15-859(J): Metric Embeddings and Methods **Homework 2**
Date: Oct 3, 2003 **Due:** Oct 21, 2003

1. A α-*spanner* of a graph G is a spanning subgraph H such that

$$d_G(x, y) \leq d_H(x, y) \leq \alpha\, d_G(x, y)$$

for all $x, y \in V(G)$. Show that every graph has an $(n-1)$-spanner that is a tree.

2. Hölder's inequality says that

$$\|x\|_p \times \|y\|_q \geq \langle x, y \rangle. \qquad \text{(for } 1/p + 1/q = 1\text{)}$$

Use it and the idea behind Theorem 4.3.1 to show that Bourgain's embedding f also gives:

$$(X, d) \xhookrightarrow{O(\log n)} \ell_p^{O(\log^2 n)}.$$

3. Consider the k-dimensional hypercube $\mathbb{B}_k = (V_k, E_k)$ whose vertex set are all the k-bit vectors $\{0,1\}^k$, with an edge between two vertices if they differ in *exactly* one bit position.

   (a) How many vertices and edges does $\mathbb{B}_k$ have?

   (b) Let $S \subseteq V_k$ be the vertices with 0 as the first bit. What is $|S|$ and $|\partial S|$?

   (c) Show that the flux α of $\mathbb{B}_k$ is 2^{1-k}. Recall that

   $$\text{flux } \alpha = \min_{S \subseteq V_k} \frac{|\partial S|}{|S| \times |V_k \setminus S|}.$$

   Hint: Try a simple setting of the edge lengths in the linear programming relaxation (LP 6.1.3) from Lecture 6, and then show a cut with the same value.

   (d) **(Isoperimetric Inequality for the Hypercube)** Show that, for any subset of the vertices $S \subseteq V_k$ with $|S| \leq |V_k|/2$, $|\partial S| \geq |S|$.

   Hint: Use the relationship between the flux α and the *sparsity* Φ.

4. In Lecture 10, we saw how a random partitioning scheme which had the properties that for any metric (X, d) and parameter $\delta \geq 0$:

   - The weak diameter of each component in the partition $\leq \delta$, and
   - for any vertex $u \in X$ and parameter $\rho \leq \delta$, we have

   $$\mathbf{Pr}[B(u, \rho) \text{ is cut in the partition }] \leq \frac{\beta \times \rho}{\delta}.$$

would yield a map

$$(X, d) \xhookrightarrow{O(\beta \log \Delta)} \ell_1^{O(\log n \log \Delta)}.$$

(Specifically, this was proved for the decomposition with $\beta = 8 \log n$ given in Lectures 9-10.) Show that the *same* map also yields

$$(X, d) \xhookrightarrow{O(\beta\, (\log \Delta)^{1/p})} \ell_p^{O(\log n \log \Delta)}.$$

Hint: Try to prove the ℓ_∞ and ℓ_2 cases first. Also, you will not need Hölder's inequality.