15-859(J): Metric Embeddings and Methods **Lecturer:** Anupam Gupta
Topic: Embeddings for special graphs: Cuts and trees **Date:** November, 13th, 2003
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22.1 Introduction

In this section we will study the embeddability into ℓ_1 and distributions over dominating trees, of outerplanar graphs, which are a special kind of planar graphs. We begin with an overview of the embeddability of various classes of graphs into ℓ_1, ℓ_2, and distributions over dominating trees (see Table 1). The following table describes these results starting from general graphs and listing classes of graphs in order of decreasing complexity. For references to the original work, please refer to the lectures cited.

Class	ℓ_1	ℓ_2	Distr. on Trees
General	$\Theta(\log n)$ (Lec. 5-7)	$\Theta(\log n)$ (Lec. 9)	$\Theta(\log n)$
Graphs with β-cutting schemes	$O(\beta\sqrt{\log n})$ (HW 2)	$O(\beta\sqrt{\log n})$	$O(\log n)$ (Lec. 9)
K_r-free graphs	$O(r^2\sqrt{\log n})$	$O(r^2\sqrt{\log n})$	$O(\log n)$
Planar graphs	$O(\sqrt{\log n})$; Best l.b. 2	$\Theta(\sqrt{\log n})$	$\Theta(\log n)$
Series-Parallel graphs	$O(1)$ (Lec. 23-24)	$\Theta(\sqrt{\log n})$ (Lec. 24)	$\Theta(\log n)$ (Lec. 23)
Outerplanar graphs	1 (Lec. 22)	$\Theta(\sqrt{\log\log n})$	$O(1)$ (Lec. 22)
Trees	1 (HW 1)	$\Theta(\sqrt{\log\log n})$	1

Table 1: Distortion for embeddings of various classes of graphs into ℓ_1, ℓ_2 and distributions over dominating trees.

Notes:

1. Recall that $\ell_2 \overset{1}{\hookrightarrow} \ell_1$. Thus all results for the embeddability of metrics into ℓ_2 hold for embeddability into ℓ_1 as well.

2. Recall that planar graphs have a β-cutting procedure with $\beta = O(1)$, and K_r-free graphs have a cuttnig procedure with $\beta = O(r^2)$. Thus we get upper bounds for the embeddability of planar and K_r-free graphs based on the corresponding results for graphs with β-cutting procedures.

3. The lower bound of $\Omega(\sqrt{\log\log n})$ for the distortion for embedding trees (and outer-planar graphs) into ℓ_2 is achieved using the complete binary tree on n nodes; see Linial and Saks (2003) for a simple proof. The matching upper bound for all trees is due to Matoušek (1999).

4. The lower bound of $\Omega(\sqrt{\log n})$ for embedding series-parallel (also planar) graphs into ℓ_2 is achieved using a special kind of graphs called "diamond" graphs. Diamond graphs also give a lower bound for embedding series-parallel graphs into distributions over trees.

22.2 Embeddability of cycles into ℓ_1

We begin with a few basic concepts, and then describe how to embed cycles into ℓ_1.

Recall that a metric μ is ℓ_1-embeddable if and only if it is a cut metric[1], that is, there exist elementary cut metrics δ_S and multipliers $\alpha_S \geq 0$ such that $\mu = \sum_S \alpha_S \delta_S$.

Definition 22.2.1 *A cut S is called "Geodesic" if for all pairs of nodes, every shortest path between the pair crosses S at most once.*

Note that in order to determine whether a cut is geodesic or not, we only need to check pairs of points that both lie on the same side of the cut and make sure that the shortest path does not cross the cut.

Definition 22.2.2 *Let S be a geodesic cut, and ϵ be arbitrary. Then, "removing the cut metric" $\epsilon\delta_S$ implies that we decrease all edge lengths in S by ϵ.*

Lemma 22.2.3 *Let S be a geodesic cut and $G_{\epsilon,S}$ be the graph obtained by removing the cut metric $\epsilon\delta_S$ from graph G. Suppose that S is still a geodesic cut in $G_{\epsilon,S}$. Then we have $d_{G_{\epsilon,S}} + \epsilon\delta_S = d_G$.*

Proof: Consider two vertices on the same side of the cut. Then the shortest path length between these two vertices is the same in graphs G and $G_{\epsilon,S}$, and the distance between them under the metric $\epsilon\delta_S$ is 0, so the claim holds. On the other hand, consider two points on different sides of the cut. Then, the length of the shortest path between them reduces by exactly ϵ, which is also the distance between them in $\epsilon\delta_S$. Thus the claim holds for this pair too. ■

The previous lemma and the fact that ℓ_1 is additive give the following corollary.

Corollary 22.2.4 *If $G_{\epsilon,S}$ is ℓ_1-embeddable, so is G.*

Now our idea behind embedding cycles and outer-planar graphs into ℓ_1 is as follows. We will iteratively remove geodesic cut metrics from the graph until all vertices shrink to a single point and it becomes a trivial metric. The trivial metric is clearly embeddable into ℓ_1. Therefore, by the above lemma, the original graph is also embeddable into ℓ_1 isometrically.

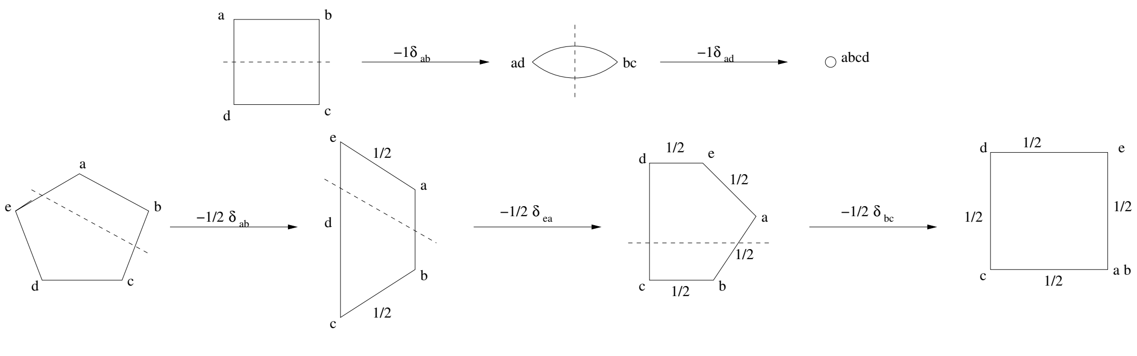

Figure 22.2.1: Embedding cycles into ℓ_1

[1]See lecture 6 for details.

We begin with some illustrative examples. Figure 22.2.1 demonstrates an application of the above lemma to a 4-cycle and a 5-cycle. In general, every cycle can be reduced in length by removing "antipodal" cuts from it; loosely speaking, antipodal cuts are cuts that divide the cycle into two roughly equal parts. For example, the cut (a, b) in the 5-cycle in Figure 22.2.1 is an antipodal cut.

In general, an ℓ_1-embedding of an n-cycle can be given succintly as follows – the ith vertex has the ith, $(i+1)$th and $(i+2)$th (modulo n) coordinates set to $\frac{1}{2}$ and the rest set to 0. An example, for the 5-cycle is given by the following matrix:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

22.3 Outer-planar graphs embed into ℓ_1 isometrically

Note that without loss of generality, we can assume that the graph is 2-connected. If not, then there exists a cut vertex, say v. Consider the two connected parts of the graph V_1 and V_2, such that $V_1 \cap V_2 = \{v\}$. Then, the shortest path metric on graph G is simply the gate-sum of the shortest path metrics on V_1 and V_2. Now, since ℓ_1-embeddability is closed under gate-sums, we only need to check the embeddability of V_1 and V_2 individually, to obtain the embeddability of G.

There are many characterizations of outer-planar graphs. It will be convenient for us to work with an inductive definition. Below we give a few of these characterizations, including the inductive definition.

1. Outer-planar graphs are planar graphs where all vertices lie on a single face (without loss of generality, this is the outer face).
2. A graph is outer-planar iff it is a (partially) triangulated cycle (polygon).
3. If we take the planar dual of a graph and remove the vertex corresponding to the outer face, then we get a tree iff the graph is outer-planar.
4. All graphs excluding K_4 and $K_{2,3}$ minors are outer-planar.
5. An inductive definition is given as follows.
   (a) A cycle is an outer-planar graph
   (b) If G is an outer-planar graph, the graph obtained by picking an edge on the outer face and adding a path between its two endpoints is also outer-planar.

   This is called an "ear-composition" of the graph (the paths added in step 2 being the "ears"). The opposite process is called an "ear-decomposition".

We use induction on the ear-decomposition of an outer-planar graph in order to prove its ℓ_1-embeddability. In particular, we remove cut-metrics from the ears in the graph, until it eventually becomes a cycle. Then using the fact that every cycle is ℓ_1-embeddable, we get that any outer-planar graph is also ℓ_1-embeddable.

We use induction on the number of "ears" added in the ear-composition. Assume (as the induction hypothesis) that all graphs with fewer than k ears are ℓ_1-embeddable. Let graph G contain $k+1$ ears, and let P be the last ear, added to edge $e = (u, v)$.

Note that, without loss of generality, we can assume $l(e) \leq l(P)$, otherwise we could remove the edge e without affecting the metric, and obtain a graph with fewer ears.

Now we claim that there is a geodesic cut that contains some edges of P. In particular, pick the edge f adjacent to u in path P. Among all edges f' in P such that the total length of edges in between f and f' is at most $l(P)/2$, pick the one that is farthest from f. Now it is obvious that the edges f and f' form a geodesic cut. Reduce the length of these edges until the cut $\{f, f'\}$ still remains a geodesic. This reduces the length of the path P.

Now, if $l(P) \leq l(e)$, we remove the edge e without affecting the metric, and apply the induction hypothesis to get the result.

If $l(P)$ still exceeds $l(e)$, we repeat the above procedure of finding a geodesic cut until P becomes shorter than e. This concludes the proof.

Theorem 22.3.1 (Folklore) *Outer-planar graphs are isometrically embeddable into ℓ_1.*

An example of the above induction is shown in the following figure.

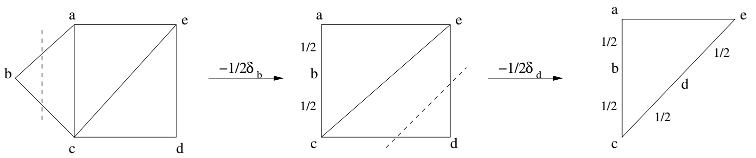

Figure 22.3.2: Embedding outer-planar graphs into ℓ_1

22.4 Embeddability of outer-planar graphs into distributions over dominating trees

In this section we show that any outer-planar graph can be embedded into a distribution over dominating trees with expected distortion at most 8. We first consider the embeddability of cycles into trees. This material is from the paper of Gupta, Newman, Rabinovich and Sinclair (1999).

Lemma 22.4.1 *Every cycle embeds into distributions over trees with expected distortion at most 2.*

Proof: For a cycle C, let $L = \sum_{e \in C} l(e)$. We pick an edge e with probability $\frac{l(e)}{L}$ and remove it. This gives us a path. The expected distance between any two adjascent nodes u and v is $l(e)(1 - \frac{l(e)}{L}) + (L - l(e))\frac{l(e)}{L} = l(e)(2 - 2\frac{l(e)}{L}) \leq 2l(e)$. Thus expected distortion is at most 2. ■

Now in order to extend this to outer-planar graphs, we again use induction on the ear-composition of the graph. The base case for cycles is already implied by the above theorem.

For now we assume that for every ear P attached to edge e, $l(P) \geq 2l(e)$, or, P is a "loose" ear. Given this assumption, we show that outer-planar graphs can be embedded into trees with distortion at most 4. Later we will show how to remove this assumption, while increasing the distortion by an additional factor of at most 2.

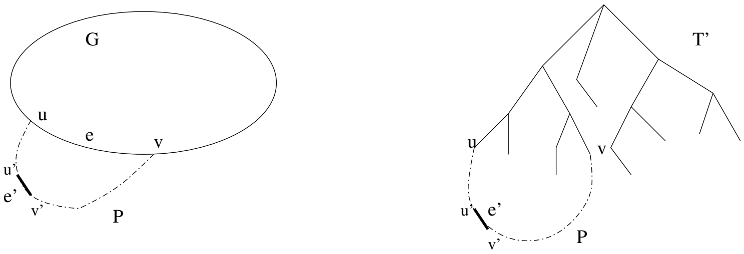

Figure 22.4.3: Embedding outer-planar graphs into distributions over trees

Suppose, for the inductive step, that graph G was obtained by adding an ear P to edge e in a graph G'. By the inductive hypothesis, G' embeds into a distribution, say $\mathcal{D}'$, with distortion at most 4. We construct distribution $\mathcal{D}$ as follows. Pick a tree T' from $\mathcal{D}'$. Pick an edge $e' \in P$ with probability $\frac{l(e')}{L}$, where $L = \sum_{e' \in P} l(e')$. Let u and v correspond to the end points of edge e. Insert the path P without the edge e' between the two end points in the tree. This gives us the tree T. (see Figure 22.4.3)

Lemma 22.4.2 *$d_T \geq d_G$, and $E[d_T] \geq 4d_G$.*

Proof: Let $e' = (u', v')$.

$$\begin{aligned} E[d_t(u', v')] &\leq l(e') \times \mathbf{Pr}\big[e' \text{ is not removed}\big] + (L + \mathbf{E}[d_{T'}(u, v)]) \times \mathbf{Pr}\big[e' \text{ is removed}\big] \\ &= \left(1 - \frac{l(e')}{L}\right) l(e') + \frac{l(e')}{L} \left(L + \mathbf{E}[d_{T'}(u, v)]\right) \end{aligned}$$

Now the expected distance between u and v in $d_{T'}$ is at most $4l(e)$ by the inductive hypothesis. Therefore, the above sum is at most $l(e') + l(e')[1 + 4\frac{l(e)}{L}]$. But $l(e)/L$ is at most 1/2 by our assumption. Therefore, the sum becomes $4l(e')$ and we are done. ■

Next we describe how to get around the above assumption that all ears are loose.

Lemma 22.4.3 *Given an outer-planar graph G, there exists an outer-planar graph G', such that $d_{G'}/2 \leq d_G \leq d_{G'}$, and all ears in G' are loose.*

Proof: Again we use induction on the ear-composition. Suppose we add an ear P to the edge e. If the ear is not loose, that is, $l(P) \leq 2l(e)$, we rescale all edges of P by an appropriate factor thereby reducing the length of P, such that $l(P) = l(e)$. Now we remove the edge e and continue.

Note that we reduce the length of every edge by a factor of at most 2 since an edge belongs to an ear only once. Thus, the length of every path in the graph reduces by a factor of at most 2. By rescaling all edges by a factor of 2, we get the result. ■

The above two lemmas together give us the following theorem:

Theorem 22.4.4 *Every outer-planar graph embeds into a distribution over dominating trees with distortion at most 8.*

References

Gupta, A., Newman, I., Rabinovich, Y., and Sinclair, A. (1999). Cuts, trees and ℓ_1-embeddings of graphs. In *40th Annual Symposium on Foundations of Computer Science (New York, 1999)*, pages 399–408. IEEE Computer Soc., Los Alamitos, CA.

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