Optimal Stopping Theorems

Original Studies of Martingales came from $\underbrace{\text{"games of chance"}}_{\text{gambling}}$.

Suppose X_t = amount we have after time t.

and suppose playing fair game. (say double or nothing)

then if we bet Y_t at time t, ~~...~~ (could depend on X_{t-1})

e.g. $Y_t \leq X_{t-1}$

then have $X_{t-1} - Y_t$ w.p. $\frac{1}{2}$ ⟵ (lose Y_t)

$X_{t-1} + Y_t$ w.p. $\frac{1}{2}$ ⟵ (get back $2Y_t$)

X_{t-1} in expectation, at time t.

Indeed $\mathbb{E}X_t = \mathbb{E}X_0$ for any time t.

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Suppose we want to say what $\mathbb{E}X_T$ is for a random time T?

Say T here is its ~~elf~~ a random variable.

When can we say this is true that $\mathbb{E}X_T = \mathbb{E}X_0$?

Clearly not for any T!

Say set T = first time when we have \$0.

then $X_T = 0 \Rightarrow \mathbb{E}X_T = 0$.

but X_0 may be non zero!

So need some conditions on T.

Suppose we're considering the case where

$X_t = f_t(Z_0, Z_1, \dots Z_t)$ for some functions f_t. with $E[|X_t|] < \infty$

and hence (X_t) is a martingale wrt (Z_t).

Now: define T to be a stopping time wrt (Z_t) if

the event $\{T = i\}$ can be inferred from the r.v.s. $Z_0, Z_1 \dots Z_i$.

E.g. sps. Z_0 = starting point.

$Z_i = \pm 1$ independently, and $X_i = Z_0 + Z_1 \dots + Z_i$

then the ~~event~~ stopping time $T = \{\text{last time that } X_i = 17\}$

is not a stopping time, since we

cannot tell from $Z_0 \dots Z_i$ if

some $X_j = 17$ for $j > i$.

but $T = \{\text{first time that } X_i = 17\}$

or $T = \{\text{third time that } X_i = 17\}$ would be perfectly fine stopping times.

Basically: a stopping time is "decided by the r.v.s. seen until that stopping time".

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Optional Stopping Theorem

Many forms. This one is easiest (see Wikipedia or Prob Texts for more).

Suppose:

(1) $E[T] < \infty$ and

(2) $E[|X_{t+1} - X_t| \mid Z_0 \dots Z_t] \leq c$ for some constant c $\forall t$.

$\Rightarrow E[X_T] = E[X_0]$.

Example:

$\cancel{X_0} = k$ $X_t = Z_0 + \cdots + Z_t$.

$Z_i = \pm 1$ wp $1/2$.

let T be first time we hit 0 or n.

(1) $E[|T|] < \infty$.

Indeed. starting from any point in $[0, n]$, hit 0 or n wp $\geq 2^{-n}$.

$\Rightarrow E[T] \leq 2^n < \infty$.

(2) $E[|X_{t+1} - X_t| \mid Z_0 \ldots Z_t] \leq 1$.

$\Rightarrow E[X_T] = E[X_0] = k$.

Now: if $p_0 = Pr(\text{hit } 0 \text{ before } n)$

$\Rightarrow E[X_T] = 0 \cdot p_0 + n \cdot (1 - p_0) = k$.

$\Rightarrow p_0 = 1 - \frac{k}{n}$ $\Rightarrow Pr(\text{hit } n \text{ first}) = \frac{k}{n}$.

(same as last time, but different proof).

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Allows us to prove Wald's Inequality.

Suppose $Z_1, Z_2 \ldots$ are iid random vars

and T is a stopping time wrt (Z_t)

then $E\left[\sum_{t=1}^{T} Z_t\right] = E[T] \cdot E[Z_1]$.

sum has a random # of terms

E[stopping time]

E[each time].

E.g. Suppose you run an algo that

(a) costs a random amount Z_t for time t. (with $Z_t \sim Z \ \forall t$ (iid))

(b) ~~stops at each~~ succeeds at ~~time~~ each run w.p. p. independently.

You run algo until it succeeds.

How much did you pay? T = first time success.

$$\text{Wald} \Rightarrow E\left[\sum_{t=1}^{T} Z_t\right] = E[T] \cdot E[Z]$$

$$= (1/p) \cdot E[Z]$$

Magic of Martingales.

Basically: consider processes that are time 'dependent'

X_t depends on $X_0, X_1 \dots X_{t-1}$

But has ~~$E[X_t] = E[X_t$~~

$$E[X_t \mid X_0 \dots X_{t-1}] = X_{t-1}.$$

May require tweaking ~~too~~ sometimes.

Eg. Sps. $Z_t = \{0,1\}$ wp $\frac{1}{2}$ each. $X_t = \sum Z_t$.

then $E[X_t \mid Z_0 \dots Z_{t-1}] = \frac{1}{2} + X_{t-1}$ (not X_{t-1}).

But then $(X_t - t/2)$ is martingale.

So may need to "center" random vars....

See probability texts, [MR] [MU] [AS] or other books.

Extensions of Azuma–Hoeffding / Method of Bounded Differences

(1) Freedman's Inequality

(2) Talagrand's Inequality.

Let's do Talagrand's Inequality (reformulation).

Suppose have random vars. $Z_1, Z_2 \dots Z_n$.

And a function $f(X_1, \dots X_n)$. We want to show f is concentrated.

- know that if $\underbrace{|f(x_1 \dots x_n) - f(y_1 \dots y_n)|}_{f(\bar{x}) - f(\bar{y})} \le \sum_{i=1}^{n} c_i \mathbb{1}(x_i \ne y_i)$

c_i-Lipschitz

Method of Bounded Differences.

$$\Rightarrow \Pr(|f - Ef| \ge \lambda) \le 2\exp\left(-\frac{\lambda^2}{\sum_i c_i^2}\right).$$

- Talagrand says: sps $\forall x, \exists\ c_i(x)$ st.

$$f(\bar{x}) \le f(\bar{y}) + \sum_{i=1}^{n} c_i(x) \cdot \mathbb{1}(x_i \ne y_i)$$

now depends on x.

Another convenient packaging of Talagrand's Ineq.

And $\forall x.\ \sum_{i=1}^{n} c_i^2(x) \le C$

$$\Rightarrow \Pr(|f - Mf| \ge \lambda) \le 4\exp(-\lambda^2/C^2).$$

median of the function value!

Clearly at least as strong as the above thm. But often much stronger!

~~Let's see an example...~~