A generating function represents an entire, infinite sequence as a single mathematical object that can be manipulated algebraically. The strength of the representation lies in the fact that many operations can be carried out a generating function, including differentiation, integration, multiplication, and others, even though the underlying sequence may be defined in purely symbolic or combinatorial terms. This makes generating functions an elegant and powerful counting technique. In fact, in the text *Concrete Mathematics* [**?**], generating functions are described as "the most important idea in this whole book."

If $a_0, a_1, a_2, \ldots$ is a sequence, then we define the *formal power series* $A(z)$ as

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k=0}^{\infty} a_k z^k. \tag{1}$$

The adjective *formal* means that z is an abstract indeterminate, and we are not concerned about numerical convergence of the series for any particular value of z.

The product of generating functions is defined as

$$\begin{aligned} A(z)B(z) &= \left(a_0 + a_1 z + a_2 z^2 + \cdots\right)\left(b_0 + b_1 z + b_2 z^2 + \cdots\right) && (2)\\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, z + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, z^2 + \cdots && (3)\\ &= c_0 + c_1 z + c_2 z^2 + \cdots && (4)\\ &= C(z) && (5)\end{aligned}$$

where

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}. \tag{6}$$

Such a sum is sometimes called a discrete *convolution*.

Here's an application of this identity. From the binomial theorem we have

$$\begin{aligned} (1+z)^r &= \sum_{k \geq 0} \binom{r}{k} z^k && (7)\\ (1+z)^s &= \sum_{k \geq 0} \binom{s}{k} z^k && (8)\end{aligned}$$

and therefore

$$\begin{aligned} (1+z)^r (1+z)^s &= (1+z)^{r+s} && (9)\\ &= \sum_{k \geq 0} \binom{r+s}{k} z^k. && (10)\end{aligned}$$

Therefore we have, by the convolution identity,

$$\binom{r+s}{k} = \sum_{i=0}^{k} \binom{r}{i}\binom{s}{k-i}. \tag{11}$$

We've seen this before—it's the number of ways of choosing k turns from a total of r avenues and s streets.

As another example, using the binomial theorem we have

$$\begin{aligned} (1-z)^r(1+z)^r &= (1-z^2)^r & (12)\\ &= \sum_{k\geq 0}(-1)^k\binom{r}{k}z^{2k}. & (13) \end{aligned}$$

But now applying the convolution identity (6) on the left, we obtain

$$\sum_{i=0}^{k}(-1)^i\binom{r}{i}\binom{r}{k-i} = \begin{cases} 0 & \text{if } k \text{ is odd;} \\ (-1)^{k/2}\binom{r}{k/2} & \text{if } k \text{ is even.} \end{cases} \tag{14}$$

This might look a bit strange; let's check some small examples. Taking $k = 2$,

$$\begin{aligned} \binom{r}{0}\binom{r}{2} - \binom{r}{1}\binom{r}{1} + \binom{r}{2}\binom{r}{0} &= 2\binom{r}{2} - r^2 & (15)\\ &= -r & (16)\\ &= -\binom{r}{1} & (17) \end{aligned}$$

which checks out. Taking $k = 3$, we get

$$\binom{r}{0}\binom{r}{3} - \binom{r}{1}\binom{r}{2} + \binom{r}{2}\binom{r}{1} - \binom{r}{3}\binom{r}{0} = 0 \tag{18}$$

which also checks.

1 Math Counts

To see the usefulness of generating functions for counting, suppose that we have two disjoint sets $\mathcal{A}$ and $\mathcal{B}$. Also, suppose that there are a_n ways of selecting n elements from $\mathcal{A}$ and b_n ways of selecting n elements from $\mathcal{B}$. Then the number ways c_n of selecting a *total* of n items from either $\mathcal{A}$ or $\mathcal{B}$ is

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \tag{19}$$

To express this in terms of generating functions, we have that the generating function for selecting items from either $\mathcal{A}$ or $\mathcal{B}$ is

$$C(z) = A(z)B(z). \tag{20}$$

We saw this earlier with choice trees and polynomials; the generating function idea extends it to choice trees of "infinite depth."

2 A Fruity Example

Here's a fun example (from notes by Albert R. Meyer and Clifford Smyth) that illustrates how generating functions can solve some seemingly very messy counting problems.

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

If we try to count the number of ways directly, it looks complicated. For example, with a basket of five fruits, there are six possibilities:

apples	0	0	0	0	0	5
bananas	4	4	2	2	0	0
oranges	1	0	2	3	4	0
pears	0	1	1	0	1	0

It's hard to see any clear pattern here that would extend to bigger baskets.

Let's give generating functions a shot. Since the number of bananas must be even, the sequence $\langle b_n \rangle$ is $\langle 1, 0, 1, 0, 1, 0, \ldots \rangle$, and so the generating functions for bananas is

$$B(z) = \sum_{n \geq 0} b_n z^n = 1 + z^2 + z^4 + \cdots = \frac{1}{1 - z^2}. \tag{21}$$

Similarly, the generating function for apples is

$$A(z) = \sum_{n \geq 0} a_n z^n = 1 + z^5 + z^{10} + \cdots = \frac{1}{1 - z^5}. \tag{22}$$

The generating functions $O(z)$ and $P(z)$ for oranges and pears are even easier:

$$O(z) = 1 + z + z^2 + z^3 + z^4 \tag{23}$$

$$P(z) = 1 + z. \tag{24}$$

Now, recall from our manipulations with geometric series that $O(z) = (1 - z^5)/(1 - z)$. So, when we multiply all of these functions together, we get

$$A(z)B(z)O(z)P(z) = \frac{1}{1 - z^5} \frac{1}{1 - z^2} \frac{1 - z^5}{1 - z}(1 + z) \tag{25}$$

$$= \frac{1}{(1 - z)^2}. \tag{26}$$

Now, we need to re-expand this as a series. To do this, we use a little differentiation:

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{(1-z)} \tag{27}$$

$$= \frac{d}{dz}\sum_{n=0}^{\infty} z^n \tag{28}$$

$$= \sum_{n=0}^{\infty} \frac{d}{dz} z^n \tag{29}$$

$$= \sum_{n=1}^{\infty} n z^{n-1} \tag{30}$$

$$= \sum_{n=0}^{\infty} (n+1) z^n. \tag{31}$$

We've determined that the number of ways of filling a basket with n fruits that satisfies our fruity constraints is $n+1$. That wasn't so bad after all! Note that our special case above checks out: there are six ways to take five fruits.

3 Basic Properties of Generating Functions

Let's now look at some of the basic ways we can manipulate generating functions, which give us a bag of tricks to be used for counting problems. Let $A(z)$ and $B(z)$ denote two generating functions for the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ respectively, so that

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k=0}^{\infty} a_k z^k \tag{32}$$

$$B(z) = b_0 + b_1 z + b_2 z^2 + \cdots = \sum_{k=0}^{\infty} b_k z^k. \tag{33}$$

We can add the two functions, and multiply by a scalar; thus $\alpha A(z) + \beta B(z) = C(z)$ is the generating function for the sequence $\langle c_n \rangle = \langle \alpha a_n + \beta b_n \rangle$. We can also easily get the function for a sequence shifted m places to the right by multiplying by z^m:

$$z^m A(z) = \sum_n a_n z^{n+m} = \sum_n a_{n-m} z^n \tag{34}$$

which corresponds to the sequence shifted to the right:

$$\underbrace{0,\ 0,\ \ldots, 0}_{m},\ \ a_0,\ a_1,\ a_2 \ldots \tag{35}$$

Shifting to the left is simple as well:

$$\frac{A(z) - a_0 - a_1 z - \ldots a_{m-1} z^{m-1}}{z^m} = \sum_{n \geq m} a_n z^{n-m} = \sum_{n \geq 0} a_{n+m} z^n \tag{36}$$

which corresponds to the sequence

$$a_m, \; a_{m+1}, \; a_{m+2} \cdots \tag{37}$$

where the first m coefficients are dropped. Another useful technique is to replace the variable z by cz, where c is a constant, yielding

$$A(cz) = \sum_n a_n c^n z^n \tag{38}$$

which is the generating function for the sequence $\langle a_n c^n \rangle$. If we want to replace $\langle a_n \rangle$ by $\langle n a_n \rangle$ then the thing to do is differentiate and multiply by z:

$$zA'(z) = z \sum_n n a_n z^{n-1} = \sum_n n a_n z^n \tag{39}$$

This highlights our perspective that generating functions are *formal* power series; we are not concerned with the numerical convergence of the series, and whether the derivative is defined, etc. In a similar fashion, we can take the integral and use

$$\int_0^x z^n \, dt = \frac{1}{n+1} z^{n+1} \tag{40}$$

to obtain

$$\int_0^x A(t) \, dt = \sum_{n \geq 1} \frac{1}{n} a_{n-1} z^n \tag{41}$$

We've seen above that multiplication corresponds to convolution of the coefficients:

$$A(z)B(z) = \sum_n \left(\sum_k a_k b_{n-k} \right) z^n \tag{42}$$

As a useful special case of this, consider what happens when we multiply by $1/(1-z)$ which has the geometric series expansion $1/(1-z) = 1 + z + z^2 + z^3 + \cdots$:

$$\frac{1}{1-z} A(z) = \sum_n \left(\sum_k a_{n-k} \right) z^n = \sum_n \left(\sum_{k \leq n} a_k \right) z^n \tag{43}$$

This corresponds to a new sequence given by the cumulative sums of the old sequence.

4 Solving Recurrences

Generating functions provide a powerful way of solving recurrences. The basic idea is to write a sequence $\langle a_n \rangle$ that satisfies some recurrence in terms of a generating function $A(z)$, which after some manipulation can be written in closed form. Then, the coefficients a_n can (often) be read off by expanding this closed form in a power series. This sounds circular, but it's not.

In more detail, here are the steps you should take to solve a recurrence using generating functions.

$$\begin{aligned}
\alpha A(z) + \beta B(z) &= \sum_n (\alpha a_n + \beta b_n) z^n && (44)\\
z^m A(z) &= \sum_n a_{n-m} z^n, \quad m \geq 0 && (45)\\
\frac{A(z) - a_0 - a_1 z - \ldots a_{m-1} z^{m-1}}{z^m} &= \sum_{n \geq 0} a_{n+m} z^n, \quad m \geq 0 && (46)\\
A'(z) &= \sum_n (n+1) a_{n+1} z^n && (47)\\
zA'(z) &= \sum_n n a_n z^n && (48)\\
\int_0^x A(t)\, dt &= \sum_{n \geq 1} \frac{1}{n} a_{n-1} z^n && (49)\\
A(z)B(z) &= \sum_n \left(\sum_k a_k b_{n-k} \right) z^n && (50)\\
\frac{1}{1-z} A(z) &= \sum_n \left(\sum_{k \leq n} a_k \right) z^n && (51)
\end{aligned}$$

Figure 1: Basic identities for generating functions.

1. Using the recurrence, write a single equation that expresses a_n in terms of the coefficients $a_m, m < n$, folding the base cases in to get an equation valid for all n, assuming $a_{-1} = a_{-2} = \cdots = 0$.
2. Multiply both sides by z^n, and sum over all n. On the lefthand side we have $A(z)$. The righthand side should be manipulated so that it can be expressed as another function of $A(z)$. (This is often the most creative part of the process.)
3. Solve the equation for $A(z)$, getting a closed form.
4. Expand this closed form into a power series, to get a closed form for a_n.

To illustrate, let's consider the recurrence

$$\begin{aligned}
a_0 &= a_1 = 1 && (52)\\
a_n &= a_{n-1} + 2a_{n-2} + (-1)^n, \quad \text{for } n \geq 2 && (53)
\end{aligned}$$

We can express this as a single equation by writing

$$a_n = a_{n-1} + 2a_{n-2} + (-1)^n [n \geq 0] + [n = 1] \tag{54}$$

where we use the notation

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true;} \\ 0 & \text{if } P \text{ is false.} \end{cases} \tag{55}$$

This single equation handles the base cases

$$\begin{aligned} a_0 &= 0 + 2 \cdot 0 + (-1)^0 + 0 = 1 && (56) \\ a_1 &= 1 + 2 \cdot 0 + (-1)^1 + 1 = 1. && (57) \end{aligned}$$

This carries out Step 1; to execute Step 2, we multiply by z^n and sum, to get

$$\begin{aligned} A(z) = \sum_n a_n z^n &= \sum_n \left(a_{n-1} + 2a_{n-2} + (-1)^n[n \geq 0] + [n = 1]\right) z^n && (58) \\ &= zA(z) + 2z^2 A(z) + \sum_{n \geq 0} (-1)^n z^n + z && (59) \\ &= zA(z) + 2z^2 A(z) + \frac{1}{1+z} + z. && (60) \end{aligned}$$

Now we carry out Step 3, solving for $A(z)$ to get

$$\begin{aligned} A(z) &= \frac{\frac{1}{1+z} + z}{1 - z - 2z^2} && (61) \\ &= \frac{1 + z + z^2}{(1+z)(1 - z - 2z^2)} && (62) \\ &= \frac{1 + z + z^2}{(1+z)^2(1-2z)}. && (63) \end{aligned}$$

At this point, we have a closed form for $A(z)$. If we can express the righthand side as a power series, we'll be done.

After some work (which we'll explain later), the expansion can be shown to have the form

$$a_n = \frac{7}{9} 2^n + \left(\frac{1}{3} n + \frac{2}{9}\right)(-1)^n \tag{64}$$

and this is the solution to the recurrence.

5 Another example: Fibonacci

We can apply this same strategy to get a generating function for the Fibonacci numbers. Recall that the recurrence relation is

$$F_n = \begin{cases} 0 & n \leq 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n > 1. \end{cases} \tag{65}$$

To execute Step 1, we need to write this in terms of a single equation. This is easily done:

$$F_n = F_{n-1} + F_{n-2} + [n = 1] \tag{66}$$

Note that the base cases check out: $F_0 = 0$, $F_1 = 1$ and $F_2 = 1$. To carry out Step 2, we multipy by z^n and sum:

$$\begin{aligned}
\sum_n F_n z^n &= \sum_n F_{n-1} z^n + \sum_n F_{n-2} z^n + z && (67)\\
&= z \sum_n F_{n-1} z^{n-1} + z^2 \sum_n F_{n-2} z^{n-2} + z && (68)\\
&= zF(z) + z^2 F(z) + z && (69)
\end{aligned}$$

Step 3 is then easily carried out, since we can solve for $F(z)$ to get

$$F(z) = \frac{z}{1 - z - z^2}. \tag{70}$$

We can view this in terms of tilings, described below.

It's not really necessary to use the somewhat funky bracketing notation $[\cdot]$. Instead, we could just reason that since $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, we have that

$$\begin{aligned}
\sum_{n=2}^{\infty} F_n z^n &= \sum_{n=2}^{\infty} F_{n-1} z^n + \sum_{n=2}^{\infty} F_{n-2} z^n && (71)\\
&= z \sum_{n=1}^{\infty} F_n z^n + z^2 \sum_{n=0}^{\infty} F_n z^n. && (72)
\end{aligned}$$

Now, to massage the lefthand side into the generating function, we need to add in—and subtract out—the missing terms $F_0 + F_1 z$. This gives us

$$\begin{aligned}
\sum_{n=2}^{\infty} F_n z^n = F(z) - F_0 - F_1 z &= z \sum_{n=1}^{\infty} F_n z^n + z^2 \sum_{n=0}^{\infty} F_n z^n && (73)\\
&= zF(z) - zF_0 + z^2 F(z) && (74)
\end{aligned}$$

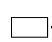
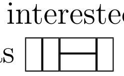
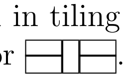
from which we obtain

$$F(z) = \frac{z}{1 - z - z^2} \tag{75}$$

by appealing to the base cases $F_0 = 0$ and $F_1 = 1$.

6 Tiling by Dominoes

A generating function is best thought of as a *symbolic* object, which can be manipulated with formal mathematical operations, without worrying about their numerical properties. To better appreciate this point, it's helpful to consider an example involving tilings by dominoes.

Suppose that we have a bunch of dominoes of identical shape; we can either stand a domino on end like this , or lay it on its side, like this . We're interested in tiling a $2 \times n$ strip with dominoes. For example, we could tile a 2×5 region as  or .

Let $\mathcal{T}$ denote the collection of all tilings of a $2 \times n$ region, for all $n \geq 0$. Then we can write $\mathcal{T}$ symbolically as

$$\mathcal{T} = | + \square + \square\square + \boxminus + \square\square\square + \square\boxminus + \boxminus\square + \cdots \tag{76}$$

Here the symbol $|$ denotes the unique tiling of the 2×0 region which uses no tiles, and the mathematical operator $+$ really means "union." We can define a kind of multiplication on tiles, where $\square \times \boxminus = \square\boxminus$ means pasting a vertical tile on the left of two horizontal tiles. Note that this multiplication is not commutative, so that $\square\boxminus = \square \times \boxminus \neq \boxminus \times \square = \boxminus\square$. The trivial tiling $|$ acts like the multiplicative identity—it doesn't change a tiling.

Now, with this symbolic multiplication in hand we can rearrange terms in the (infinite) description of $\mathcal{T}$ to obtain

$$\begin{aligned}
\mathcal{T} &= | + \square + \square\square + \boxminus + \square\square\square + \square\boxminus + \boxminus\square + \cdots && (77)\\
&= | + \square(| + \square + \square\square + \boxminus + \cdots) + \boxminus(| + \square + \square\square + \boxminus + \cdots) && (78)\\
&= | + \square\,\mathcal{T} + \boxminus\,\mathcal{T} && (79)
\end{aligned}$$

To spell this symbolic equation out longhand, we would get the following long addition formula:

$$\begin{array}{cccccccccccccccc}
 & | & + & \square & + & \square\square & + & \boxminus & + & \square\square\square & + & \square\boxminus & + & \boxminus\square & + & \cdots \\
- & \square & - & \square\square & - & \square\square\square & - & \square\boxminus & - & \square\square\square\square & - & \square\square\boxminus & - & \square\boxminus\square & - & \cdots \\
- & \boxminus & - & \boxminus\square & - & \boxminus\square\square & - & \boxminus\boxminus & - & \boxminus\square\square\square & - & \boxminus\square\boxminus & - & \boxminus\boxminus\square & - & \cdots \\
\hline
 & | & & & & & & & & & & & & & &
\end{array}$$

Now, we can proceed by a leap of faith to solve equation (79) for $\mathcal{T}$, obtaining

$$\mathcal{T} = \frac{|}{| - \square - \boxminus} \tag{80}$$

What do we mean by the fraction $\frac{|}{|-\square-\boxminus}$? It should be thought of as a shorthand for a formal geometric series:

$$\begin{aligned}
\frac{|}{| - \square - \boxminus} &= | + (\square + \boxminus) + (\square + \boxminus)^2 + (\square + \boxminus)^3 + \cdots && (81)\\
&= | + (\square + \boxminus) + (\square\square + \square\boxminus + \boxminus\square + \boxminus\boxminus) + && (82)\\
&\quad (\square\square\square + \square\square\boxminus + \square\boxminus\square + \square\boxminus\boxminus + \boxminus\square\square + \boxminus\square\boxminus + \boxminus\boxminus\square + \boxminus\boxminus\boxminus) + \cdots
\end{aligned}$$

Now, when we tile a $2 \times n$ region, there are two basic "moves" we can make. Either we can can place down a vertical tile $\square$, or we can place down two horizontal tiles on top of each other, $\boxminus$. Note that if we are only interested in counting the number of tilings, then the order in which these "moves" are made does not matter. If we represent $\square$ by the variable z, and we represent $\boxminus$ by z^2, then it is plausible that the above expression is equivalent to

$$T(z) = \frac{1}{1 - z - z^2}. \tag{83}$$

In this way, we get an algebraic representation of symbolic object—the set of tilings of a $2 \times n$ strip.

Note the resemblance to the generating function for Fibonaccis; we'll explore this connection in more depth in the next lecture.