15-859(J): Metric Embeddings and Methods **Homework 1**
Date: Sept 12, 2003 **Due:** Sept 25, 2003

1. Given (X, d_1) and (Y, d_2) with $X \cap Y = \{z\}$, the *gate-sum* of the two metrics is the metric $(X \cup Y, d)$, where $d|_{X\times X} = d_1$, $d|_{Y\times Y} = d_2$, and $d(x, y) = d_1(x, z) + d_2(z, y)$ for $x \in X, y \in Y$.
   (a) If both d_1 and d_2 are embeddable into ℓ_1, is d?
   (b) If both d_1 and d_2 are embeddable into ℓ_2, is d?
2. Define a *spanning-tree metric* to be a metric (X, d) whose critical graph is a tree. Show an isometric embedding of any spanning-tree metric into ℓ_1. How many dimensions do you need?
3. Suppose two metrics (X, d_1) and (X, d_2) isometrically embed into ℓ_1.
   (a) For $\alpha \in \mathbb{R}^+$, does $(X, \alpha\, d_1)$ isometrically embed into ℓ_1?
   (b) Does $(X, d_1 + d_2)$?
   (c) Prove that the set of all n-point ℓ_1 metrics, viewed as vectors in $\mathbb{R}^k$ with $k = \binom{n}{2}$, forms a convex cone.
4. Prove or give counterexamples for the following statements:
   (a) Metrics isometrically embeddable into ℓ_2 form a convex cone in $\mathbb{R}^{\binom{n}{2}}$.
   (b) Metrics isometrically embeddable into ℓ_∞ form a convex cone in $\mathbb{R}^{\binom{n}{2}}$.
5. Use Fréchet's embedding $(X, d) \overset{1}{\hookrightarrow} \ell_\infty^n$ to show that
   (a) Any metric embeds into ℓ_1 with distortion at most n.
   (b) Any metric embeds into ℓ_2 with distortion at most $\sqrt{n}$.
6. The following result was shown in class:

$$(X, d) \overset{D}{\hookrightarrow} \ell_\infty^{O(Dn^{2/D} \log n)}.$$

   Adapt the proof (slightly) to show that

$$(X, d) \xhookrightarrow{O(\log^2 n)} \ell_1^{O(\log^2 n)}.$$

   Hint: Use Chernoff bounds.
7. **(Dimension Reduction – Part I)** Prove that any n-point metric (X, d) which is embeddable into ℓ_1 with *any* number of dimensions can also be embedded into ℓ_1 with $\binom{n}{2}$ dimensions.

   Hint: The answer uses the fact that ℓ_1 metrics are cut metrics, Question 3c above, and some linear algebra.