Zero Tests for Exp-Log Constants

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Recognizing Zero

It is not always easy to recognize zero:

Example

$4arctan(1/5) - arctan(1/239) - \pi/4 = 0$?

where

$$\begin{aligned} arctan(x) &= (i/2)log((i+x)/(i-x)) \\ i &= (-1)^{1/2} \\ \pi &= log(1)/2i \end{aligned}$$

Problem Statement

Let $\mathcal{E}$ be a set of expressions standing for real or complex numbers.

Definition

The Zero Problem for $\mathcal{E}$ is:

> Given a partially evaluated expression E in $\mathcal{E}$, decide whether or not $V(E) = 0$

Definition of $\mathcal{E}$

Let $\mathcal{E}$ be the smallest set of expressions containing $\mathbb{Q}$ and so that

1. If A and B are in $\mathcal{E}$ then so are
   - $(A + B)$,
   - $(A - B)$,
   - $(A * B)$ and
   - (A/B)
2. If A is in $\mathcal{E}$ then so are
   - $Exp(A)$ and
   - $Log(A)$
3. If A is in $\mathcal{E}$ and $n \in \mathbb{N}$ then $(A)^{1/n}$ is in $\mathcal{E}$.

$\mathcal{E}$ is called the set of exp-log expressions.

Example

Examples of expressions in $\mathcal{E}$:

- $((1/3)^{1/5} - (3/5)^{1/8})$
- $Exp(Exp(Exp(100)))$

Definition of *partial evaluation of $\mathcal{E}$*

Definition

- Each node n of the expression tree for $\mathcal{E}$ has associated with it a box $b(n) \subset \mathbb{C}$ so that a single valued branch of the appropriate analytic function maps an open set containing the product of the children's boxes into the parent's box $b(n)$
- Each box has the form

$$\{x + iy : \mid x - x_0 \mid \leq 10^{-k}, \mid y - y_0 \mid \leq 10^{-k}\}$$

  where $x_0, y_0 \in \mathbb{Q}$, $k \in \mathbb{N}$
- The boxes for the logarithmic or radical nodes are so small that they can only contain part of one branch of the appropriate function.

We assume also that we are given an approximation algorithm, which will refine a partial evaluation so that all the boxes are as small as desired.

Definition of Order for $\mathcal{E}$

- Suppose E is a partially evaluated expression in $\mathcal{E}$.
- Let $V(E)$ be the value determined by E with its partial evaluation.

Definition

Define the order of E to be the number of distinct exponential or logarithmic subexpressions of E (possibly including E itself).

- Note that identical subexpressions are assumed to have identical partial evaluations.

Basic Zero Test

Given partially evaluated expression E in $\mathcal{E}$.
In parallel until conclusion:

- P1 Approximate $V(E)$ with error bounds. If it is proved that $V(E) \neq 0$ then halt.
- P2(E) If $E_\eta \equiv 0$ then halt, conclude $V(E) = 0$
  Else
  1. Form fundamental sequence $(x_1, y_1), \ldots, (x_k, y_k)$ of exponential points associated with E
  2. Search for integers $a_1, \ldots, a_k$ not all zero so that $a_1 x_1 + \ldots a_k x_k = 0$
  3. If the search succeeds, then use $a_1, \ldots, a_k$ to reduce E to $E*$ so that $V)E) = V(E*)$ but $order(E*) < order(E)$
  4. Continue with **P2**($E*$)

Definition of E_η

Suppose that the distinct exponential and logarithmic subexpressions of E are $H_1, \dots, H_k$ ordered by size.

Definition

E_η is obtained by replacing $H_1, \dots, H_k$ by variables $W_1, \dots, W_k$.

- For each i, let the domain of variable W_i be the approximating box associated with the expression H_i in the partial evaluation of E.
- With this interpretation, E_η is an expression standing for an algebraic function of several variables defined by radicals.
- $E_\eta \equiv 0 \Rightarrow V(E) = 0$

Definition of fundamental sequence

- Definition of the fundamental sequence of exponential points $(x_1, y_1), \ldots, (x_k, y_k)$ associated with partially evaluated E.
- Suppose that the distinct exponential and logarithmic subexpressions of E are $H_1, \ldots, H_k$, ordered by size.

Definition

For $i = 1 \ldots k$

- If $H_i = Exp(G_i)$, then $x_i = V(G_i), y_i = V(H_i)$
- If $H_i = Log(G_i)$, then $x_i = V(H_i), y_i = V(G_i)$

For all i, $y_i = e^{x_i}$.

How to find $a_1, \ldots, a_k$

- How to find $a_1, \ldots, a_k \in \mathbb{Z}$ not all zero, so that $a_1 x_1 + \ldots a_k x_k = 0$
- Suppose $x_1, \ldots, x_k$ are defined by $A_1, \ldots A_k$ and $y_1, \ldots y_k$ are defined by $B_1, \ldots B_k$ in E.
- Use the PSLQ algorithm to search for $a_1, \ldots a_k$ not all zero so that $a_1 x_1 + \cdots + a_k x_k \approx 0$. If such are found, then check either
  - $(a_1 A_1 + \ldots a_k A_k)_\eta \equiv 0$, or
  - $(B_1^{a_1} \ldots B_k^{a_k})_\eta \equiv 1$

  If so, the search succeeds, returning $(a_1, \ldots, a_k)$.
  Otherwise the search continues.

The Schanuel conjecture

The Schanuel conjecture implies that

- if j is minimal such that

$$a_1 x_1 + \cdots + a_j x_j = 0$$

  where $a_1, \ldots a_j$ are integral,
- then either

$$(a_1 A_1 + \ldots a_k A_k)_\eta \equiv 0$$

- or

$$(B_1^{a_1} \ldots B_k^{a_k})_\eta \equiv 1$$

Reduction of E to $E*$

Suppose $a_1x_1 + \dots a_jx_j = 0$, but $a_j \neq 0$.

- Then $y_1^{a_1} \dots y_j^{a_j} = 1$.
- Assume $x_1, \dots x_j$ are defined by $A_1, \dots A_j$ respectively and $y_1, \dots y_j$ are defined by $B_1, \dots B_j$ respectively in E.
- If $B_j = Exp(A_j)$, then replace B_j everywhere in E by an expression equivalent to

$$(B_1^{-a_1} \dots B_{j-1}^{-a_{j-1}})^{1/a_j}$$

  and if $A_j = Log(B_j)$, then replace A_j everywhere in E by an expression equivalent to

$$-(1/a_j)(a_1A_1 + \dots + a_{j-1}A_{j-1})$$

  to get $E*$ with lower order.

A Worked Example

Example

1. $4arctan(1/5) - arctan(1/239) - \pi/4 = 0??$
2. where $arctan(x) = (i/2)log((I + x)/(i - x))$.
3. Fundamental sequence

$$
\begin{aligned}
(x_1, y_1) &= (2i\pi, 1) \\
(x_2, y_2) &= (log((i + 1/5)/(i - 1/5)), (i + 1/5)/(i - 1/5)) \\
(x_3, y_3) &= (log((i + 1/239)/(i_1/239)), (i + 1/239)/(i - 1/239))
\end{aligned}
$$

4. From PSLQ we get $x_1 + 8x_1 - 2x_3 \approx 0$.
5. We verify with corresponding multiplicative identity:
   - $y_1 y_2^8 y_3^{-2} = 1$.
   - so $x_1 + 8x_2 - 2x_3 = 0$.
   - reduce ....
   - $= 0$

Basic Zero Test

Theorem

Zero test

1. *If the basic zero test terminates, it returns the correct result.*
2. *If the Schanuel conjecture is true, the basic zero test always terminates.*

More elaborate Zero Test

A more elaborate Zero Test depending on bounds $\lambda(E), \sigma(E)$
Given partially evaluated E in $\mathcal{E}$, in parallel:

P1 Approximate $V(E)$ with error bounds.

- If $V(E) \neq 0$ proved, then halt with this conclusion
- But if $|\ V(E)\ | \leq 10^{-\lambda(E)}$ then halt and conclude $V(E) = 0$

P2(E) If $E_\eta \equiv 0$ then halt and conclude $V(E) = 0$.
Else form fundamental sequence $(x_1, y_1), \ldots (x_k, y_k)$.

1. Search for $a_1, \ldots a_k$ integers not all zero so that $a_1 x_1 + \ldots a_k x_k = 0$. If found, reduce E to $E*$ and continue with P2($E*$).
2. If no integer non zero vector $(a_1, \ldots a_k)$ is found so that $a_1 x_1 + \cdots + a_k x_k = 0$ with $|\ (a_1, \ldots, a_k)\ | < 10^{\sigma(E)}$ then halt and conclude $V(E) = 0$.

- The function $\lambda(E)$ is called a Liouville bound.
- The function $\sigma(E)$ is called a Schanuel bound.

Details of bounds

- Conjectural Liouville bound: $\lambda(E) = H(E)2^{d(E)}$, where
  - $H(E)$ is the maximum of absolute values of integers on frontier of E
  - $d(E)$ is the depth of nesting of E

  provided all arguments of Exp in E have values in the unit disc.
- A Schanuel bound $\sigma(E)$ can, in principle, be explicitly computed. However this has not yet been done.

Conclusion

Question

Given (partially evaluated) algebraic function $f(W_1, \ldots, W_k)$ built up from field operations and radicals, how should we decide whether or not $f(W_1, \ldots, W_k) \equiv 0$?

See `http://www.bath.ac.uk/~masdr/sch.ps` for discussion of the Zero test in relation to the Schanuel conjecture.