \{-1,1\}^K, y \in \{-1,1\}^L$, independently and uniformly at random.
- Choose $\lambda \in \{-1,1\}^L$ from the δ-biased distribution (i.e., each λ_i is independently -1 w.p. δ).
- Define $x_\pi \in \{-1,1\}^L$ by $(x_\pi)_i := x_{\pi(i)}$. Set $z := x_\pi \cdot y \cdot \lambda \cdot (b, \ldots, b)$.
- Check $f(x)g(y)g(z) = b$.



Next we analyze the probability that a given pair of functions f, g pass.

$$
\begin{aligned}
\mathbf{Pr}[f, g \text{ pass}] &= \mathop{\mathbf{E}}_{b,x,y,\lambda}\left[\tfrac{1}{2} + \tfrac{1}{2} b f(x) g(y) g(z)\right] \\
&= \tfrac{1}{2} + \tfrac{1}{2} \sum_{\substack{S \subseteq [K] \\ T,U \subseteq [L]}} \hat{f}(S)\hat{g}(T)\hat{g}(U)\mathbf{E}\left[b\chi_S(x)\chi_T(y)\chi_U(z)\right] \\
&= \tfrac{1}{2} + \tfrac{1}{2} \sum_{S,T,U} \hat{f}(S)\hat{g}(T)\hat{g}(U)\mathbf{E}\left[b\chi_S(x)\chi_T(y)\chi_U(x_\pi)\chi_U(y)\chi_U(\lambda)\chi_U(b,\ldots,b)\right] \\
&= \tfrac{1}{2} + \tfrac{1}{2} \sum_{S,T,U} \hat{f}(S)\hat{g}(T)\hat{g}(U) \underbrace{\mathop{\mathbf{E}}_{b}\left[b^{|U|+1}\right]}_{\mathbf{1}[|U| \text{odd}]} \mathop{\mathbf{E}}_{x}\left[\chi_S(x)\chi_U(x_\pi)\right] \underbrace{\mathop{\mathbf{E}}_{y}\left[\chi_{T \triangle U}(y)\right]}_{\mathbf{1}[T=U]} \underbrace{\mathop{\mathbf{E}}_{\lambda}\left[\chi_U(\lambda)\right]}_{(1-2\delta)^{|U|}} \\
&= \tfrac{1}{2} + \tfrac{1}{2} \sum_{\substack{S \subseteq [K] \\ |T| \text{ odd}}} (1-2\delta)^{|T|}\hat{g}(T)^2)\hat{f}(S) \underbrace{\mathop{\mathbf{E}}_{x}\left[\chi_S(x)\chi_T(x_\pi)\right]}_{(\star)}
\end{aligned}
$$

Now, note that,

$$
\begin{aligned}
(\star) &= \mathop{\mathbf{E}}_{x}\left[\prod_{i\in S} x_i \prod_{j\in T} x_{\pi(j)}\right] \\
&= \mathbf{E}\left[\prod_{i\in S} x_i \prod_{i\in[K]} x_i^{|\pi^{-1}(i)\cap T|}\right] \\
&= \prod_{i\in[K]} \mathbf{E}\left[x_i^{|\pi^{-1}(i)\cap T| + \mathbf{1}[i\in S]}\right] \\
&= \prod_{i\in[K]} \mathbf{1}\left[i\in S \iff |\pi^{-1}(i)\cap T| \text{ odd}\right] \\
&= \mathbf{1}\left[S = \pi_{\text{odd}}(T)\right]
\end{aligned}
$$

where, for each $T \subseteq [L]$, we define,

$$
\begin{aligned}
\pi(T) &:= \{a \in [K] : \pi^{-1}(a) \cap T \neq \emptyset\} \\
\pi_{\text{odd}}(T) &:= \{a \in [K] : |\pi^{-1}(a) \cap T| \text{ is odd}\}
\end{aligned}
$$

Thus, we have,

$$
\mathbf{Pr}[f, g \text{ pass}] = \tfrac{1}{2} + \tfrac{1}{2} \sum_{|T| \text{ odd}} (1-2\delta)^{|T|} \hat{g}(T)^2 \hat{f}(\pi_{\text{odd}}(T))
$$

The following two facts about $\pi(T)$ and $\pi_{\text{odd}}(T)$ will be useful.

Fact 2.1. $\pi_{\text{odd}}(T) \subseteq \pi(T)$.

Fact 2.2. *If $|T|$ is odd, then $\pi_{\text{odd}}(T) \neq \emptyset$.*

We're now ready to prove the completeness and soundness of the test.

2.1 Completeness

Suppose $f : \{-1,1\}^K \to \{-1,1\}, g : \{-1,1\}^L \to \{-1,1\}$ are matching dictators (i.e., $f(x) = x_a$ and $g(x) = x_\alpha$ where $\pi(\alpha) = a$). Then, g has Fourier support $\{\{\alpha\}\}$, f has Fourier support $\{\{a\}\}$, and $\pi_{\text{odd}}(\{\alpha\}) = \{a\}$, so,

$$
\mathbf{Pr}[f, g \text{ pass}] = \tfrac{1}{2} + \tfrac{1}{2}(1-2\delta)\hat{g}(\{\alpha\})^2 \hat{f}(\pi_{\text{odd}}(\{\alpha\})) = 1 - \delta
$$

2.2 Soundness

As usual, we'll prove soundness using the probabalistic method. Specifically, we want to come up with "random decoding" functions $\mathrm{Dec}_f : f_u \mapsto a \in [K]$ and $\mathrm{Dec}_g : g_v \mapsto \alpha \in [L]$, such that if f_u, g_u pass with sufficiently high probability, then $\pi_{vu}(\mathrm{Dec}_g(g_v)) = \mathrm{Dec}_f(f_u)$ holds with sufficiently high probability. As before, we'll decode f_u, g_v in two steps, by first decoding to some suggestion sets $\mathrm{Sugg}(f_u) \subseteq [K]$, $\mathrm{Sugg}(g_v) \subseteq [L]$, and then choosing the actual key/label uniformly at random from those suggestions. To do this, we want $|\mathrm{Sugg}(f_u)|, |\mathrm{Sugg}(g_v)|$ to be small (independent of K, L) and to have the property that if f_u, g_v pass with probability at least $\frac{1}{2} + \epsilon$, then there exist $a \in \mathrm{Sugg}(f_u), \alpha \in \mathrm{Sugg}(f_v)$ such that $\pi_{vu}(\alpha) = a$.

Suppose f, g pass with probability at least $\frac{1}{2} + \epsilon$. Then,

$$2\epsilon \leq \sum_{|T| \text{ odd}} (1-2\delta)^{|T|} \hat{g}(T)^2 \hat{f}(\pi_{\text{odd}}(T)) \leq \sum_{|T| \text{ odd}} (1-2\delta)^{|T|} \hat{g}(T)^2 \left| \hat{f}(\pi_{\text{odd}}(T)) \right|$$

Recall from Parseval's theorem that $\sum_{T \subseteq [L]} \hat{g}(T)^2 = 1$. Thus, we can think of $\hat{g}^2$ as a probability distribution on sets $T \subseteq [L]$. Then, the last line is equivalent to,

$$2\epsilon \leq \mathop{\mathbf{E}}_{T \sim \hat{g}^2} \left[\mathbf{1}\left[|T| \text{ odd}\right] (1-2\delta)^{|T|} \left| \hat{f}(\pi_{\text{odd}}(T)) \right| \right]$$

Note the expression inside the expectation is just a RV on T taking values in $[0,1]$, so by a simple averaging argument,

$$\epsilon \leq \mathop{\mathbf{Pr}}_{T \sim \hat{g}^2} \underbrace{\left[\mathbf{1}\left[|T| \text{ odd}\right] (1-2\delta)^{|T|} \left| \hat{f}(\pi_{\text{odd}}(T)) \right| \geq \epsilon \right]}_{\mathsf{GOOD}_T}$$

Now, suppose some set $T \subseteq [L]$ is good (i.e., GOOD_T occurs). Then, many good things happen. Specifically,

- $|T|$ is odd.
- $|T| \leq \frac{\ln(1/\epsilon)}{2\delta} =: B$.
- $\pi_{\text{odd}}(T) \neq 0$, by Fact 2.2.
- $|\pi_{\text{odd}}(T)| \leq B$, since $\pi_{\text{odd}}(T) \subseteq \pi(T)$ and $|\pi(T)| \leq |T| \leq |B|$.
- $\hat{f}(\pi_{\text{odd}})^2 \geq \epsilon^2$

The last point suggests a good way to decode f. Define,

$$\mathrm{Sugg}(f) := \bigcup \{S \subseteq [K] : 1 \leq |S| \leq B, \hat{f}(S)^2 \geq \epsilon^2\}$$

Note that since each $\hat{f}(S)^2 \geq \epsilon^2$, there can be at most $1/\epsilon^2$ such S, and each S has size at most B. Thus, $|\mathrm{Sugg}(f)| \leq B/\epsilon^2$.

The only remaining question is how to decode g. One immediate but naive idea would be to take the union of all good T. However, we don't have any kind of bound on the *number* of such good T. That is, it could be the case that there are many small good sets T, in which case the union could be very large. So, we're going to need to decode g in a different manner. Now, in the past, our "random decoding" has essentially consisted of *deterministically* decoding g to some small suggestion set, and then choosing the actual label uniformly at random from that. Of course, there's no need for the first step to be deterministic; we're free to have $\mathrm{Sugg}(g)$ be a random function too. So, we can simply let $\mathrm{Sugg}(g) = T$, where T is drawn from the distribution given by $\hat{g}^2$.

We've shown how to decode f and g to suggestion sets of small size. Now, suppose T is good. Then, the union in the definition of $\mathrm{Sugg}(f)$ includes $\pi_{\mathrm{odd}}(T)$, so $\pi_{\mathrm{odd}}(T) \subseteq \mathrm{Sugg}(f)$. Since $\pi_{\mathrm{odd}}(T)$ is non-empty, there exists some $a \in \pi_{\mathrm{odd}}(T)$. But then $a \in \pi(T)$, so there exists some $\alpha \in T$ with $\pi(\alpha) = a$. That is, for any good T, there exists $a \in \mathrm{Sugg}(f)$ and $\alpha \in T = \mathrm{Sugg}(g)$ such that $\pi(\alpha) = a$.

Now, putting it all together, suppose f, g pass with probability $\geq \frac{1}{2} + \epsilon$. Then, we know $\mathbf{Pr}_{T\sim\hat{g}^2}[\mathsf{GOOD}_T] \geq \epsilon$. Thus,

$$\mathop{\mathbf{Pr}}_{T\sim\hat{g}^2}[|T| \leq B, \exists a \in \mathrm{Sugg}(f), \alpha \in T : \pi(\alpha) = a] \geq \epsilon$$

Since $\mathrm{Dec}(f)$ is chosen uniformly from $\mathrm{Sugg}(f)$ and $\mathrm{Dec}(g)$ is chosen uniformly from T, we have,

$$\begin{aligned}
\mathbf{Pr}[\pi(\mathrm{Dec}(g)) = \mathrm{Dec}(f)] &\geq \mathbf{Pr}_{T\sim\hat{g}^2}[\mathsf{GOOD}_T] \cdot \frac{1}{|\mathrm{Sugg}(g)|} \cdot \frac{1}{|\mathrm{Sugg}(f)|} \\
&\geq \epsilon \cdot \frac{1}{B} \cdot \frac{\epsilon^2}{B} = \frac{\epsilon^3}{B^2} = \frac{\epsilon^3}{4\delta^2 \ln^2(1/\epsilon)}
\end{aligned}$$

References

[1] J. Håstad. Some optimal inapproximability results. *J. ACM*, 48(4):798–859, 2001.