Advanced Approximation Algorithms **CMU 15-854B, Spring 2008**

HOMEWORK 3
Due: Thursday, February 28

1. Some Generalizations of k-Center. These problems are not necessarily related to each other — so please don't read anything into their relative placement.

a) Suppose you are given an *asymmetric* metric space (V, d). You are given a set $D \subseteq V$ of clients who have to be covered. You now want to pick $F \subseteq V$ with $|F| \leq k$ to minimize

$$\max_{j \in D} d(F, j) = \max_{j \in D} \min_{f \in F} d(f, j).$$

To what factor can you approximate this problem?

b) Suppose we are given an instance of the standard *symmetric* k-center problem (as defined in Lecture 7), but now each potential center i has a cost f_i, and the objective is to open at most k centers F to minimize

$$\Phi'(F) = \sum_{i \in F} f_i + \max_{j \in V} d(F, j).$$

Give a constant factor approximation for the problem; try to make the constant as small as you can.

c) Consider a variant of the symmetric k-center problem above: now you can pick a set of centers F such that $\sum_{i \in F} f_i \leq B$ for some given budget B. You want to minimize

$$\Phi(F) = \max_{j \in V} d(F, j).$$

Give a constant-factor approximation for this problem. (Note that if $f_i = 1$ for all i, we have standard k-center.)

2. Approximability Curves, and the Location of the Gap. Suppose we have proved a c_0 vs. s_0 hardness reduction for Max-Ek-Independent-Set.

a) Consider taking the hypergraph produced by the reduction and adding a large number of isolated vertices. Via this additional reduction, what kind of c vs. s hardness results do we get?

b) Consider instead the additional reduction of adding a large number of vertices, along with all possible k-uniform hyperedges on them. Now what kind of c vs. s hardness results do we get?

c) Draw a 2-d plot with a horizontal c-axis and a vertical s-axis, both ranging from 0 to 1. Schematically draw in a point (c_0, s_0), then shade in the region of points (c, s) for which the previous two questions let us deduce hardness. (For maximum schematic realism, try to draw in the point $(c_0, s_0) = (1 - 2/k - \delta, \epsilon)$, as shown in class.)

d) How can we easily get the analogous plot for Min-Ek-Vertex-Cover? (Hint: your answer should involve at least one word from the following set: {rotate, reflect}.) Add to your diagram a differently-shaded region representing the (c, s) points *easy* for Max-Ek-Independent-Set due to a factor-k Vertex-Cover algorithm.

3. Raz's Theorem — the Parameters. Consider the following reduction, parameterized by some $d \in \mathbb{N}$, which maps a Label-Cover(K,L) instance $\mathcal{G} = (U, V, E, \pi\text{'s})$ into a Label-Cover(K^d,L^d) instance denoted $\mathcal{G}^{\otimes d}$:

"d-round Parallel Repetition". $\mathcal{G}^{\otimes d}$ has left vertices U^d, right vertices V^d, left keys K^d, and right labels L^d. For each list of d edges $(u_1, v_1), \dots, (u_d, v_d) \in E$, the instance $\mathcal{G}^{\otimes d}$ has an edge joining $\vec{u} := (u_1, \dots, u_d)$ and $\vec{v} := (v_1, \dots, v_d)$. The constraint on this edge is defined by

$$\pi_{\vec{v} \to \vec{u}}(\alpha_1, \dots, \alpha_d) = (\pi_{v_1 \to u_1}(\alpha_1), \dots, \pi_{v_d \to u_d}(\alpha_d)).$$

a) Show that $\mathrm{Opt}(\mathcal{G}^{\otimes d}) \geq \mathrm{Opt}(\mathcal{G})^d$ always. (Note: it is not true in general that equality holds.)

Fact: Raz's Parallel Repetition Theorem shows that if $\mathrm{Opt}(\mathcal{G}) \leq 1 - \epsilon$ then

$$\mathrm{Opt}(\mathcal{G}^{\otimes d}) \leq (1 - \epsilon^{32}/O(1))^{d/\log(|K| \cdot |L|)}.$$

In 2006 Holenstein improved the 32 to a 3. A couple of months ago Anup Rao improved the above to

$$\mathrm{Opt}(\mathcal{G}^{\otimes d}) \leq (1 - \epsilon^2/O(1))^d$$

(and then last month Raz showed this was best possible...).

b) As we saw on Homework 1 #3b, the PCP Theorem easily implies 1 vs. $1 - \epsilon_0$ NP-hardness for Label-Cover([2], [7]), for some universal constant $\epsilon_0 > 0$; and further, that this is true when the left-degree is 6 and the right-degree is 3. Using Parallel Repetition, show that for any η there is an $n^{O(\log(1/\eta))}$ time hardness reduction establishing 1 vs. η hardness for Label-Cover(K, L), where $|K|, |L| \leq \mathrm{poly}(1/\eta)$, and where the instances are left- and right-regular with degree at most $\mathrm{poly}(1/\eta)$.

4. Bicriteria Hardness for Max-Coverage. Show that given a Max-Coverage instance with the promise that there are m sets covering all elements, there is no polynomial time algorithm which can guarantee covering a $1 - 2^{-r-1}$ fraction of elements, *even if it is allowed to choose $r \cdot m$ sets* — unless NP $\subseteq$ DTIME($n^{O(\log r)}$). Deduce (carefully) that there is no factor $\Omega(\log n)$ algorithm for Min-Set-Cover unless NP $\subseteq$ DTIME($n^{O(\log \log n)}$)

Do at least one of the following two:

5. Weighted vs. Unweighted. In class we proved $1 - 3/k$ vs. $1/k$ hardness for weighted Max-Ek-Independent-Set. Our instances had vertex set $V \times \{0,1\}^L$, and the weights on these vertices were $(1 - 3/k)$-biased.

a) Using the parameters from problem 2b, show that we can take $|L| \leq \mathrm{poly}(k)$, that the number of vertices in our instances is at most $n^{O(\log k)}$, and that the number of hyperedges is at most $2^{\mathrm{poly}(k)}$ times the number of vertices.

b) Show how to get all of these results to hold for *unweighted* Max-Ek-Independent-Set.

6. Technical Theorem about Intersecting Families. Taking the statement of bonus problem 6b from Homework 2 as a black box, prove Theorem 2.6 from Lecture 8's scribe notes.