Lecture 21
SURFACES

This chapter focus on two important and practical classes of surfaces: quadric surfaces and NURB surfaces.

§1. Quadric Surfaces

The zero set of a polynomial $P(\mathbf{X}) = P(X_1, \ldots, X_n) \in \mathbb{Q}[X_1, \ldots, X_n]$ is a **hypersurface** in n-dimensional affine space (either real or complex). When $P(\mathbf{X})$ is homogeneous, it is called a **form** and defines a hypersurface in $(n-1)$-dimensional projective space. When $\deg(P) = 2$, the polynomial is **quadratic** and the corresponding hypersurface is **quadric**. We shall reserve the term "surface" for 3-dimensional affine or projective hypersurfaces. When the context is clear, we will say "quadric" for quadric surfaces. Quadric surfaces are thus the simplest nonlinear non-planar geometry we can study.

The intersection of two quadric surfaces is a key problem in computer graphics and geometric modeling. Levin [7] initiated a general approach of reducing such intersection to finding a ruled quadric in the pencil of the input quadrics. Many recent papers have expanded upon this work, e.g., [6, 10, 4]. A general reference is Wenping's survey [5, Chap. 31].

Consider the quadratic polynomial

$$\begin{aligned} P(X, Y, Z) = \\ aX^2 + bY^2 + cZ^2 + 2fXY + 2gYZ + 2hZX + 2pX + 2qY + 2rZ + d. \end{aligned}$$

This polynomial can be written as a matrix product:

$$P(X, Y, Z) = \mathbf{X}^T \cdot A \cdot \mathbf{X} = 0 \tag{1}$$

where $\mathbf{X} = (X, Y, Z, 1)^T$ and A is the symmetric 4×4 matrix,

$$A = \left[\begin{array}{ccc|c} a & f & h & p \\ f & b & g & q \\ h & g & c & r \\ \hline p & q & r & d \end{array}\right]. \tag{2}$$

The principal 3×3 submatrix of A which is indicated in (2) is conventionally denoted A_u, and called the **upper submatrix of** A.

NOTATION: For any $n \times n$ matrix A, and $\mathbf{X} = (X_1, \ldots, X_n)^T$, let[1] $Q_A = Q_A(\mathbf{X})$ denote the **quadratic form**

$$Q_A(\mathbf{X}) = \mathbf{X}^T A \mathbf{X} \tag{3}$$

associated with A. For $n = 4$, we usually write $\mathbf{X} = (X, Y, Z, W)^T$. The notation in (3) is also used in the affine case, in which case $\mathbf{X} = (X_1, \ldots, X_{n-1}, 1)^T$ and $Q_A(\mathbf{X}) = Q_A(X_1, \ldots, X_{n-1})$. The corresponding quadric surface (affine or projective) is denoted $Q_A(\mathbf{X}) = 0$.

Classification of Quadrics in $\mathbb{A}^n(\mathbb{R})$. Classification of quadric surfaces is a highly classical subject (see [2]). There are four versions of this problem, depending on whether we consider the real $\mathbb{R}$ or complex $\mathbb{C}$ field, and whether we consider the affine $\mathbb{A}^n$ or projective $\mathbb{P}^n$ space. We focus on the real affine case. Our approach follows Burington [3] For instance, for $n = 2$, there is the well-known classification of conics. The classification for $n = 3$ is shown in Table 1. The main thrust of this section is to prove the correctness of this table.

Various invariants of the matrix A can be used in making such classifications. Some candidates to consider are:

[1] More generally, if $\mathbf{Y} = (Y_1, \ldots, Y_n)$ then $Q_A(\mathbf{X}, \mathbf{Y}) = \mathbf{X}^T A \mathbf{Y}$ is a bilinear form.

ρ	rank of A
ρ_u	rank of A_u
Δ	determinant of A
Δ_u	determinant of A_u
$\mathtt{sign}(\Delta)$	sign of Δ
$\mathtt{sign}(\Delta_u)$	sign of Δ_u

Δ and Δ_u are called the **discriminant** and **subdiscriminant** of the quadric. But there are other possibilities. For instance Levin [7] uses an oft-cited table from Breyer [1, p. 210–211], based on a different set of invariants: $\rho, \rho_u, \mathtt{sign}(\Delta), k$. Here $k = 1$ if the non-zero eigenvalues of A_u have the same sign, and $k = 0$ otherwise. The exact constitution of such a classification varies, depending on which "degenerate" cases are omitted (published classifications have 17, 18 or 19 classes, and if the imaginary ones are omitted, correspondingly smaller number is obtained). Our classification in Table 1 do not omit degenerate classes and has 20 classes.

The notion of invariance depends on the choice of the class of transformations, and whether we consider projective or affine space, and whether we consider the complex or real field. The most general transformation considered in this chapter are those defined by an arbitrary non-singular $n \times n$ real matrix M. These are called (real) **projective transformations**, and they transform space under the action

$$\mathbf{X} \mapsto M\mathbf{X}.$$

We also say that they **transform polynomials** under the action

$$P(\mathbf{X}) \mapsto P(M\mathbf{X}). \tag{4}$$

Suppose a quadratic polynomial $Q_A(\mathbf{X}) = \mathbf{X}^T A\mathbf{X}$ is transformed to $Q_B(\mathbf{X}) = Q_A(M\mathbf{X})$, for some matrix B. Thus $\mathbf{X}^T B\mathbf{X} = (M\mathbf{X})^T A(M\mathbf{X}) = \mathbf{X}^T(M^T AM)\mathbf{X}$. Hence

$$B = M^T AM. \tag{5}$$

Using the same terminology, we say that M **transforms** A to $M^T AM$. Two matrices A and B related as in (5) via some non-singular matrix M are said to be **congruent**, or $A \mapsto M^T AM$ is a **congruence transformation**. Further, when $A = A^T$, we have

$$(M^T AM)^T = M^T A^T M = M^T AM$$

This proves that symmetric matrices are closed under congruence transformation. It is easy to see that matrices, including symmetric ones, are thereby partitioned into congruence classes. If M_i transforms A_i to A_{i+1} for $i \geq 1$, then $M_1 M_2 \cdots M_i$ transforms A_1 to A_{i+1}. Thus the composition of transformations amounts to matrix multiplication.

The ranks ρ, ρ_u are clearly invariants of A under congruence transformations. Also, $\mathtt{sign}(\Delta)$ is an invariant since $\det(A) = \det(B)\det(M^T M) = \det(B)\det(M)^2$. But $\Delta = \det(A)$ is not invariant unless $|\det(M)| = 1$.

The (real) **affine transformations**, are those projective transformation represented by real matrices M of the form

$$M = \left[\begin{array}{c|c} M_u & \mathbf{t} \\ \hline \mathbf{0} & d \end{array}\right] \tag{6}$$

where $d \neq 0$, M_u is any invertible matrix and $\mathbf{t}$ is a $(n-1)$-column vector. Two matrices A, B are **affinely congruent** if there is an affine transformation M such that $A = M^T BM$. Affine transformations treats the last (homogenization) coordinate as special. The **rigid transformation** are affine transformations with $\det(M_u) = \pm 1$ and $d = \pm 1$. Rigid transformations preserve distances between pairs of points, and are exemplified by rotation, translation or reflection. Thus Δ and Δ_u are invariants under rigid transformations.

	Name	Canonical Equation	σ	σ_u	Notes
	Nonsingular Surfaces				
1	Imaginary Ellipsoid	$X^2+Y^2+Z^2=-1$	(4,0)	(3,0)	(**)
2	Ellipsoid	$X^2+Y^2+Z^2=1$	(3,1)	(3,0)	R_0, C
3	Hyperboloid of 2 sheet	$X^2+Y^2-Z^2=-1$	(3,1)	(2,1)	R_0, C
4	Hyperboloid of 1 sheet	$X^2+Y^2-Z^2=1$	(2,2)	(2,1)	R_2, C
5	Elliptic Paraboloid	$X^2+Y^2-Z=0$	(3,1)	(2,0)	R_0
6	Hyperbolic Paraboloid	$X^2-Y^2-Z=0$	(2,2)	(1,1)	R_2, L
	Singular Nonplanar Surfaces				
7	Imag. Elliptic Cone (Point)	$X^2+Y^2+Z^2=0$	(3,0)	(3,0)	(*)
8	Elliptic Cone	$X^2+Y^2-Z^2=0$	(2,1)	(2,1)	R_1
9	Elliptic Cylinder	$X^2+Y^2=1$	(2,1)	(2,0)	R_1
10	Imag. Elliptic Cylinder	$X^2+Y^2=-1$	(3,0)	(2,0)	(**)
11	Hyperbolic Cylinder	$X^2-Y^2=-1$	(2,1)	(1,1)	R_1, L
12	Parabolic Cylinder	$X^2=Z$	(2,1)	(1,0)	R_1, L
	Singular Planar Surfaces				
13	Intersecting Planes	$X^2-Y^2=0$	(1,1)	(1,1)	
14	Intersect. Imag. Planes (Line)	$X^2+Y^2=0$	(2,0)	(2,0)	(*)
15	Parallel Planes	$X^2-1=0$	(1,1)	(1,0)	
16	Imag. Parallel Planes	$X^2+1=0$	(2,0)	(1,0)	(**)
17	Single Plane	$X=0$	(1,1)	(0,0)	
18	Coincident Planes	$X^2=0$	(1,0)	(1,0)	
19	Invalid	$1=0$	(1,0)	(0,0)	(**)
20	Trivial	$0=0$	(0,0)	(0,0)	(*)

Table 1: Classification of Quadric Surfaces in $\mathbb{A}^3(\mathbb{R})$

We emphasize that Table 1 is a classification for *affine* and *real* space. For instance, the elliptic cylinder and elliptic cone has, as special case, the usual cylinder and cone. In projective space, there would be no difference between a cone and a cylinder (e.g., Rows 8 and 9 would be projectively the same). Similarly, there would be no distinction between an imaginary and a real ellipsoid in complex space (i.e., Rows 1 and 2 would collapse).

Under "Notes" (last column) in Table 1, we indicate by $(**)$ those surfaces with no real components, and by $(*)$ those "surfaces" that degenerate to a point or a line or fills the entire space. It is convenient[2] to call those quadrics in Table 1 that are starred, either (*) or (**), as **improper**; otherwise they are **proper**. We indicate by R_1 the singly ruled quadric surfaces, and by R_2 the doubly ruled quadric surfaces. Also R_0 is for the non-ruled quadric surfaces. The planar surfaces are automatically ruled, so we omit any indication. The letter L are the parameterization surfaces used in Levin's algorithm. The letter C indicate the so-called **central quadrics**.

Non-singular Quadrics. To gain some insight into Table 1, we consider the non-singular quadrics. In general, a surface $P(\mathbf{X}) = P(X_1, \ldots, X_n) = 0$ is **singular** if it contains a singular point $\mathbf{p}$, i.e., one in which $P(\mathbf{p}) = 0$ and $(\partial P/\partial X_i)(\mathbf{p}) = 0$ for $i = 1, \ldots, n$. Otherwise the surface is **non-singular**. Non-singular quadrics are those whose matrix A have rank 4 (Exercise). In Table 1, these are represented by the first 6 rows. These can further be grouped into three pairs: 2 ellipsoids, 2 hyperboloids and 2 paraboloids. In the following discussion, we assume the reader is familiar with the basic properties of conics (ellipses, hyperbolas and parabolas).

1. There is nothing to say for the imaginary ellipsoids. The real ellipsoids is possibly the easiest quadric to understand, perhaps because it is the only bounded surface in our shortlist. Ellipsoids are basically squashed spheres. But in our canonical equation $X^2 + Y^2 + Z^2 = 1$, it is just a regular sphere.

2. The hyperboloids (either 1- or 2-sheeted) have equations $X^2 + Y^2 - Z^2 = \pm 1$. The Z-axis is an axis of symmetry in this canonical form: for any value of $Z = z_0$, the (X, Y)-values lies in the circle[3] $X^2 + Y^2 = z_0^2 \pm 1$. In $\mathbb{A}^3(\mathbb{R})$, the circle $X^2 + Y^2 = Z^2 - 1$ would be purely imaginary for $|Z| < 1$. Hence this quadric surface is clearly divided into two connected components (separated by the plane $Z = 0$). Thus

[2] This is not standard terminology. See, e.g., [5, p.778, Chap. 31].

[3] In general, we would have an ellipse $(aX)^2 + (bY)^2 = (cz_0)^2 \pm 1$, and hence these hyperboloids are also known as *elliptic hyperboloids*.

it is "two sheeted". On the other hand, $X^2 + Y^2 = Z^2 + 1$ has solutions for all values of Z and it is clearly seen to be a connected set, or "one sheeted".

3. Finally, consider the paraboloids: in the elliptic case $X^2 + Y^2 = Z$ is clearly confined to the upper half-space $Z \geq 0$. In the hyperbolic case, we see a hyperbola $X^2 - Y^2 = z_0$ in every plane $Z = z_0$. When $z_0 > 0$, the hyperbola lies in a pair of opposite quadrants defined by the pair of lines $X + Y = 0$ and $X - Y = 0$. When $z_0 < 0$, the hyperbola lies in the other pair of opposite quadrants. At $z_0 = 0$, the hyperbola degenerates into the pair of lines $X + Y = 0, X - Y = 0$. It is easy to see a saddle point at the origin.

The Inertia Invariant. The reader will note that Table 1 did not use the invariants $\rho, \rho_u, \texttt{sign}(\Delta), \texttt{sign}(\Delta_u)$. These information are implicit in σ, σ_u, defined to be the **inertia** of A and A_u. We now explain this concept.

Let A be any real symmetric matrix A of order n. It is not hard to see that rank and sign of determinants are invariants of the congruence relation. But they are only partial invariants in that they do not fully characterize congruence. Instead, we let us look at the eigenvalues of A. The eigenvalues of A are all real. It is easy to see that these eigenvalues (like the determinant) are not preserved by congruence transformations. But the number of positive and negative eigenvalues turns out to be invariant. This is the substance of Sylvester's law of inertia.

Let $\sigma^+ = \sigma^+(A)$ and $\sigma^- = \sigma^-(A)$ denote the numbers of positive and negative eigenvalues of A. The pair

$$\sigma = \sigma(A) = (\sigma^+(A), \sigma^-(A))$$

is called the **inertia** of A. Note that rank of A is given by $\rho = \sigma^+ + \sigma^-$.

Theorem 1 (Sylvester's Law of Inertia). *Let A, A' be real symmetric matrices. Then A and A' are congruent iff $\sigma(A) = \sigma(A')$.*

We shall prove a generalization of this theorem. Note that the diagonal sign matrix

$$\texttt{diag}(\underbrace{1, \ldots, 1}_{\sigma^+}, \underbrace{-1, \ldots, -1}_{\sigma^-}, 0, \ldots, 0)$$

has inertia (σ^+, σ^-). Hence the inertia theorem implies that every real symmetric matrix is congruent to such a sign diagonal matrix. Note that this theorem applies only to symmetric matrices. For instance, the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not congruent to any diagonal matrix (Exercise).

On the other hand:

Corollary 2. *Inertia is a complete set of invariants for congruence: A, B are congruent iff they have the same inertia.*

Proof. If they are congruent, then they have the same inertia. Conversely, suppose A, B have the same inertia (s, t). Then it is easy to see that they are each congruent to $\texttt{diag}(a_1, \ldots, a_s, b_1, \ldots, b_t, 0, \ldots, 0)$ where $a_i = 1$ and $b_j = -1$ for all i and j. Since congruence is an equivalence relation, this shows that A and B are congruent. **Q.E.D.**

Since our surface is unaffected if A is multiplied by any non-zero constant, and as the eigenvalues of $-A$ are just the negative of the eigenvalues of A, we regard the inertia values (s, s') and (s', s) as equivalent. By convention, we choose $\sigma^+ \geq \sigma^-$. For instance, for rank 4 matrices, we have the following possible inertia: $(4, 0), (3, 1), (2, 2)$.

We should note that, in fact, A is never congruent to $-A$ except for the trivial case $A = \mathbf{0}$. This remark is depends on our assumption that only real matrices are used in the definition of congruence. If we allow complex matrices then $A = M^T(-A)M$ with the choice $M = \mathbf{i}I$.

Canonical Equation. In Column 3 of Table 1, we see the canonical equation for each type of surface. We shall show that these are canonical forms for affine transformations. Most, but not all, canonical equations are **diagonal**, i.e., have the form

$$Q_A = aX^2 + bY^2 + cZ^2 + d = 0. \tag{7}$$

It is well-known that quadratic equation can be made diagonal by applying a projective transformations (this is shown in the next section). If we require $|\det(M)| = 1$ for our transformation matrices, then a, b, c, d would be general real numbers. For instance, if $a, b, c, -d$ are all positive, then we often see this in the equivalent form $(X/\alpha)^2 + (Y/\beta)^2 + (Z/\gamma)^2 = r^2$, representing an ellipsoid. But with general projective transformations, we can further ensure that $a, b, c, d \in \{-1, 0, +1\}$. Such are the canonical equations in Table 1.

From the canonical equations in Column 3, we can almost read off the inertia values σ, σ_u. There is an observation that will help our reading. Consider the hyperbolic paraboloid $X^2 - Y^2 - Z = 0$ in Row 6. Its equation is not diagonal (7). The matrix for this quadric is A_0, implicitly defined by the following equation. The matricies M_0 and A_1 are also implicitly defined:

$$M_0^T A_0 M_0 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & e & -e \\ & & e & e \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & & -1 \\ & & -1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & e & e \\ & & -e & e \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = A_1, \qquad e = \frac{1}{\sqrt{2}}. \tag{9}$$

Since A_0 is the matrix of the hyperbolic paraboloid, and A_1 has inertia $(2, 2)$, this proves that inertia σ is $(2, 2)$ as given in Row 6. The same trick enables us to determine σ for all the other cases. Note the inverse of M_0 is its transpose: $M_0^{-1} = M_0^T$ and M_0 is not an affine transformation.

Projective Classification. The classification of surfaces in projective space $\mathbb{P}^3(K)$ (for $K = \mathbb{R}$ or $K = \mathbb{C}$) is relatively simple. In the projective case, we replace the equation in (7) by

$$aX^2 + bY^2 + cZ^2 + dW^2 = 0 \tag{10}$$

where W is the homogenizing variable. In projective space, the W-coordinate is no different than the other three coordinates. If $K = \mathbb{C}$, then we can choose complex transformations so that each a, b, c, d are either 0 or 1. Hence, *the rank ρ of the matrix A is a complete set invariant for complex projective surfaces.* Next consider the case $K = \mathbb{R}$. In this case, we can make $a, b, c, d \in \{0, \pm 1\}$. Consider the classification for non-singular surfaces under general projective transformations. Non-singular means $abcd \neq 0$. Then there are three distinct values for the inertia σ:

- $\sigma = (4, 0)$. This is purely imaginary.
- $\sigma = (3, 1)$. One has opposite sign than the others. Wlog, let $d = -1$ and the others positive. We get the equation of the ellipsoid
$$X^2 + Y^2 + Z^2 = W^2.$$
- $\sigma = (2, 2)$. Say $a = d = +1$ and $b = c = -1$. Then the canonical equation becomes $X^2 - Y^2 = Z^2 - W^2$ or
$$(X + Y)(X - Y) = (Z + W)(Z - W). \tag{11}$$
Note that (11) gives rise to an alternative canonical form for such surfaces: $X'Y' - Z'W' = 0$.

In contrast to the projective classification, the affine classification in Table 1 makes a distinction between the X, Y, Z-coordinates and the W-coordinate. This introduces, in addition to σ, another invariant σ_u. In analogy to Corollary 2, we have:

THEOREM 3.
(i) The pair (σ, σ_u) is a complete set of invariants for real affine transformations.
(ii) This set of four numbers $\sigma^+, \sigma^-, \sigma_u^+, \sigma_u^-$ is minimal complete: if any of the numbers is omitted, then the resulting set is not complete.

Proof. (Partial) (i) One direction is trivial: if A, B are affine congruent, then they clearly have the same inertia pair (σ, σ_u). We will defer the proof of the other direction until we have have introduced the affine diagonalization algorithm.

(ii) It is enough to look at Table 1. If any of the four numbers are omitted, we can find two rows in the table which agree on the remaining three numbers but differ on the omitted number. We can even restrict our rows to the proper quadrics for this purpose. **Q.E.D.**

We are now ready to prove the main classification theorem.

THEOREM 4. *The classification of quadrics under real affine transformations in Table 1 is complete and non-redundant.*

Proof. In view of Theorem 3, this amounts to saying that the list of inertia pairs (σ, σ_u) in Table 1 is complete and non-redundant. Non-redundancy amounts to a visual inspection that the inertia pairs in the table are distinct. As for completeness, we need to show that there are no other possibilities beyond the 20 rows displayed. For instance, the pair $(\sigma, \sigma_u) = ((3,1),(1,0))$ is absent from the table, and we have to account for such absences. Our plan is to examine the possible σ_u associated with each value of σ. Assume that our quadratic polynomial $Q_A(X, Y, Z, W)$ is equivalent to (10) under some projective transformation.

- $\sigma = (4,0)$: clearly, $\sigma_u = (3,0)$ is the only possibility.
- $\sigma = (3,1)$: In this case, we can have $d > 0$ or $d < 0$. This corresponds to $\sigma_u = (2,1)$ or $\sigma_u = (3,0)$. But in any case, we can apply the transformation matrix M_0 in (8) to create the matrix with $\sigma_u = (2,0)$, illustrated as:

$$\begin{aligned} M_0^T A M_0 &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & e & -e \\ & & e & e \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & e & e \\ & & -e & e \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}, \qquad e = \frac{1}{\sqrt{2}}. \end{aligned}$$

  Such application of the matrix M_0 has the effect of annhilating one of the diagonal entries of A_u. All three values of σ_u appear in Table 1 (Rows 2,3,5).
- $\sigma = (2,2)$: Note that if the inertia of A is $(2,2)$ then the inertia of A_u must be $(2,1)$ (Row 4). But we can also apply M_0 to annihilate one diagonal entry of A_u, giving $\sigma_u = (1,1)$ (Row 6).
- $\sigma = (3,0)$: The inertia of A_u is either $(3,0)$ or $(2,0)$ when inertia of A is $(3,0)$. This is shown in Rows 7, 10. In this case, no application of M_0 is possible.
- $\sigma = (2,1)$: Without applying the M_0, we get $\sigma_u = (2,1), (2,0)$ or $(1,1)$ (Rows 8,9,11). Using M_0, we get $\sigma_u = (1,0)$ (Row 12).

- $\sigma = (2,0)$: The only possibilities are $\sigma_u = (2,0), (1,0)$, (Rows 14,16).
- $\sigma = (1,1)$: The possibilities are $\sigma_u = (1,1), (1,0), (0,0)$ (Rows 13,15,17).
- $\sigma = (1,0)$: The possibilities are $\sigma_u = (1,0), (0,0)$ (Rows 18,19).
- $\sigma = (0,0)$: The possibilities are $\sigma_u = (0,0)$ (Row 20).

Q.E.D.

Examples. Let us see some simple techniques to recognize the type (i.e., classification) of a quadric.

- Consider the quadric surface whose equation is

$$Q_A(X, Y, Z) = X^2 + YZ = 0. \tag{12}$$

What is the type of this surface? Similar to the transformation (8), if we set

$$X = X', \quad Y = Y' - Z', \quad Z = Y' + Z' \tag{13}$$

the equation is transformed to

$$Q_{A'}(X', Y', Z') = X'^2 + Y'^2 - Z'^2 = 0. \tag{14}$$

Thus $\sigma = \sigma_u = (2,1)$, and so our surface is an elliptic cone (row 8, Table 1). Actually, we ought to verify that (13) is invertible. See Exercise.

- Consider next the quadric $X^2 + Y^2 + YZ = 0$. To determine its type, use the substitution

$$X = X', \quad Y = Y', \quad Z = Z' - Y'$$

to transformed this quadric to the form $X'^2 + Y'Z' = 0$. The later has already been shown to be an elliptic cone.

- What if there are linear terms? Consider $X^2 + Y + Z = 0$. Linear terms do not affect σ_u (since they live outside the upper submatrix), and so we can read off $\sigma_u = (1,0)$. To determine σ, we view the equation projectively as $X^2 + (Y+Z)W = 0$. This can first be transformed to $X^2 + Y'W = 0$ for some new variable Y'. Then, using the projective transformation as in (8), the equation is further transformed to to $X^2 + Z'^2 - W'^2 = 0$. Thus $\sigma = (2,1)$. Thus the surface is a parabolic cylinder.

- A key trick is to "complete the square" to kill off off-diagonal terms. Consider $X^2 + 2XY - 2XZ + Y^2 - 2Z^2 = 0$. Here, we have a square term (X^2) as well as linear terms ($2XY - 2XZ$) in the variable X. Collecting these terms in X together, we rewrite it as

$$X^2 + 2XY - 2XZ = (X + (Y - Z))^2 - (Y - Z)^2 = X'^2 - (Y - Z)^2.$$

We rename $X + (Y - Z)$ as X'. This is called **completing the square for** X. The original equation becomes

$$X'^2 - (Y - Z)^2 + Y^2 - 2Z^2 = X'^2 - 2YZ - 3Z^2 = 0.$$

Completing the square for Z,

$$3Z^2 + 2YZ = 3[Z^2 + (2/3)YZ] = 3[(Z + Y/3)^2 - Y^2/9] = 3(Z + Y/3)^2 - Y^2/3$$

The original equation now becomes $X'^2 - 3Z'^2 + Y^2/3 = 0$. Since we only used affine transformations, we see that $\sigma = \sigma_u = (2,1)$, and the surface is an elliptic cone.

- What if there constant terms? Consider $X^2+Y^2-2Z=1$. As far σ_u is concerned, the constant term has no effect. In this case, we can just read off $\sigma_u=(2,0)$ immediately. To compute σ, the constant term amounts to the introduction of W^2. The equation becomes $X^2+Y^2-2ZW-W^2=0$. But we can complete the square for W to get $X^2+Y^2-(W+Z)^2+Z^2=0$. Thus $\sigma=(3,1)$. We conclude that the surface is an elliptic paraboloid.

Exercises

Exercise 1.1: Prove that $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ is not congruent to a diagonal matrix. Suppose $A=M^TBM$ where $M=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and B is diagonal. Then, by suitable modification of M, we can assume that $B=\mathtt{diag}(1,\pm 1)$. But in fact, $B=\mathtt{diag}(1,-1)$ leads to a contradiction: for any non-zero vector $v=(x,y)$, we have $f(x,y)=v^TAv=(x+y)^2>0$. But we can choose v such that $Mv=(0,1)$. Then $f(x,y)=(Mv)^TB(Mv)=-1<0$. So we may assume $B=\mathtt{diag}(1,1)=I$. Then $M^TBM=M^TM=$ is a symmetric matrix, and this cannot be equal to A which is non-symmetric. ♢

Exercise 1.2: (a) Verify the transformation (13) is invertible.
(b) Show that the following transformations $(X,Y,Z)\mapsto(X,X,Z)$ and $(X,Y,Z)\mapsto(X+Y+Z,X-Y-Z,Y+Z)$ are not invertible.
(c) Show that $X_i\mapsto cX_i$ and $X_i\mapsto X_i+cX_j$ (for $c\neq 0$ and $i\neq j$) are invertible. In this notation, it is assumed that X_k $(k\neq i)$ is unchanged. Call these transformations **elementary**.
(d) Show that every invertible transformations is a composition of elementary operations. HINT: view transformations as matrices, and give a normal form for such matrices under elementary transformations. ♢

Exercise 1.3: What is the real affine classification of the following surfaces?
(a) $X^2+Y^2-2X=0$
(b) $X^2+Y^2-2X+XZ=0$
(c) $X^2+Y^2-2X+XY=0$
(d) $X^2+Y^2-2X+XY=1$
(e) $X^2+Y^2-2X+XY+YZ=0$
(f) $X^2-Y^2-2X+Y+Z=1$
(g) $X^2-Y^2-2X+XY+Z=0$ ♢

Exercise 1.4: In the previous exercise, construct the transformation matrices that bring each of quadric into its normal form. ♢

Exercise 1.5: Let $Q(\mathbf{X})=\mathbf{X}^TA\mathbf{X}=0$ be a quadric hypersurface where $\mathbf{X}=(X_1,\ldots,X_{n-1},1)$ and $A=(a_{ij})_{i,j=1}^n$ is symmetric. Prove that the hypersurface is singular iff $\det(A)=0$. ♢

Exercise 1.6: Show that a non-singular quadric surface $Q_A(X,Y,Z)=0$ is elliptic, hyperbolic, parabolic iff $\det(A_u)>0,<0,=0$ (respectively). ♢

Exercise 1.7: Let $A=\begin{bmatrix}A_u & t\\ t^T & c\end{bmatrix}$ be the matrix for a central quadric. Show that the eigenvectors of A_u gives the three axes of the quadric. Also, the center of the quadric is $-(A_u)^{-1}t$. ♢

Exercise 1.8: Extend our classification of quadrics in $\mathbb{A}^3(\mathbb{R})$ to $\mathbb{A}^4(\mathbb{R})$. ♢

End Exercises

§2. Projective Diagonalization

The last section gave some ad hoc tricks for recognizing the type of a quadric. To do this systematically, we need an algorithm to diagonalize quadratic polynomials. Note that it is a small step to move from a diagonal form to the canonical quadrics. In some applications, we want something more: not only do we want to know the type, we actually need to know the transformation matrix M that transforms a quadric $Q_A(\mathbf{X})$ into a diagonal form $Q_A(M\mathbf{X})$. For instance, it is easy to parametrize the canonical quadrics, and knowing M, we can convert this into a parametrization of the original quadric. We now provide such an algorithm.

In fact, we shall describe a diagonalization algorithm for a quadratic form in any dimension n,

$$Q_A(\mathbf{X}) = Q_A(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} X_i X_j$$

where $a_{ij} = a_{ji}$ and $\mathbf{X} = (X_1, \ldots, X_n)^T$. Although we may think of X_n as the homogenization variable, but the present algorithm makes no such distinction. There are two simple ideas in this algorithm:

- Completing the Square for a Diagonal Term: if $Q_A(\mathbf{X})$ contains a **diagonal term** $a_{11}X_1^2$ ($a_{11} \neq 0$) then we can transformation $Q_A(\mathbf{X})$ to $Q_A(M\mathbf{X}) = cX_1^2 + Q_B(X_2, \ldots, X_n)$ for some $c \neq 0$ and Q_B.
- Elimination of a Hyperbolic Term: if $Q_A(\mathbf{X})$ contains a **hyperbolic term** $a_{1j}X_1X_j$ ($1 \neq j$, $a_{1j} \neq 0$) then we can transform $Q_A(\mathbf{X})$ to $Q_A(M\mathbf{X}) = cX_1^2 + Q_C(X_1, X_2, \ldots, X_n)$ for some $c \neq 0$ and Q_C. In this case, Q_C may contain terms in X_1. But, if $a_{11} = 0$ in the first place, then Q_C has only terms that are linear in X_1. Therefore, we have introduced a diagonal term in X_1, which can be completed by the previous method.

The input of our algorithm is the real symmetric matrix A. The output is a pair $(M, \mathtt{diag}(a_1, \ldots, a_n))$ of real matrices where M is invertible satisfying

$$A = M^T \mathtt{diag}(a_1, \ldots, a_n) M \tag{15}$$

and all elements in M and the a_i's belong to the rational field generated by elements of A. This field will be denoted $\mathbb{Q}(A)$. We shall use induction on n to do this work. All the computation can be confined to a subroutine which takes an input matrix A, and outputs a pair (M, B) of real matrices such that $A = M^T BM$ where

$$\begin{aligned} Q_B(\mathbf{X}) &= Q_A(M\mathbf{X}) \\ &= a_1 X_1^2 + Q_{B'}(X_2, \ldots, X_n), \end{aligned} \tag{16}$$

for some real a_1. Here, as a general notation, B' is the submatrix of B obtained by deleting the first row and first column. Call this subroutine **Diagonalization Step**. Using such a subroutine, we construct the recursive diagonalization algorithm:

> Projective Diagonalization Algorithm
> Input: A real symmetric matrix A of order n
> Output: Pair of matrices (M, B) where $B = \mathtt{diag}(a_1, \ldots, a_n)$
> and $A = M^T BM$.
> 1. Call Diagonalization Step on input A.
> This returns a pair (M_1, B_1).
> 2. If $n = 1$, we return (M_1, B_1).
> 3. Recursively call Projective Diagonalization with matrix B_1'.
> This returns a pair (M_2, B_2) of $(n-1) \times (n-1)$ matrices.
> Suppose the top-left entry of B_1 is a.
> Let $B_3 = \left[\begin{array}{c|c} a & \\ \hline & B_2 \end{array}\right]$, and $M_3 = \left[\begin{array}{c|c} 1 & \\ \hline & M_2 \end{array}\right]$.
> 4. Return $(M, B) = (M_3 M_1, B_3)$.

The correctness of this algorithm is straightforward. We only verify that step 4 returns the correct value (M_3M_1, B_3):

$$\begin{array}{rcl} B_1 & = & \left[\begin{array}{c|c} a & \\ \hline & B_1' \end{array}\right] \\ & = & \left[\begin{array}{c|c} 1 & \\ \hline & M_2^T \end{array}\right] \left[\begin{array}{c|c} a & \\ \hline & B_2 \end{array}\right] \left[\begin{array}{c|c} 1 & \\ \hline & M_2 \end{array}\right]. \\ & = & M_3^T B_3 M_3 \\ A & = & M_1^T B_1 M_1 \\ & = & (M_3M_1)^T B_3 (M_3 M_1) \end{array}$$

This last equation is just the specification for diagonalizing A.

Diagonalization Step. This is a non-recursive algorithm. It maintains a pair of two matrices (M, B) which satisfies the invariant $A = M^T BM$. We initialize $M = \mathbf{1}$ (identity matrix) and $B = A$. Thus, (M, B) satisfies the invariant $A = M^T BM$ initially. Our algorithm consists of applying at most three transformations of the pair (M, B) by some invertible N as[4] follows:

$$(M, B) \mapsto (NM, N^{-T} BN^{-1}) \tag{17}$$

It is easy to verify that our invariant is preserved by such updates. The transformation (17) amounts to transforming the quadric $Q_B(\mathbf{X})$ to $Q_C(\mathbf{X})$ where $Q_C(N\mathbf{X}) = Q_B(\mathbf{X})$ or $N^T CN = B$. Thus $Q_A(\mathbf{X}) = Q_B(M\mathbf{X}) = Q_C(NM\mathbf{X})$. Let $B = (b_{ij})_{i,j=1}^n$.

The algorithm consists of the following sequence of steps, which we have organized so that it can be directly implemented in the order described.

STEP 0 If $n = 1$ or if $b_{12} = b_{13} = \cdots = b_{1n} = 0$, we can return $(M, B) = (\mathbf{1}, A)$ immediately. So assume otherwise. If $b_{11} = 0$, go to STEP 1; else, go to STEP 3.

STEP 1 We now know $b_{11} = 0$. First the first $k > 1$ such that $b_{kk} \neq 0$. If such a k is not found, we go to STEP 2. Otherwise, we transpose the variables X_1 and X_k. This amounts to applying the matrix $P_k = (p_{ij})_{i,j=1}^n$ which is the identity matrix except that $p_{1i} = p_{i1} = 1$ and also $p_{11} = p_{ii} = 0$:

$$P_k = \begin{bmatrix} 0 & & \cdots & 1 & & & \\ & 1 & & & & & \\ \vdots & & \ddots & \vdots & & & \\ & & 1 & & & & \\ 1 & & \cdots & 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}. \tag{18}$$

Note that P_k is its own transpose and its own inverse. Thus P_k transforms $Q_B(\mathbf{X})$ to $Q_C(\mathbf{X}) = Q_B(P_k\mathbf{X})$ whose associated matrix

$$C = P_k \cdot B \cdot P_k$$

simply interchanges the 1st and kth rows, and also the 1st and kth columns. We update the pair (M, B) by the matrix $N = P_k$ as in (17). Go to STEP 3.

[4] Note that we write N^{-T} as shorthand for $(N^{-1})^T = (N^T)^{-1}$.

STEP 2 We now know that $b_{1k} \neq 0$ for some $k \geq 2$, and moreover $b_{11} = b_{kk} = 0$. Our goal is to create a non-zero diagonal term from a hyperbolic term $b_{1k}X_1X_k$. Putting

$$Y_1 = (X_1 + X_k)/2, \qquad Y_k = (X_1 - X_k)/2, \tag{19}$$

we have $2X_1X_k = Y_1^2 - Y_k^2$, $X_1 = (Y_1 + Y_k)$ and $X_2 = (Y_1 - Y_k)$. We use the convention that "$(X_1 \ldots \widehat{X_k} \ldots X_n)$" indicates the absence of X_k in the sequence $(X_1, \ldots, X_n)$. Then

$$\begin{aligned} Q_B(\mathbf{X}) &= 2b_{1k}X_1X_k + 2X_1\left(\sum_{j=2;j\neq k}^n b_{1j}X_j\right) + 2X_k\left(\sum_{j=2;j\neq k}^n b_{kj}X_j\right) + Q_{B''}(X_2, \ldots, \widehat{X_k}, \ldots, X_n) \\ &= b_{1k}(Y_1^2 - Y_k^2) + 2(Y_1 + Y_k)\left(\sum_{j=2;j\neq k}^n b_{1j}X_j\right) + 2(Y_1 - Y_k)\left(\sum_{j=2;j\neq k}^n b_{kj}X_j\right) + Q_{B''} \\ &= Q_C(Y_1, X_2, \ldots, X_{k-1}, Y_k, X_{k+1}, \ldots, X_n) \qquad \text{(implicit definition of } Q_C) \end{aligned}$$

The coefficients of Y_1^2 and Y_k^2 are non-zero in $Q_C(Y_1, X_2, \ldots, X_{k-1}, Y_k, X_{k+1}, \ldots, X_n)$. The transformation (19) is given by the matrix $R_k = (r_{ij})_{i,j=1}^n$ which is similar to the identity matrix except that $r_{11} = r_{1k} = r_{k1} = e$ and $r_{kk} = -e$ where $e = 1/2$:

$$R_k = \begin{bmatrix} e & & \cdots & & e & & & \\ & 1 & & & & & & \\ \vdots & & \ddots & & \vdots & & & \\ & & & 1 & & & & \\ e & & \cdots & & -e & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}. \tag{20}$$

The inverse R_k^{-1} is obtained from R_k by replacing each occurence of e by 1 (and $-e$ becomes -1). Also, the R_k and R_k^{-1} are their own transposes. Observe that $R_k\mathbf{X} = (\frac{X_1+X_2}{2}, X_2, \ldots, X_{k-1}, \frac{X_1-X_2}{2}, X_{k+1}, \ldots, X_n)^T$ and $R_k^{-1}\mathbf{X} = ((X_1 + X_2), X_2, \ldots, X_{k-1}, (X_1 - X_2), X_{k+1}, \ldots, X_n)^T$. Hence $Q_B(\mathbf{X}) = Q_C(R_k\mathbf{X})$, or $Q_B(R_k^{-1}\mathbf{X}) = Q_C(\mathbf{X})$. We may therefore update (M, B) with the matrix $N = R_1$ as in (17). Go to STEP 3.

STEP 3 At this point $b_{11} \neq 0$. All the previous steps are preparations to arrive at this point. The basic idea is to "complete the square" for X_1. We have

$$\begin{aligned} Q_B(\mathbf{X}) &= b_{11}X_1^2 + 2X_1\left(\sum_{j=2}^n b_{1j}X_j\right) + Q_{B'}(X_2, \ldots, X_n) \\ &= \frac{1}{b_{11}}\left(b_{11}X_1 + \left(\sum_{j=2}^n b_{1j}X_j\right)\right)^2 - \frac{1}{b_{11}}\left(\sum_{j=2}^n b_{1j}X_j\right)^2 + Q_{B'}(X_2, \ldots, X_n) \\ &= \frac{1}{b_{11}}Y_1^2 + Q_{C'}(X_2, \ldots, X_n) \qquad (C' \text{ implicitly defined here with } Y_1 = \textstyle\sum_{j=2}^n b_{1j}X_j\ ) \\ &= Q_C(Y_1, X_2, \ldots, X_n) \qquad (Q_C \text{ implicitly defined here}). \end{aligned}$$

Thus, with the matrix

$$T = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{21}$$

then $Q_C(N\mathbf{X}) = Q_B(\mathbf{X})$. Hence $Q_C(\mathbf{X}) = Q_B(N^{-1}\mathbf{X}) = X_1^2 + Q_C(X_2, \ldots, X_n)$. The inverse N^{-1} is easy to describe:

$$N^{-1} \quad = \quad \begin{bmatrix} 1/b_{11} & -b_{12}/b_{11} & \cdots & -b_{1n}/b_{11} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

So we update (M, B) using the matrix N as in (17). This concludes the description of Diagonalization Step.

Sign Diagonalization. In general, let us call a matrix a **sign matrix** if every entry of the matrix is a sign (i.e., $0, \pm 1$). In particular, the matrix $\mathtt{diag}(a_1, \ldots, a_n)$ is a **sign diagonal matrix** if each $a_i \in \{0, \pm 1\}$. Let (M, B) be the output of our diagonalization algorithm where $B = \mathtt{diag}(a_1, \ldots, a_n)$. Thus, $A = M^T BM$ where the entries of B and M belongs to $\mathbb{Q}(A)$ (rational in entries of A).

We can modify this output so that B is a sign diagonal matrix as follows: let $c_i = \sqrt{|a_i|}$ for $i = 1, \ldots, n$. Define $S = \mathtt{diag}(1/c_1, \ldots, 1/c_n)$ and $S^{-1} = \mathtt{diag}(c_1, \ldots, c_n)$. Then the output $(S^{-1}M, SBS)$ would have the desired property. Let $\sqrt{\mathbb{Q}(A)}$ denote those elements that are square roots of elements in $\mathbb{Q}(A)$. Thus the elements of S belong to $\sqrt{\mathbb{Q}(A)}$. We do not build this "sign diagonalization" into the main algorithm because of its need for square roots. Further more, if we allow $\sqrt{-1}$, then we can make all the signs to be positive or 0. Thus, we have given a constructive proof of the following theorem.

Theorem 5. *Let A be a symmetric real matrix.*
(i) There exists an invertible matrix M whose elements belong to $\mathbb{Q}(A)$ such that $M^T AM$ is diagonal.
(ii) There exists an invertible matrix M whose elements belong to $\sqrt{\mathbb{Q}(A)}$ such that $M^T AM$ is sign diagonal.
(iii) There exists an invertible matrix M whose elements belong to $\mathbb{C}$ such that $M^T AM$ is non-negative sign diagonal. In particular if A is non-singular, $M^T AM = \mathbf{1}$.

§3. Affine Diagonalization

The canonical form for symmetric matrices under projective transformations is the diagonal matrix. What is the corresponding canonical form under affine transformations? We observe that diagonal matrices may not be achievable: e.g., the canonical hyperbolic paraboloid $X^2 - Y^2 - Z = 0$ cannot be diagonalized using affine transformations. Part of our task is to determine what the "affine canonical form" take.

We shall describe the diagonalization process for a quadratic polynomial in $n-1$ dimensions,

$$Q_A(X_1, \ldots, X_{n-1}, 1) = \mathbf{X}^T A\mathbf{X}$$

where A is $n \times n$ and $\mathbf{X} = (X_1, \ldots, X_{n-1}, 1)^T$, $n \geq 2$. As usual, the (i, j)th entry of A is denoted a_{ij}. The two techniques of projective diagonalization still works, but a third technique is needed.

- Completing the square can be extended to the affine case. Let

$$Q_A(\mathbf{X}) = a_{11}X_1^2 + 2X_1 \left(\sum_{i=2}^{n-1} a_{1i}X_i \right) + 2X_1 a_{1n} + Q_B(X_2, \ldots, X_{n-1})$$

Completing the square for X_1, we get

$$Q_A(\mathbf{X}) = \frac{1}{a_{11}} \left(a_{11}X_1 + \left(\sum_{i=2}^{n-1} a_{1i}X_i \right) + a_{1n} \right)^2 + Q_C(X_2, \ldots, X_{n-1})$$

where

$$Q_C(X_2, \ldots, X_{n-1}) = -\frac{1}{a_{11}} \left(a_{1n} + \sum_{i=2}^{n-1} a_{1i}X_i \right)^2 + Q_C(X_2, \ldots, X_{n-1})$$

- Hyperbolic terms can again be eliminated by reduction to square terms: assuming $a_{11} = a_{22} = 0$ and $a_{12} \neq 0$, we have

$$Q_A(\mathbf{X}) = 2a_{12}X_1X_2 + 2X_1\left(\sum_{i=3}^{n-1} a_{1i}X_i\right) + 2X_2\left(\sum_{i=3}^{n-1} a_{2i}X_i\right) + Q_B(X_3, \ldots, X_{n-1}).$$

  Then putting $X_1 = (X_1' + X_2')$ and $X_2 = (X_1' - X_2')$, we get

$$\begin{aligned} Q_A(\mathbf{X}) &= 2a_{12}(X_1'^2 - X_2'^2) \\ &\quad +2(X_1' + X_2')\left(\sum_{i=3}^{n-1} a_{1i}X_i\right) + 2(X_1' - X_2')\left(\sum_{i=3}^{n-1} a_{2i}X_i\right) + Q_B(X_3, \ldots, X_{n-1}). \end{aligned}$$

  Thus we can now complete the square for either X_1' or X_2'.

- Consolidation of Linear and Constant Terms: if $a_{11}, a_{12}, \ldots, a_{1,n-1} = 0$ and $a_{1n} \neq 0$, then we have no technique for eliminating the linear term $a_{1n}X_1$. We shall live with such terms. The key observation is that *we do not more than one such term*. Suppose that we can no longer apply any of the above transformations. Then $Q_A(\mathbf{X})$ is equal to a linear term in some variables (say) $X_1, \ldots, X_k$, plus the sum of squares in the remaining variables, $X_{k+1}, \ldots, X_{n-1}$:

$$Q_A(\mathbf{X}) = a_{nn} + 2\left(\sum_{i=1}^{k} a_{in}X_i\right) + \sum_{i=k+1}^{n} a_{ii}X_i^2$$

  where a_{nn} is arbitrary but $a_{in} \neq 0$ for each $i = 1, \ldots, k$. There are two possibilities: if $k = 0$ then there are no linear terms. This gives the **affine diagonal form**,

$$Q_A(\mathbf{X}) = a_{nn} + \sum_{i=1}^{n} a_{ii}X_i^2$$

  Otherwise, suppose $k \geq 1$. Then by the substitution

$$X_1 = \frac{1}{2a_{1n}}\left(X_1' - a_{nn} - 2\sum_{i=2}^{k} a_{in}X_i\right)$$

  we obtain the **almost diagonal form**,

$$Q_A(\mathbf{X}) = X_1' + \sum_{i=k+1}^{n} a_{ii}X_i^2$$

Note that the above transformations are affine. We have thus proved:

Theorem 6. *Every quadratic polynomial $Q_A(X_1, \ldots, X_{n-1})$ can be transformed by a rational affine transformation to an affine diagonal form*

$$Q_B(X_1, \ldots, X_{n-1}) = a_n + \sum_{i=1}^{n-1} a_iX_i^2 \tag{22}$$

or an almost diagonal form

$$Q_C(X_1, \ldots, X_{n-1}) = X_{n-1} + \sum_{i=1}^{n-2} a_iX_i^2. \tag{23}$$

Using transformation matrices whose elements come from $\sqrt{\mathbb{Q}(A)}$, we can make the a_i's to be signs.

The matrices B and C corresponding to the quadratic polynomials (22) and (23) are

$$B = \left[\begin{array}{ccc|c} a_1 & & & \\ & \ddots & & \\ & & a_{n-1} & \\ \hline & & & a_n \end{array}\right] \tag{24}$$

and

$$C = \left[\begin{array}{ccc|cc} a_1 & & & & \\ & \ddots & & & \\ & & a_{n-2} & & \\ \hline & & & & \frac{1}{2} \\ & & & \frac{1}{2} & \end{array}\right], \tag{25}$$

respectively. Also, a quadratic polynomial is said to be in **canonical affine form** if it is in either of these forms. If we are willing to use square-roots, then we can make each a_i into a sign $(0, \pm 1)$; such are the canonical equations shown in Table 1.

Lemma 7. *Let A and A_u have rank r and r_u, respectively. The following are equivalent:*
(i) A is affine congruent to an almost diagonal matrix.
(ii) $r - r_u = 2$.
Similarly, the following are equivalent:
(i)′ A is affine congruent to a diagonal matrix.
(ii)′ $r - r_u \le 1$.

Proof. By Theorem 6, A is affine congruent to B where B is either diagonal or almost diagonal.
(a) (i) implies (ii): Suppose B is almost diagonal. Then $r_u = \mathtt{rank}(A_u) = \mathtt{rank}(B_u) = \mathtt{rank}(B_{uu})$ where B_{uu} is the upper submatrix of B_u. Further $r = \mathtt{rank}(A) = \mathtt{rank}(B) = \mathtt{rank}(B_{uu}) + 2$. This shows $r - r_u = 2$. This proves (i) implies (ii).
(b) (i)′ implies (ii)′: If B is diagonal, then $r = \mathtt{rank}(B) \le \mathtt{rank}(B_u) + 1 = r_u + 1$. Thus $r - r_u \le 1$.
To show the converse of (a), suppose suppose (ii) holds. So (ii)′ does not hold. Then (b) says that (i)′ cannot hold. Hence (i) must hold. A symmetrical argument shows the converse of (b). **Q.E.D.**

The conditions (ii) $r - r_u \le 1$ and (ii)′ $r - r_u = 2$ characterize the diagonal and almost diagonal matrices. Since these two conditions are mutually exclusive, we have shown:

Corollary 8. *A diagonal matrix and an almost diagonal matrix are not affine congruent.*

We can now complete a deferred proof. Our proof of Theorem 3 was incomplete because we did not prove the following result:

Lemma 9. *If A, B have the same inertia pair (σ, σ_u), then A and B are affine congruent.*

Proof. We may assume that A, B are already in affine canonical form. By Lemma 7, either A and B are both diagonal, or they are both almost diagonal. By a further transformatiom, we can assume A and B are sign matrices. Let $\sigma = (\sigma^+, \sigma^-)$ and $\sigma_u = (\sigma_u^+, \sigma_u^-)$. Recall that we distinguish matrices up to multiplication by a non-zero constant.
CLAIM: There is a choice of $B \in \{A, -A\}$ such that (a) B has exactly σ^+ positive entries, and (b) B_u has exactly σ_u^+ positive entries. *Proof*: If A is diagonal, and $\sigma^+ > \sigma^-$, then we choose B such that it has exactly σ^+ positive diagonal entries. If $\sigma^+ = \sigma^-$, then we pick B such that the last diagonal entry in B is negative or zero. If B is almost diagonal, then we pick B such that that B has σ^+ positive diagonal entries. This proves our claim.
First assume A, B are diagonal. Since B_u and A_u have the same number of positive and same number of negative entries, and they are sign matrices, we can find an affine permutation transformation M such that

$(B)_u = (M^T AM)_u$. But since A and B have the same rank, we conclude that, in fact, $B = M^T AM$. This proves that A and B are affine congruent.

If A, B are almost diagonal, the proof is similar. **Q.E.D.**

Affine Diagonalization Algorithm. We consider the affine analogue of the projective diagonalization algorithm. The state of our algorithm is captured, as before, by the pair of matrices (M, B) that will eventually represent our output. Again (M, B) is initially $(\mathbf{1}, A)$ and we successively transform (M, B) by "updating with an invertible matrix N", as in (17).

In addition, we keep track of a pair of integers (s, t) initialized to $(1, n)$ and satisfying $1 \le s \le t \le n$. The shape of the matrix B has the following structure:

$$B = \left[\begin{array}{c|c|c||c} D & 0 & 0 & 0 \\ \hline 0 & C & 0 & c \\ \hline 0 & 0 & 0 & d \\ \hline\hline 0 & c^T & d^T & e \end{array}\right], \qquad \text{where} \begin{cases} D = \mathtt{diag}(a_1, \ldots, a_{s-1}), a_i\text{'s arbitrary} \\ C = (t-s) \times (t-s), \\ c = (t-s) \times 1, \\ d = (d_t, \ldots, d_{n-1})^T, \text{each } d_j \ne 0 \\ e \in \mathbb{R}. \end{cases} \tag{26}$$

Intuitively, the first $s - 1$ variables $(X_1, \ldots, X_{s-1})$ have already been put in diagonal form; last $n - t$ variables $(X_t, \ldots, X_{n-1})$ are known to be linear. The variable X_s is the current focus of our transformation.

Note that when $s = t$, the matrix C and vector c are empty. In that case

$$Q_B(\mathbf{X}) = \sum_{i=1}^{t-1} a_i X_i^2 + 2 \sum_{j=t}^{n-1} d_j X_j + e$$

where $d = (d_t, d_{t+1}, \ldots, d_{n-1})^T$. With at most one more transformation, we would be done. More precisely:
(a) If $t = n$, B is diagonal.
(b) If $t \le n - 1$ then we apply the transformation $X_{n-1} \mapsto \frac{1}{2d_{n-1}} \left(2X'_{n-1} - e - 2\sum_{j=t}^{n-2} d_j X_j\right)$ or $2X'_{n-1} \mapsto e + 2\sum_{j=t}^{n-1} d_j X_j$. This transforms

$$Q_B(\mathbf{X}) = \sum_{i=1}^{t-1} a_i X_i^2 + 2 \sum_{j=t}^{n-1} d_j X_j + e \quad \mapsto \quad \sum_{i=1}^{t-1} a_i X_i^2 + 2X'_{n-1} = Q_C(X_1, \ldots, X_{n-2}, X'_{n-1}).$$

Denoting the transformation matrix by

$$T = \left[\begin{array}{c|c|c|c} \mathbf{1}_{t-1} & & & \\ \hline & \mathbf{1}_{n-t-1} & & \\ \hline & d_t \quad d_{t+1} \quad \cdots \quad d_{n-2} & d_{n-1} & e/2 \\ \hline & & & 1 \end{array}\right] \tag{27}$$

where $\mathbf{1}_m$ is the $m \times m$ identity matrix. Thus $Q_B(\mathbf{X}) = Q_C(T\mathbf{X})$ where $Q_C(X_1, \ldots, X_{n-1})$ is almost diagonal.

Affine Diagonalization Algorithm

Input: A real symmetric matrix A of order n

Output: Pair of matrices (M, B) where $B = \texttt{diag}(a_1, \ldots, a_n)$ and $A = M^T BM$.

INITIALIZATION:

$(M, B) \leftarrow (\mathbf{1}, A)$ and $(s, t) \leftarrow (1, n)$.

– so B initially has the form of (26) –

MAIN LOOP:

while $(s < t)$ do

1. If $b_{ss} \neq 0$, complete the square for X_s.
   Update (M, B) (as in Equation (21)).
   $s \leftarrow s + 1$. Break.
2. Else if $(\exists k = s + 1, \ldots, t - 1)$ with $b_{kk} \neq 0$ then
   Apply the matrix $P_{s,k}$ (as in (18)) to exchange the X_s and X_k. Break.
3. Else if $(\exists k = s + 1, \ldots, t - 1)$ with $b_{sk} \neq 0$ then
   Apply the matrix $R_{s,k}$ (as in (20)) to remove the cross term $X_s X_k$. Break.
4. Else (we know $b_{si} = 0$ for $i = s, \ldots, n - 1$)
   Set $t := t - 1$, and if $s < t$ then apply the matrix $P_{s,t}$ to exchange X_s and X_t. Break.

CONSOLIDATING LINEAR AND CONSTANT TERMS:

If $(t = n$ or $(t = n - 1$ and $e = 0))$ then return (M, B).

Else apply the matrix (27).

Example: Find the affine canonical form for the following quadratic polynomial:

$$Q(W, X, Y, Z) = WX - 2W + 2Y - 2Z + 1. \tag{28}$$

For simplicity, we omit the tracking of the matrix M. Eliminating the hyperbolic term WX gives

$$(W'^2 - X'^2) - 2(W' + X') + 2Y - 2Z + 1.$$

Completing the square for W' gives

$$W''^2 - X'^2 - 2X' + 2Y - 2Z.$$

Completing the square for X' gives

$$W''^2 - X''^2 + 2Y - 2Z + 1.$$

Simplifying linear and constant terms,

$$W''^2 - X''^2 + 2Y' = 0.$$

REMARKS:

1. Note that we classify only *real* matrices A since we use inertia of A in our classification. But we could generalize A to be Hermitian, i.e., complex matrices satisfying $A = A^*$ (its conjugate transpose). In this case, the eigenvalues of A are still real, and we again have the law of inertia.
2. The affine diagonalization algorithm contains the projective diagonalization algorithm as a subalgorithm: all one has to do is to ignore the part that has to do with the last homogenization component.

Exercises

Exercise 3.1: Give the matrix M that transforms $Q_A(W, X, Y, Z)$ in (28) to $Q_B(W, X, Y, Z) = W^2 - X^2 + 2Y$. ♢

Exercise 3.2: Find the affine canonical form of $XY + YZ + 2X + 2Y + 2Z = 1$. Also determine the matrix which achieves this transformation. ♢

Exercise 3.3: Refine our algorithm so that we keep track of three integers, $1 \le r \le s \le t \le n$. With s, t as before, we want $r-1$ to be the number of identically zero rows in B. Moreover, we assume that these corresponds to the variables $X_1, \ldots, X_{r-1}$. Thus, the diagonal entries $a_{rr}, \ldots, a_{s-1,s-1}$ are non-zero. ♢

Exercise 3.4: Extend the affine diagonalization process to Hermitian matrices. ♢

Exercise 3.5: Suppose our transformations M are restricted to be upper block diagonal of the form

$$\left[\begin{array}{c|c|c|c} M_{11} & M_{12} & \cdots & M_{1k} \\ \hline & M_{22} & \cdots & M_{2k} \\ \hline & & \ddots & \\ \hline & & & M_{kk} \end{array}\right]$$

where each M_{ii} is an invertible $n_i \times n_i$ matrix. Give a complete set of invariants under such congruence. ♢

End Exercises

§4. Parametrized and Ruled Quadric Surfaces

A surface S with locus $L(S) \subseteq \mathbb{R}^3$ is **rational** if there exists a function

$$\mathbf{q} : \mathbb{R}^2 \to L(S) \cup \{\infty\}, \qquad \mathbf{q}(U,V) = (X(U,V), Y(U,V), Z(U,V))^T \tag{29}$$

where $X(U,V), Y(U,V), Z(U,V) \in \mathbb{R}(U,V)$ are rational functions, and with finitely many exceptions, each point in $L(S)$ is given by $\mathbf{q}(U,V)$ for some U, V. We call $\mathbf{q}$ a **rational parametrization** of S. We are concerned here with a special kind of rational surface, one that is swept by a line moving in space. More precisely, a **ruled surface** is one with a rational parametrization of the form

$$\mathbf{q}(U,V) = \mathbf{b}(U) + V\mathbf{d}(U). \tag{30}$$

Here, $\mathbf{b} \in \mathbb{R}(U,V)^3$ is called the **base curve** and $\mathbf{d} \in \mathbb{R}(U,V)^3$ is called the **director curve**. Observe that for each $u \in \mathbb{R}$, we have a line $\{\mathbf{b}(u) + v\mathbf{d}(u) : v \in \mathbb{R}\}$, called a **ruling**. In general, a ruling for a surface is a family of lines whose union is the locus of the surface, and with finitely point exceptions, each point in the surface lies on a unique line. The Gaussian curvature[5] is everywhere non-positive on such a surface. Two special cases may be mentioned:
(1) When the base curve degenerates to a single point $\mathbf{b}_0 \in \mathbb{R}^3$: $\mathbf{b}(u) = \mathbf{b}_0$ for all $u \in \mathbb{R}$. The ruled surface is a cone pointed at $\mathbf{b}_0$.
(2) When the director curve degenerates to a single point $\mathbf{d}_0 \in \mathbb{R}^3$: $\mathbf{d}(u) = \mathbf{d}_0 \neq \mathbf{0}$ for all $u \in \mathbb{R}$. The ruled surface is a cylinder whose axis is parallel to $\mathbf{d}_0$.

A surface is **singly-ruled** (resp., **doubly-ruled**) if there is a unique set (resp., exactly two sets) of rulings. For instance, the cylinder is singly-ruled. But a plane has infinitely many sets of rulings. The 1-sheeted hyperboloid $X^2 + Y^2 - Z^2 = 1$ has 2 ruled parametrizations,

$$\mathbf{q}(\theta, V) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} + V \begin{bmatrix} \pm\sin\theta \\ \mp\cos\theta \\ 1 \end{bmatrix}. \tag{31}$$

[5] The Gaussian curvature at a point p of a differentiable surface is given by $\kappa_1\kappa_2$ where κ_1, κ_2 are the principal curvatures at p.

It has no other sets of rulings (Exercise), so it is doubly-ruled. To see that equation (31) is a rational parametrization, we replace $\cos\theta$ and $\sin\theta$ by the rational functions,

$$\cos\theta = \frac{1-U^2}{1+U^2}, \qquad \sin\theta = \frac{2U}{1+U^2}, \qquad (U \in \mathbb{R}). \tag{32}$$

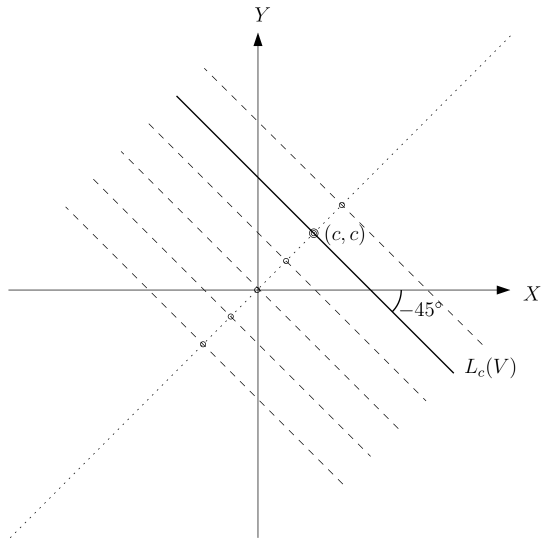

Figure 1: Rulings of the Hyperbolic Paraboloid projected to the (X, Y)-plane.

The hyperbolic paraboloid $X^2 - Y^2 - Z = 0$ also has 2 ruled parametrizations

$$\mathbf{q}(U,V) = \begin{bmatrix} U \\ \pm U \\ 0 \end{bmatrix} + V \begin{bmatrix} 1 \\ \mp 1 \\ 4U \end{bmatrix}. \tag{33}$$

This case is important in Levin's algorithm, so let us understand the parametrization. Assume $\mathbf{q}(U,V) = (U+V, U-V, 4UV)$ (the other parametrization $\mathbf{q}(U,V) = (U+V, -U+V, 4UV)$ is similarly treated). Observe that for each $c \in \mathbb{R}$, the projection of the ruling $\{\mathbf{q}(c,V) : V \in \mathbb{R}\}$ into the (X,Y)-plane is a line $L_c(V)$ passing through the point (c,c) with a constant slope $-45°$. Since the lines $L_c(V)$ are parallel and pairwise disjoint, we conclude that the rulings are pairwise disjoint.

Table 2 gives a list of 10 quadric surfaces. These are all ruled quadrics, as demonstrated by the ruled parameterization given in the last column. The proper quadrics which are not accounted for by Table 2 are the ellipsoid, 2-sheeted hyperboloid and the elliptic paraboloid.

Let us prove that these 3 quadrics are non-ruled. The ellipsoid is clearly non-ruled, since it is bounded and cannot contain any line. The 2-sheeted hyperboloid $X^2 + Y^2 - Z^2 = -1$ has parametrization

$$[\sinh U \cos V, \sinh U \sin V, 1 \pm \cosh U]$$

but this is not a ruled parametrization. In fact, this surface also cannot contain any line: This surface is disjoint from the plane $Z = 0$. So if it contain any line at all, the line must lie is some plane $Z = c$ ($|c| \geq 1$). But our hyperboloid intersects such a plane in an ellipse, which also cannot contain any line. A similar argument proves that the elliptic paraboloid $X^2 + Y^2 - Z = 0$ (which has a parametrization $[V\cos U, V \sin U, V^2]$) cannot contain any line. Combined with the parametrized quadrics listed in Table 2, we have proved:

	Name	Equation	σ	σ_u	Ruled Parametrization
	Nonsingular Surfaces				
1	Hyperboloid of 1 sheet	$X^2+Y^2-Z^2=1$	(2,2)	(2,1)	$[\cos\theta \pm V\sin\theta, \sin\theta \mp V\cos\theta, V]$
2	Hyperbolic Paraboloid	$X^2-Y^2-Z=0$	(2,2)	(1,1)	$[U+V, \pm(U-V), 4UV]$
	Singular Nonplanar Surfaces				
3	Elliptic Cone	$X^2+Y^2-Z^2=0$	(2,1)	(2,1)	$[V\cos\theta, V\sin\theta, V]$
4	Elliptic Cylinder	$X^2+Y^2=1$	(2,1)	(2,0)	$[\cos\theta, \sin\theta, V]$
5	Hyperbolic Cylinder	$X^2-Y^2=-1$	(2,1)	(1,1)	$[\frac{1}{2}(U-\frac{1}{U}), \frac{1}{2}(U+\frac{1}{U}), V]$
6	Parabolic Cylinder	$X^2=Z$	(2,1)	(1,0)	$[U, V, U^2]$
	Singular Planar Surfaces				
7	Intersecting Planes	$X^2-Y^2=0$	(1,1)	(1,1)	$[U, U, V]$
8	Parallel Planes	$X^2-1=0$	(1,1)	(1,0)	$[1, U, V]$
9	Coincident Planes	$X^2=0$	(1,0)	(1,0)	$[0, U, V]$
10	Single Plane	$X=0$	(1,1)	(0,0)	$[0, U, V]$

Table 2: Ruled Quadric Surfaces

Theorem 10.
(i) A proper quadric is non-ruled iff it does not not contain a real line, and these are the ellipsoid, 2-sheeted hyperboloid and elliptic paraboloid.
(ii) Table 2 is a complete list of all proper ruled quadrics.

We now address the form of the ruled parametrizations (30). In particular, what kind of functions arise in the definition of the base curve $\mathbf{b}(U)$ and director curve $\mathbf{d}(U)$? Each is a vector function, e.g., $\mathbf{b}(U) = (X_1(U), X_2(U), X_3(U))$ where the $X_i(U)$'s are rational functions, i.e., $X_i(U) = P_i(U)/Q_i(U)$ where $P_i(U), Q_i(U)$ are relatively prime polynomials. The degree of $\mathbf{b}(U)$ is the maximum of the degrees of the polynomials $P_i(U), Q_i(U)$, $(i = 1, 2, 3)$. Similarly for the degree of $\mathbf{d}(U)$. There are three steps in this determination:
1. By an examination of Table 2, we see that in the base curve $\mathbf{b}(U)$ and director curve $\mathbf{d}(U)$ of canonical quadrics are **quadratic** rational functions of U. Quadratic means the degree is at most 2. Note that the appearance of $\cos\theta, \sin\theta$ in the table should be interpreted as the rational functions in (32).
2. We now examine the transformation from $Q_A(\mathbf{X})$ to the canonical form $Q_C(\mathbf{X})$ where C is a sign matrix that is either diagonal or almost diagonal. We want to determine the transformation matrix N such that

$$Q_A(N\mathbf{X}) = Q_C(\mathbf{X}). \tag{34}$$

By applying our algorithm to A, we obtain (M, B) such that

$$A = M^T BM, \qquad B = \mathtt{diag}(a_1, \ldots, a_n) \quad \text{or} \quad \mathtt{diag}(a_1, \ldots, a_{n-2}; d).$$

Let

$$S = \mathtt{diag}(c_1, \ldots, c_n)$$

where $c_i = \sqrt{|a_i|}$ for $i = 1, \ldots, n$. In case of the almost diagonal matrix, we use $c_{n-1} = c_n = \sqrt{|d|}$ instead. Then we have $B = SCS$ where $C = \mathtt{diag}(s_1, \ldots, s_n)$ or $C = \mathtt{diag}(s_1, \ldots, s_{n-2}; s)$ is a sign matrix. Thus $A = (SM)^T C(SM)$ and we can choose $N = SM$ in (34).
3. Suppose that $\mathbf{x} = (x(U,V), y(U,V), z(U,V))^T$ is a parametization of the canonical quadric $Q_C(X,Y,Z) = 0$. From $C = S^{-1}(M^{-1})^T AM^{-1}S^{-1}$, we see that

$$\mathbf{q}(U,V) = M^{-1}S^{-1} \cdot \mathbf{x}$$

is a parametrization for the quadric $Q_A(X,Y,Z) = 0$. Thus:

Theorem 11. *Every ruled quadric $Q_A(X,Y,Z)$ in real affine space has a rational quadratic parametrization*

$$\mathbf{q}(U,V) = (X(U,V), Y(U,V), Z(U,V))^T$$

of degree 2.

EXAMPLE: Give the ruling for the quadric

$$Q_A : XY + 2YZ + Z - 2 = 0. \tag{35}$$

We first analyze Q_A: with

$$(X, Y) \mapsto (X' + Y', X' - Y') \tag{36}$$

we obtain

$$X'^2 - Y'^2 + 2X'Z - 2Y'Z + Z - 2 = 0.$$

With

$$(X', Y') \mapsto (X'' - Z, Y'' - Z) \tag{37}$$

we obtain

$$X''^2 - Y''^2 + Z - 2 = 0.$$

With

$$Z \mapsto -Z' + 2 \tag{38}$$

we obtain

$$X''^2 - Y''^2 - Z' = 0.$$

Hence Q_A is a hyperbolic paraboloid. One ruled parametrization of the canonical hyperbolic paraboloid is

$$\begin{bmatrix} X'' \\ Y'' \\ Z' \end{bmatrix} = \mathbf{q}(U, V) = \begin{bmatrix} U \\ U \\ 0 \end{bmatrix} + V \begin{bmatrix} 1 \\ -1 \\ 4U \end{bmatrix}. \tag{39}$$

From (36), (37) and (38), we see that

$$\begin{aligned} X'' &= (X + Y + 2Z)/2 \\ Y'' &= (X - Y + 2Z)/2 \\ Z' &= -Z + 2 \end{aligned}$$

Thus

$$\begin{aligned} U + V &= (X + Y + 2Z)/2 \\ U - V &= (X - Y + 2Z)/2 \\ 4UV &= -Z + 2 \end{aligned}$$

or

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 2U + 8UV - 4 \\ 2V \\ 2 - 4UV \end{bmatrix} \tag{40}$$

Exercises

Exercise 4.1: Consider the surface $X^2 + Y + Z = 1$. Determine the ruling of this surface. ♢

Exercise 4.2:
(i) Show that ruled parameterizations (31) are the only rulings of the canonical hyperboloid of 1-sheet.
(ii) Do the same for ruled parameterizations (33) of the canonical hyperbolic paraboloid. (i) To see that this is indeed a ruling, we note that for each V $X = \cos U - V \sin U$, $Y = \sin U + V \sin U$ lies on a circle of radius $1 + V^2$ in the plane $Z = U$. So if $V \neq V'$ then the points $\mathbf{q}(U, V)$ and $\mathbf{q}(U, V')$ lies on different circles. For fixed V, we also see that (X, Y) corresponds to distinct points on the circle. (ii) We must show that if L is a line in the hyperboloid, then L is one of the rulings in our two families. ♢

Exercise 4.3: The following is another parametrization of the hyperboloid of 1 sheet: $[U+V, UV-1, U-V, UV+1]$. What is the relation to the ruled parameterization in our table? ♢

Exercise 4.4: Consider the intersection of two orthogonal right-circular cylinders along the Y- and Z-axes:

$$Q_0 = X^2 - Y^2 - 1, \quad Q_1 = Z^2 + X^2 - 1$$

Compute the projecting cone of their intersection. ♢

Exercise 4.5: Write a display program to visualize parametric space curves. For instance, display the curve given by (40) and (44). ♢

End Exercises

§5. Parametrization of Quadric Space Intersection Curves

Consider the intersection of two quadrics:

$$Q_0 = \mathbf{X}^T A_0 \mathbf{X} = 0, \qquad Q_1 = \mathbf{X}^T A_1 \mathbf{X} = 0, \tag{41}$$

with $\mathbf{X} = (X, Y, Z, 1)^T$. They intersect in a space curve that is known[6] as a **quadric space intersection curve** (QSIC). This is a **quartic**, i.e., degree 4 curve. By definitely, the degree of a space curve is the maximum number of intersection points with any plane. In this section, we describe a well-known method of parametrizing a QSIC from Levin (1976). There are some subtleties in this parametrization, however, which we will point out.

The classification of QSIC is a classic problem. We have three general binary criteria for this classification:

- A QSIC is **reducible** if it contains a component of degree 3 or less; otherwise it is **irreducible**. The linear or quadratic components can be complex, but cubic components are always real.
- A QSIC can be **planar** or **nonplanar**. A planar QSIC lies in one or two planes, and such QSIC's are necessarily reducible.
- A QSIC is **singular** if it contains a singular point. Reducible QSIC's must be singular. An irreducible and singular QSIC has one double singular point that may be classified as acnode, crunode or cusp. Such a curve is rationally parametrizable. An acnode may be the only real point of the QSIC.

A QSIC that is non-singular and irreducible is **non-degenerate** and otherwise **degenerate**. A complete classification for $\mathbb{P}^3(\mathbb{C})$ may be based on the **Segre characteristic** [6]. This classification scheme does not differentiate important morphological features in $\mathbb{P}^3(\mathbb{R})$. This is addressed in [11, 6]. Tu et al [9] considered the classification in $\mathbb{P}^3(\mathbb{R})$ by analysing the Jordan Canonical form of the pencil matrix.

If λ is a real variable, a **matrix pencil** has the form

$$R(\lambda) := (1-\lambda)A_0 + \lambda A_1 \tag{42}$$

where A_0, A_1 are symmetric real matrices. Thus, $R(\lambda)$ is a symmetric matrix whose elements are linear in λ. The parametrized family $\{\mathbf{X}^T R(\lambda)\mathbf{X} = 0 : \lambda \in \mathbb{R}\}$ of surfaces is called a **surface pencil**. We say the pencil is **trivial** if there is a constant matrix R_0 such that $R(\lambda) = \lambda R_0$. When $R(\lambda)$ is trivial, its surface pencil has only one surface, $\mathbf{X}^T R_0 \mathbf{X} = 0$.

Let C be the QSIC of the two surfaces (41). In $\mathbb{A}^3(\mathbb{R})$, C may have empty locus. When it is non-empty, it also known as the **base curve** of the corresponding pencil. Every member of the surface pencil contains the base curve: for if $\mathbf{X} \in C$ then $\mathbf{X}^T A_i \mathbf{X} = 0$ for $i = 0, 1$ and so $\mathbf{X}^T((1-\lambda)A_0 + \lambda A_1)\mathbf{X} = 0$. The intersection of any two distinct surfaces in the pencil is equal to C (Exercise).

[6] It seems as reasonable to read "QSIC" as quadric *surface* intersection curve.

Can C be empty in $\mathbb{A}^3(\mathbb{C})$? To understand this, we consider a new diagonalization problem. We say that a pair (A_0, A_1) of symmetric matrices is **simultaneously diagonalizable** if there exists an invertible matrix M such that

$$M^T A_0 M = \mathbf{1}, \qquad M^T B_0 M = \mathtt{diag}(\lambda_1, \ldots, \lambda_n) \tag{43}$$

for some $\lambda_1, \ldots, \lambda_n$.

Theorem 12. *Let A_0, A_1 be symmetric matrices of order n. If A_0 is non-singular and $\det(zA_0 - A_1)$ has n distinct roots $\lambda_1, \ldots, \lambda_n$, then (A_0, A_1) is simultaneously diagonalizable.*

Proof. By Theorem 5, we can choose M_0 such that $M_0^T A_0 M_0 = \mathbf{1}$. The matrix M_0 may be complex. Then $M_0^T(zA_0 - A_1)M_0 = z\mathbf{1} - A_1'$ where $A_1' = M_0^T A_1 M_0$. But

$$\det(z\mathbf{1} - A_1') = \det(M_0^T(zA_0 - A_1)M_0) = \det(M_0)^2 \det(zA_0 - A_1)$$

and thus the eigenvalues of A_1' are the roots of $\det(zA_0 - A_1)$. These are, by assumption, are distinct and equal to $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be the associated eigenvectors: $A_1'\mathbf{x}_i = \lambda_i \mathbf{x}_i$ and we may assume the scalar product $\mathbf{x}_i^T \mathbf{x}_i = 1$ for all i. Putting $M_1 = [\mathbf{x}_1 | \cdots | \mathbf{x}_n]$, we obtain $A_1' M_1 = M_1 \mathtt{diag}(\lambda_1, \ldots, \lambda_n)$. Note that

$$\begin{aligned}
\lambda_i \mathbf{x}_i^T \mathbf{x}_j &= \mathbf{x}_j^T(\lambda_i \mathbf{x}_i) \\
&= \mathbf{x}_j^T A_1' \mathbf{x}_i \\
&= (\mathbf{x}_j^T A_1' \mathbf{x}_i)^T \\
&= \mathbf{x}_i^T (A_1')^T \mathbf{x}_j \\
&= \mathbf{x}_i^T (A_1' \mathbf{x}_j) \\
&= \lambda_j \mathbf{x}_i^T \mathbf{x}_j.
\end{aligned}$$

If $i \neq j$, the equality $\lambda_i \mathbf{x}_i^T \mathbf{x}_j = \lambda_j \mathbf{x}_i^T \mathbf{x}_j$ implies that $\mathbf{x}_i^T \mathbf{x}_j = 0$. Thus $M_1^T M_1 = \mathbf{1}$, and $M_1^T A_1' M_1 = \mathtt{diag}(\lambda_1, \ldots, \lambda_n)$.

$$\begin{aligned}
M_1^T M_0^T (zA_0 - A_1) M_0 M_1 &= M_1^T (z\mathbf{1} - A_1') M_1 \\
&= z\mathbf{1} - \mathtt{diag}(\lambda_1, \ldots, \lambda_n).
\end{aligned}$$

The matrix $M = M_0 M_1$ achieves the desired simultaneous diagonalization of (A_0, A_1). **Q.E.D.**

The approach of Levin can be broken into two observations:

- The first observation is that it is easy to intersect a ruled quadric with a general quadric. We just plug in a ruling $\mathbf{b}(U) + V\mathbf{d}(U)$ of the ruled quadric into the general quadric $Q(X, Y, Z) = 0$ to obtain $Q'(U, V)$. Since Q' is of degree 2 in V, we can solve for V in terms of U. Thus the QSIC is parametrized by U.
- The second observation is that in any non-trivial pencil there is a ruled quadric surface.

EXAMPLE. To illustrate the first observation, recall the hyperboloid paraboloid (35) whose ruled parametrization is given by (40). Suppose we want to intersect the ruled surface with the sphere,

$$Q_B : X^2 + Y^2 + Z^2 - 3 = 0.$$

We might observe that this intersection is non-empty since $(-1, 1, 1)$ lies on both quadrics. Plugging the ruled parametrization into the second quadric:

$$\begin{aligned}
X^2 + Y^2 + Z^2 - 3 &= (2U + 8UV - 4)^2 + (2V)^2 + (2 - 4UV)^2 - 3 \\
&= 4U^2 + 64U^2V^2 + 16 + 32U^2V - 16U - 64UV + 4V^2 + 4 - 16UV + 16U^2V^2 - 3 \\
&= (80U^2 + 4)V^2 + (32U^2 - 80U)V + (4U^2 - 16U + 17) \\
&= 4(20U^2 + 1)V^2 + 16(2U^2 - 5U)V + (4U^2 - 16U + 17) \\
&= 0.
\end{aligned}$$

Solving for V:

$$\begin{aligned} V &= \frac{1}{4(20U^2+1)}\left(8(5U-2U^2) \pm \sqrt{8^2(2U^2-5U)^2 - 4(20U^2+1)(4U^2-16U+17)}\right) \\ &= \frac{1}{2(20U^2+1)}\left(4(5U-2U^2) \pm \sqrt{16(2U^2-5U)^2 - (20U^2+1)(4U^2-16U+17)}\right) \\ &= \frac{1}{2(20U^2+1)}\left(4(5U-2U^2) \pm \sqrt{-16U^4+56U^2-16U+17}\right) \end{aligned} \quad (44)$$

To prove the second observation, we follow the "geometric argument" in [11, Appendix]. In fact, this argument is quite general and applies to a quadric in any dimension. We first do this argument in projective space.

There is a preliminary concept. By definition, the locus of a line L in $\mathbb{P}^n(\mathbb{C})$ is a set of points of the form $\{a\mathbf{X}_0 + b\mathbf{X}_1 : (a,b) \in \mathbb{P}^1(\mathbb{C})\}$ where $\mathbf{X}_0, \mathbf{X}_1 \in \mathbb{P}^n(\mathbb{C})$ are distinct points. We say the line is **real** if the restriction of its locus to $\mathbb{A}^n(\mathbb{R})$ is a line in the usual sense. For instance, if $\mathbf{X}_0 = (\mathbf{i}, 0, 1, 1)^T$ and $\mathbf{X}_1 = (1, 1, 0, 1)^T$ then L is not a real line: we may verify that $a\mathbf{X}_0 + b\mathbf{X}_1$ is non-real for all $a, b \in \mathbb{C} \setminus \{0\}$.

Theorem 13. *If $R(\lambda) = \lambda A_0 + (1-\lambda)A_1$ is a non-trivial pencil, there exists a $\lambda_0 \in \mathbb{R}$ such that the $R(\lambda_0)$ is a ruled quadric surface.*

Proof. Our first task is to find a real line L that passes through two points $\mathbf{X}_0, \mathbf{X}_1$ (not necessarily real) in the intersection of the quadrics. If the intersection has two real points then the line passing through them will be our L. Suppose the intersection does not have two real points. Pick a complex point $\mathbf{X}_0 = \mathbf{U} + \mathbf{i}\mathbf{V}$ in the intersection such that $\mathbf{U}, \mathbf{V}$ are distinct real vectors. Then $\mathbf{X}_1 = \mathbf{U} - \mathbf{i}\mathbf{V}$ is also in the intersection (using the fact that for any real polynomial $Q(X,Y,Z)$, we have $Q(x,y,z) = 0$ iff $Q(\overline{x}, \overline{y}, \overline{z}) = 0$ where $x, y, z \in \mathbb{C}$ and $\overline{x} = u - \mathbf{i}v$ is the complex conjugate of $x = u + \mathbf{i}v$. But note that the line L through $\mathbf{X}_0, \mathbf{X}_1$ contains two real points $\mathbf{U} = (\mathbf{X}_0 + \mathbf{X}_1)/2$ and $\mathbf{V} = (\mathbf{X}_0 - \mathbf{X}_1)/(2\mathbf{i})$. We conclude L is a real line.

Next pick a real point $\mathbf{X}^* = a\mathbf{X}_0 + b\mathbf{X}_1$ on L that is distinct from $\mathbf{U}, \mathbf{V}$. We may assume $a = b = 1$. Our goal is to show that $\mathbf{X}^*$ lies on some surface $Q_S = 0$ where $S = \lambda_0 A_0 + \lambda_1 A_1$ and $\lambda_0, \lambda_1 \in \mathbb{R}$. Observe that

$$\begin{aligned} \mathbf{X}^{*T}(\lambda_0 A_0 + \lambda_1 A_1)\mathbf{X}^* &= \mathbf{X}_0^T(\lambda_0 A_0 + \lambda_1 A_1)\mathbf{X}_1 + \mathbf{X}_1^T(\lambda_0 A_0 + \lambda_1 A_1)\mathbf{X}_0 \\ &= \lambda_0(\mathbf{X}_0^T A_0 \mathbf{X}_1 + \mathbf{X}_1^T A_0 \mathbf{X}_0) + \lambda_1(\mathbf{X}_0^T A_1 \mathbf{X}_1 + \mathbf{X}_1^T A_1 \mathbf{X}_0) \\ &= \lambda_0 d_0 + \lambda_1 d_1. \end{aligned}$$

If $d_0 = d_1 = 0$ then $\mathbf{X}^*$ lies on the surface $Q_S = 0$ for for all $\lambda_0, \lambda_1 \in \mathbb{R}$. Otherwise, say $d_1 \neq 0$, and we may choose

$$(\lambda_0, \lambda_1) = (1, -d_0/d_1).$$

We need to verify that λ_0, λ_1 are real. It suffices to show that d_0, d_1 are real. This is clear when $\mathbf{X}_0, \mathbf{X}_1$ are real; otherwise, it follows from

$$d_i = (\mathbf{X}_0 + \mathbf{X}_1)^T A_i (\mathbf{X}_0 + \mathbf{X}_1) = (2\mathbf{U})^T A_i (2\mathbf{U}).$$

The surface $Q_S = 0$ thus contains three points X_0, X_1, X^* of the line L. The intersection of any quadric with a line has at most two points, unless the line is contained in the quadric. We conclude that the surface R contains L. By Theorem 10, Q_S is ruled. **Q.E.D.**

This general theorem gives us more than we need: in case the two surfaces has no real intersection, our application do not really care to find a ruled surface in the pencil. But we can obtain more restrictive conditions on the nature of the ruled quadric in the pencil, essentially in the form that Levin actually proved [7, Appendix]:

Theorem 14. *The intersection of two real affine quadric surfaces lies in a hyperbolic paraboloid, a cylinder (either hyperbolic or parabolic), or a degenerate planar surface.*

In outline, Levin's method first finds a ruled quadric S in the pencil of A_0, A_1. The points of S are rationally parametrized by $\mathbf{q}(U,V) \in \mathbb{R}^3$. For any fixed u_0, $\mathbf{q}(u_0,V)$ is a line parametrized by V. Plugging $\mathbf{q}(u_0,V)$ into one of the input quadrics, say $Q_0(X,Y,Z)$, we obtain a quadratic equation in V. Solving this equation, we obtain $v_0 = V(u_0)$ as an closed expression involving a square root. Thus we have found one point $\mathbf{q}(u_0, V(u_0))$ (possibly an imaginary one) on the intersection curve. Note that this method is suitable for display of the intersection curve. Thus, the QSIC is parametrized by the U-parameter. We shall show that the parametrization of the QSIC takes the form:

$$\mathbf{p}(U) = \mathbf{a}(U) \pm \mathbf{d}(U)\sqrt{s(U)} \tag{45}$$

where the bold face letters indicate vector valued polynomials and $s(U)$ is a degree 4 polynomial.

Any ruled surface S in a pencil is called **parametrization surface** of the pencil. It has the parameterization $S = \{\mathbf{q}(u,v) : u,v \in \mathbb{R}\}$ given by

$$\mathbf{q}(u,v) = \mathbf{b}(u) + v\mathbf{d}(u)$$

where $\mathbf{b}(u)$ is the base curve and $\mathbf{d}(u)$ the director curve. Recall that we showed for a ruled quadric, $\mathbf{b}(u)$ and $\mathbf{d}(u)$ are rational functions of degree ≤ 2 in u. Levin calls u the primary parameter and v the secondary parameter. For a fixed u_0, $\mathbf{q}(u_0,v)$ is a line in S. Thus the intersection of S with the surface $Q_0 = 0$ is reduced to the intersection of a ruling $\mathbf{q}(u_0,v)$ with the surface $Q_0 = 0$. [If S coincides with Q_0, then use Q_1 instead.] As noted in the outline above, this intersection can be solved numerically by plugging $\mathbf{q}(u_0,v)$ into the quadratic equation $Q_0 = 0$, yielding a quadratic equation in the secondary parameter. Solving, we obtain a value $v = v(u_0)$. But we can also find v symbolically, simply by plugging $\mathbf{q}(u,v)$ into the equation $Q_0 = 0$. This gives

$$c_2 v^2 + 2c_1 v + c_0 = 0 \tag{46}$$

where

$$c_2 = c_2(u) = \mathbf{d}(u)^T A_0 \mathbf{d}(u), \quad c_1 = c_1(u) = \mathbf{b}(u)^T A_0 \mathbf{d}(u), \quad c_0 = c_0(u) = \mathbf{b}(u)^T A_0 \mathbf{b}(u)$$

Let

$$v_0(u) = \frac{-c_1 \pm \sqrt{s}}{c_2}$$

where $s = c_1^2 - c_2 c_0$. Thus we obtain

$$\mathbf{p}(u) = \mathbf{q}(u, v_0(u))$$

which has the form (45) above. This point $\mathbf{p}(u)$ might be imaginary and there are also two possible values for $v(u_0)$.

The algorithm [7, p. 560] goes as follows: first compute the real roots λ_i $(i = 1,2,3,4)$ of the equation $R(\lambda) = \det(A_0 - \lambda A_1) = 0$. If for any i, $R(\lambda_i)$ is improper, then there is no intersection. Otherwise, $R(\lambda_i)$ corresponds to a ruled surface for some i. We may now proceed to intersect this surface with one of the original surface. Since λ_i is an algebraic number, this computation will require careful approximation in the sense of Exact Geometric Computation. Details of this process has not been analyzed.

Exercises

Exercise 5.1: If $\lambda \neq \lambda'$ then the intersection of the surfaces $\mathbf{X}^T R(\lambda)\mathbf{X} = 0$ and $\mathbf{X}^T R(\lambda')\mathbf{X} = 0$ is equal to the base curve. ♢

Exercise 5.2: Let C be the intersection of the quadrics in (41). Assume A_0, A_1 have rank 3 or 4. For each of the spaces $S = \mathbb{P}^3(\mathbb{C}), \mathbb{P}^3(\mathbb{R}), \mathbb{A}^3(\mathbb{C}), \mathbb{A}^3(\mathbb{R})$, either show that C is non-empty, or give an instance where C is empty. ♢

Exercise 5.3: Show that the line L with locus $\{a\mathbf{X}_0 + b\mathbf{X}_1 : a,b \in \mathbb{P}^1(\mathbb{C})\}$ where $\mathbf{X}_0 = (\mathbf{i},0,1)^T$ and $\mathbf{X}_1 = (1,1,0)^T$ is not a real line. ♢

Exercise 5.4: Give an algebraic version and proof of Theorem 13. HINT: The roots of the pencil $R(\lambda)$ are continuous function of λ. ♢

End Exercises

§6. Morphology of Degenerate QSIC

We now describe the approach of Farouki, Neff and O'Connor [6]. Their focus is on how to parametrize the degenerate QSIC. This decision is reasonable because in the non-degenerate case, Levin's method is adequate.

If the QSIC is a non-singular irreducible QSIC, we say it is **non-degenerate**; all other cases are **degenerate**. The degenerate QSIC's are of 5 types:

1. a singular quartic space curve
2. a cubic space curve and a straight line
3. two irreducible conic curves
4. an irreducible conic curve and two straight lines
5. four straight lines

The first case is irreducible, but the rest are reducible. In any case, we have

Theorem 15. *A QSIC is rationally parametrizable iff it is degenerate.*

Hence, it is useful to detect degeneracy and to exploit this parametrization property.

For any pencil $R(\lambda) = \lambda A_0 + (1-\lambda)A_1$, let

$$\Delta(\lambda) = \det(R(\lambda)) = \sum_{i=0}^{4} \Delta_i \lambda^i$$

be the **determinant** of the pencil $R(\lambda)$. Recall that the discriminant of any univariate polynomial $p(X)$ is denoted $\mathrm{disc}(p)$ and it defined as a resultant

$$\mathrm{disc}(p) := \mathtt{res}(p(X), p'(X))$$

where $p'(X)$ indicates differentiation by X. The **discriminant** of the pencil $R(\lambda)$ is defined to be $\mathrm{disc}(\Delta(\lambda))$.

Lemma 16. *A QSIC is degenerate iff* $\mathrm{disc}(\Delta(\lambda))$ *vanishes.*

Proof. **Q.E.D.**

For a generic point $\mathbf{X} = (X, Y, Z)^T$, and real variable t, consider the parametric line $t\mathbf{X} = (tX, tY, tZ)^T$ that passes through the origin and $\mathbf{X}$. If $t\mathbf{X}$ lies on the quadric Q_i, then we have $Q_i(tX, tY, tZ) = 0$. Now consider the surface defined by

$$R(X, Y, Z) = \mathtt{res}_t(Q_0(t\mathbf{X}), Q_1(t\mathbf{X})) = 0. \tag{47}$$

By the usual interpretation of resultants, this means for any point $P = (X, Y, Z)^T$ in the surface $R(X, Y, Z) = 0$, there exists a $t \in \mathbb{R}$ such that tP lies in the QSIC. In view of this, $R(X, Y, Z)$ is called the **projection cone** of the QSIC.

EXAMPLE. Compute the projection cone in the case of a sphere

$$Q_0 : X^2 + Y^2 + Z^2 = 4$$

and a cylinder

$$Q_1 : X^2 + Y^2 = 2X.$$

$$\begin{aligned}
R(X,Y,Z) &= \det\begin{bmatrix} X^2+Y^2+Z^2 & & -4 & \\ & X^2+Y^2+Z^2 & & -4 \\ X^2+Y^2 & -2X & & \\ & X^2+Y^2 & -2X & \end{bmatrix} \\
&= -4\det\begin{bmatrix} X^2+Y^2+Z^2 & & -4 \\ X^2+Y^2 & -2X & \\ & X^2+Y^2 & -2X \end{bmatrix} \\
&= -4[4X^2(X^2+Y^2+Z^2) - 4(X^2+Y^2)^2] \\
&= 16[Y^4 + X^2Y^2 - X^2Z^2]. \qquad (48)
\end{aligned}$$

In this example from Farouki et al [6], the QSIC is a figure-of-8, which is singular at the point $(2,0,0)$ where the two loops of the "8" meet.

Lemma 17. *$R(X,Y,Z) \in \mathbb{R}[X,Y,Z]$ is homogeneous of degree 4.*

Proof. The proof is like in the proof of Bezout's theorem. Introduce a new variable s as follows:

$$\begin{aligned}
R(sX,sY,sZ) &= \det\begin{bmatrix} s^2a_2 & sa_1 & a_0 & \\ & s^2a_2 & sa_1 & a_0 \\ s^2b_2 & sb_1 & b_0 & \\ & s^2b_2 & sb_1 & b_0 \end{bmatrix} \\
s^2R(sX,sY,sZ) &= \det\begin{bmatrix} s^3a_2 & s^2a_1 & sa_0 & \\ & s^2a_2 & sa_1 & a_0 \\ s^3b_2 & s^2b_1 & sb_0 & \\ & s^2b_2 & sb_1 & b_0 \end{bmatrix} \\
&= s^6\det\begin{bmatrix} a_2 & a_1 & a_0 & \\ & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & \\ & b_2 & b_1 & b_0 \end{bmatrix}.
\end{aligned}$$

From $R(sX,sY,sZ) = s^4R(X,Y,Z)$, we conclude that R is homogeneous of degree 4. **Q.E.D.**

Suppose the plane curve $R(X,Y,1)=0$ has a rational parametrization. If $(X(U),Y(U))$ is a parametrization of the plane curve $R(X,Y,1)$, then we can obtain a parametrization of the QSIC by plugging $(tX(U),tY(U),t)^T$ into any one of the original curve, say $Q_A(\mathbf{X}) = 0$. This gives a quadratic polynomial in t, which we can solve, say $t = t(U)$. This gives a parametrization of the QSIC of the form

$$(t(U)X(U), t(U)Y(U), t(U))^T.$$

We should also find a parametrization of the plane curve $R(X,Y,0)$ and repeat this procedure to get a complete parametrization of the QSIC.

EXAMPLE (contd). In the projection cone $R(X,Y,Z) = Y^4 + X^2Y^2 - X^2Z^2 = 0$ from (48), we prefer to set $Y = 0$ and $Y = 1$. The curve $R(X,1,Z)$ has parametrization $Z = \pm X\sqrt{X^2+1})$ I.e., we plug $(tU, t, \pm U\sqrt{U^2+1})^T$ into the cylinder $X^2+Y^2 = 2X$ to get

Errors! Also set $Z = 1$ i OK

$$t^2U^2 + t^2 = 2tU$$

thus $t = 2U/(1+U^2)$. Thus the QSIC becomes

$$(2U^2/(1+U^2), 2U/(1+U^2), \pm U\sqrt{U^2+1})^T$$

Similarly, the curve $R(X,0,Z)$ has parametrization $Z = \pm X\sqrt{X^2-1})$. Then $t = 2/U$, and the QSIC is

$$(2, 2/U, \pm U\sqrt{U^2-1})^T$$

ALTERNATIVE APPROACH: The sphere is

$$Q_A : X^2 + Y^2 + Z^2 = 4$$

but we may reexpress the cylinder $Q_B : X^2 + Y^2 = 2X$ as

$$Q_B : (X-1)^2 + Y^2 = 1$$

The ruled parametrization of Q_B is $(X,Y,Z) = (1+\cos\theta, \sin\theta, V)$. So plugging into Q_A, we get $2X+Z^2 = 4$ or $V = \sqrt{2-2\cos\theta}$. Let

$$P(\theta) = (1+\cos\theta, \sin\theta, \pm\sqrt{2-2\cos\theta})^T.$$

E.g. $P(0) = (2,0,0)$. This is a singular point of the QSIC.

Since R is homogeneous, it is basically equivalent to the bivariate polynomial

$$f(X,Y) = \Phi(X,Y,1).$$

We can now factor $f(X,Y)$ using elementary techniques. First consider what the linear factors of $f(X,Y)$ give us: ...

§7. Approach of Dupont, Lazard, et al

The ruled quadrics are given one of the canonical forms:

	Equation	Parametrization
(2,2)	$aX^2 + bY^2 - cZ^2 - dW^2 = 0$	$\left[\frac{u+av}{a}, \frac{uv-b}{b}, \frac{u-av}{\sqrt{ac}}, \frac{uv+b}{\sqrt{bd}}\right]$
(2,1)	$aX^2 + bY^2 - cZ^2 = 0$	$\left[uv, \frac{u^2-abv^2}{2b}, \frac{u^2+abv^2}{2\sqrt{bc}}, s\right]$
(2,0)	$aX^2 + bY^2 = 0$	$[0,0,u,v]$
(1,1)	$aX^2 - bY^2 = 0$	$\left[u, \pm\frac{ab}{b}u, v, s\right]$
(1,0)	$X^2 = 0$	$[0,u,v,s]$

REMARKS: An different approach to intersecting quadrics is proposed in [8]: by the simultaneous transformation of two quadrics into a canonical form in which their intersection is easily computed. See [5, Chapter 31] for discussion of this approach.

§8. Steiner Surfaces

A **rational parametrization** of a surface S is a map $\mathbf{q} : \mathbb{R}^2 \to \mathbb{R}^3$ of the form

$$\mathbf{q}(U,V) = (x(U,V), y(U,V), z(U,V))^T$$

where x, y, z rational functions in U, V such that the locus of S is, with finitely many exceptions, equal to $\{\mathbf{q}(U,V) : U, V \in \mathbb{R}\}$. The **degree** of this rational parametrization is the maximum degree of the rational functions. The rational parametrization is **faithful** if, with finitely many exceptions, $\mathbf{q}(U,V)$ is 1-1 on the surface.

A surface with rational parametrization of degree 2 is called a **Steiner surface**. Such surfaces are a generalization of quadrics, since we know that every quadric has a rational paramerization of degree 2. It is to see that Steiner surfaces have algebraic degree at most 4: consider the polynomials,

$$x_1(U,V)X - x_2(U,V), y_1(U,V)Y - y_2(U,V), z_1(U,V)Z - z_2(U,V)$$

where $x(U,V) = x_1(U,V)/x_2(U,V)$, etc, and the $x_i(U,V)$, etc, are polynomials of degree ≤ 2.

Exercises

Exercise 8.1: The Segre Characteristic is based on the theory of invariants. Suppose λ_i $(i = 1, \ldots, k)$ are the distinct complex eigenvalues of a matrix Q. Let λ_i have multiplicity $\mu_i \geq 1$. We can refine this notion of multiplicity as follows: let λ_i have multiplicity $\mu_{i,j}$ in... ♢

Exercise 8.2: A set of real symmetric matrices $\{A_i\}_i$ is simultaneously diagonalizable by an orthogonal matrix iff they are pairwise commutative, $A_i A_j = A_j A_i$. ♢

End Exercises

§9. Cut Curves of Quadric Surfaces

Consider the surfaces defined by $p, q \in \mathbb{Z}[X, Y, Z]$ where

$$\begin{aligned} p &= Z^2 + p_1 Z + p_0 \\ q &= Z^2 + q_1 Z + q_0 \end{aligned}$$

where $p_i, q_i \in \mathbb{Z}[X, Y]$. In case of quadric surfaces, $\deg(p_i) = 2 - i$, but part of this analysis does not require this restriction.

The **cut curve** of these two surfaces is the plane curve defined by the polynomial

$$f = \mathtt{res}_Z(p, q) \tag{49}$$

$$= \begin{bmatrix} 1 & p_1 & p_0 & 0 \\ 0 & 1 & p_1 & p_0 \\ 1 & q_1 & q_0 & 0 \\ 0 & 1 & q_1 & q_0 \end{bmatrix} \tag{50}$$

$$= (p_0 q_1 - p_1 q_0) \cdot (q_1 - p_1) + (p_0 - q_0)^2 \tag{51}$$

$$= (p_0 q_1 - p_1 q_0) \cdot \mathtt{psc}_1(p, q; Z) + (p_0 - q_0)^2 \tag{52}$$

Our main goal is to analyze the singular points of this curve.

Let $(a, b) \in \mathtt{Zero}(f)$. We call this a **top/bot point** if there exist $c \neq c'$ such that $\{(a, b, c), (a, b, c')\} \subseteq \mathtt{Zero}(p, q)$. We call it a **genuine point** if there exist c such that (a, b, c) is a tangential intersection of $p = 0$ and $q = 0$.

Theorem 18. *(i) Every singularity of $f = 0$ is either a top/bot point or a genuine point.*
(ii) Conversely, every top/bot point or genuine point is a singularity of $f = 0$.

Proof. (i) Without loss of generality, assume the singularity of f is $\mathbf{0} = (0, 0)$.

Note that if $\mathtt{psc}_1(p, q; Z)(\mathbf{0}) = 0$, then $p(0, 0, Z)$ and $q(0, 0, Z)$ has at least two roots in common, and so $\mathbf{0}$ is a top/bot singularity. Hence assume $\mathtt{psc}_1(p, q; Z) = (q_1 - p_1)$ does not vanish at $\mathbf{0}$.

To show that $\mathbf{0}$ is genuine amounts to showing that the tangent planes

$$p_X(\mathbf{0})X + p_Y(\mathbf{0})Y + p_Z(\mathbf{0})Z = 0, \qquad q_X(\mathbf{0})X + q_Y(\mathbf{0})Y + q_Z(\mathbf{0})Z = 0$$

are the same. [Note $\mathbf{0}$ here is $(0, 0, 0)$.] This amounts to

$$0 = (p_Y q_Z - p_Z q_Y)(\mathbf{0}) = (p_X q_Z - p_Z q_X)(\mathbf{0}) = (p_Y q_X - p_X q_Y)(\mathbf{0}). \tag{53}$$

We check that

$$\begin{aligned} p_0 &= p|_{Z=0}, & p_1 &= p_Z|_{Z=0}, \\ q_0 &= q|_{Z=0}, & q_1 &= q_Z|_{Z=0} \end{aligned}$$

Therefore

$$\begin{aligned} f &= \mathtt{res}_Z(p,q) \\ &= (pq_Z - p_Z q)|_{Z=0}\mathtt{psc}_1(p,q;Z) + (p-q)^2|_{Z=0} \end{aligned}$$

Now, using the fact that $(p_X)|_{Z=0} = (p|_{Z=0})_X$, we can compute

$$\begin{aligned} f_X &= ((pq_Z - p_Z q)_X)|_{Z=0}\mathtt{psc}_1(p,q;Z) + (pq_Z - p_Z q)|_{Z=0}(\mathtt{psc}_1(p,q;Z))_X + (2(p-q)\cdot(p_X - q_X)|_{Z=0} \\ &= ((pq_Z - p_Z q)_X)|_{Z=0}\mathtt{psc}_1(p,q;Z) \\ &= (p_X q_Z - p_Z q_X)|_{Z=0}\mathtt{psc}_1(p,q;Z). \end{aligned}$$

By assumption, $f_X(\mathbf{0}) = 0$ and so we conclude that

$$(p_X q_Z - p_Z q_X)|_{Z=0}(\mathbf{0}) = 0.$$

This is the same as

$$(p_X q_Z - p_Z q_X)(\mathbf{0}) = 0,$$

as required by (53). In the same way, we show

$$(p_Y q_Z - p_Z q_Y)(\mathbf{0}) = 0.$$

There is one more case in (53), namely

$$(p_Y q_X - p_X q_Y)(\mathbf{0}) = 0.$$

This requires a slightly different argument, left as an exercise.

(ii) Now we show that if p is a top/bot point, then it is a singularity of $f = 0$. Again we assume $p = (0,0)$.

Q.E.D.

Singularities of Projection of QSIC Note that none of the arguments so far required any bounds on the degrees of p_i, q_i's. The following arguments will now use the fact that p and q are quadratic. Again, let f be their cut curve.

Assume f is square free and generally aligned (i.e., its leading coefficients in X and in Y are constants). Then we claim that the cutcurve $f = 0$ has at most 2 top/bot points and at most 4 genuine points.

Theorem 19. *There are at most two top/bot points. We can compute two quadratic polynomials $t \in \mathbb{Z}[X]$ and $b \in \mathbb{Z}[Y]$ such that the top/bot points lies in the grid of zeros of t, b.*

References

[1] W. H. Breyer. *CRC Standard Mathematical Tables*. CRC Press, Boca Raton, Florida, 28 edition, 1987.

[2] T. J. I. Bromwich. The classification of quadrics. *Transactions of the American Mathematical Society*, 6(3):275–285, 1905.

[3] R. S. Burington. A classification of quadrics in affine n-space by means of arithmetic invariants. *Amer. Math. Monthly*, 39(9):527–532, 1932.

[4] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics. In *ACM Symp. on Comp. Geometry*, volume 19, pages 246 – 255, 2003.

[5] G. Farin, J. Hoschek, and M.-S. Kim, editors. *Handbook of Computer Aided Geometric Design*. North-Holland, Amsterdam, 2002.

[6] R. Farouki, C. Neff, and M. O'Connor. Automatic parsing of degenerate quadric-surface intersections. *ACM Trans. on Graphics*, 8(3):174–203, 1989.

[7] J. Z. Levin. A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces. *Comm. of the ACM*, 19(10):555–563, 1976.

[8] S. Ocken, J. T. Schwartz, and M. Sharir. Precise implementation of CAD primitives using rational parameterization of standard surfaces. In J. Hopcroft, J. Schwartz, and M. Sharir, editors, *Planning, Geometry, and Complexity of Robot Motion*, pages 245–266. Ablex Pub. Corp., Norwood, NJ, 1987.

[9] C. Tu, W. Wang, and J. Wang. Classifying the intersection curve of two quadric surfaces, 2004. To appear.

[10] W. Wang, R. Goldman, and C. Tu. Computing quadric surface intersections based on an analysis of plane cubic curves. *Graphical Models*, 64:335–367, 2003.

[11] W. Wang, R. Goldman, and C. Tu. Enhancing Levin's method for computing quadric-surface intersection. *Computer Aided Geom. Design*, 20:401–422, 2003.