15-859(J): Metric Embeddings and Methods **Homework 3**
Date: October 23, 2003 **Due:** Nov 11, 2003

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1. A distance function d is an *ultrametric* if it satisfies the ultrametric inequality:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\} \qquad \forall x, y, z$$

   Does the set of ultrametrics form a convex cone?

2. Let (X, d) be an ultrametric interpreted as a weighted complete graph. Show that every edge of this graph is in some minimum spanning tree.

3. Show that any Ultrametric has a minimum spanning tree that is a Hamiltonian path.

4. The role of a minimum spanning tree of an ultrametric as a representing tree has a counterpart in the theory of flows.

   Given an undirected graph $G = (V, E)$ with nonnegative edge capacities c, for any pair u, v of distinct vertices in V, let $f(u, v)$ denote the value of a maximum flow between u and v. (By the max-flow min-cut theorem, this is also value of a minimum-capacity cut separating u and v).

   (a) Show that
   $$f(x, y) \geq \min\{f(x, z), f(y, z)\} \qquad \forall x, y, z$$

   (b) Show that there is a "representing" tree T for all-pair flow values in G; Namely, T is a capacitated spanning tree on the nodes V such that the maximum flow between any pair of nodes in G is the same as that in T (This tree is also called a "flow-equivalent" tree).

   (c) With a linear space ($O(|V|)$) data structure, can you answer queries about max-flows between any pair of nodes in time $o(|V|)$ (say, $O(\log |V|)$)?

   Hint: Use the result of Question 3.

5. Suppose an additive metric (X, d) has a tree reconstruction that does not use any additional nodes (i.e., the representing tree has only X as its nodes). Show that the representing tree is the *unique* minimum spanning tree of (X, d).