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TWO-LEVEL OVERLAPPING SCHWARZ ALGORITHMS FOR A STAGGERED DISCONTINUOUS GALERKIN METHOD*

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Abstract. Two overlapping Schwarz algorithms are developed for a discontinuous Galerkin finite element approximation of second order scalar elliptic problems in both two and three dimensions. The discontinuous Galerkin formulation is based on a staggered discretization introduced by Chung and Engquist [*SIAM J. Numer. Anal.*, 47 (2009), pp. 3820–3848] for the acoustic wave equation. Two types of coarse problems are introduced for the two-level Schwarz algorithms. The first is built on a nonoverlapping subdomain partition, which allows quite general subdomain partitions, and the second on introducing an additional coarse triangulation that can also be quite independent of the fine triangulation. Condition number bounds are established and numerical results are presented.

Key words. domain decomposition, elliptic problems, preconditioned conjugate gradients, discontinuous Galerkin methods, staggered grids, overlapping Schwarz algorithms

AMS subject classifications. 65F10, 65N30, 65N55

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1. Introduction.

Two-level overlapping Schwarz algorithms are developed for the fast and stable solution of a staggered discontinuous Galerkin method applied to second order elliptic problems. Discontinuous Galerkin methods allow test functions which are discontinuous across element boundaries and this feature makes them more suitable for modeling problems with discontinuous coefficients, singularities, multi-scales, and multiphysics. Since the first work, by Reed and Hill [29], for hyperbolic equations, discontinuous Galerkin methods have been applied to various problems and the field has become an active research area; see, e.g., [21, 17, 30, 9, 5]. The design of the flux condition across the interelement boundary determines the accuracy of the discontinuous Galerkin approximation and the properties of the resulting linear system.

In relatively recent works by Engquist, Chung, and others [13, 14, 16, 15], a staggered discontinuous Galerkin method is developed and analyzed. A second order problem is written as a system of first order with two unknowns $\boldsymbol{U}$ and u. To approximate $\boldsymbol{U}$ and u, each triangle/tetrahedron, in a given triangulation, is subdivided and discontinuous functions $\boldsymbol{U}_h$ and u_h are built for the resulting triangulation so that on each interelement boundary one of these functions is continuous and the other discontinuous. In addition, we require these functions to satisfy a certain inf-

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sup stability. Using them, a conservative interelement flux condition is then obtained straightforwardly. Such a flux condition preserves the symmetry of the model problem and results in an optimal order of approximation. Moreover, the use of this staggered approximation provides locally and globally conservative schemes. While many discontinuous Galerkin methods have the advantage that they can be used with nonmatching meshes with hanging nodes, the discretization technique considered in this paper has not yet been developed for such meshes.

For elliptic problems, the resulting linear system arising from the staggered discontinuous Galerkin formulation is symmetric and positive definite after eliminating one set of variables locally. These desirable properties are obtained without the use of additional penalty terms. A low order discontinuous Galerkin method with similar properties is developed in [11] for nonstaggered meshes.

However, one disadvantage of the staggered discontinuous Galerkin method is that the resulting linear system is relatively large and less sparse than those from other discontinuous Galerkin formulations, because the test functions are built after a further subdivision of the given triangulation and are also partially continuous. Therefore, a fast and stable solver for the staggered discontinuous Galerkin formulation is quite desirable to increase its applicability for real-world problems.

Previous studies have addressed fast and stable solvers for discontinuous Galerkin methods. In the works by Feng and Karakashian [23, 24], two-level additive Schwarz methods were developed for second order elliptic problems and fourth order problems, and in the work by Lasser and Toselli [28] overlapping Schwarz preconditioners were developed for advection-diffusion problems. A more general framework of Schwarz preconditioners was studied in [1, 2, 3, 4], including multiplicative Schwarz preconditioners and *hp*-discontinuous Galerkin formulations. In the work by Dryja, Galvis, and Sarkis [22], balancing domain decomposition by constraints (BDDC) methods were applied to discontinuous Galerkin formulations of elliptic problems with discontinuous coefficients, where the finite element functions are continuous inside each subdomain and discontinuous across the subdomain boundaries only. Recently, two-level additive Schwarz preconditioners have also been studied by Barker et al. [6]. In their work, algorithms are developed and analyzed for several types of coarse problems and their performance compared for these different choices.

In this paper, we develop a two-level overlapping Schwarz preconditioner for the staggered discontinuous Galerkin formulation of [14] applied to elliptic problems. In all previous work on two-level Schwarz preconditioners for the discontinuous Galerkin formulation, each subdomain is assumed to be an element of a coarse regular partition or the union of a few such elements. Our algorithm, in contrast, allows for a quite general subdomain partition without such an assumption. Two types of coarse problems are introduced. The first one is related only to the subdomain partition where each subdomain is obtained as the union of elements provided in the problem domain. On each face, which is the common part of two subdomain boundaries, we introduce a face-based finite element function; its value is one on the given face and zero on the rest of the subdomain interface. For these interface values, the values in the interior of each subdomain are determined by minimizing a certain discrete energy norm. By using these face-based functions in the construction of the coarse problem, we can prove that the condition number can be bounded by $C(1 + H/\delta)(1 + H^{2-d} \max_{F_{ij}} |\theta^c_{F_{ij}}|^2_{H^1(\Omega)})$, where d is the dimension, H is the subdomain diameter, δ the overlapping width, C a positive constant independent of any mesh parameters, and $\theta^c_{F_{ij}}(x)$ a continuous, face-based finite element function described in section 4. We note that our result can be applied to quite general subdomain partitions, where each subdomain satisfies a

Poincaré inequality and a starlike property. We note that all our results are just as strong as the earlier ones for conforming finite element methods.

The second type of coarse problem is obtained by introducing an additional coarse triangulation. In this case, the subdomains again need not be a union of coarse tetrahedra/triangles. With the less strong assumption that the diameter of each subdomain is comparable to those of the coarse tetrahedra/triangles which intersect it, we can prove a condition number bound of $C(1 + H/\delta)$.

The rest of this paper is organized as follows. In section 2, the staggered discontinuous Galerkin formulation is introduced for a model elliptic problem, and in sections 3 and 4, our first two-level Schwarz algorithm is developed and analyzed. In section 5, the algorithm with the second type of the coarse problem is introduced and analyzed. In section 6, numerical experiments are reported for the proposed algorithms. Throughout this paper, C denotes a generic positive constant, which is independent of any mesh parameters.

2. The staggered discontinuous Galerkin formulation.

2.1. Variational form. We consider a scalar, elliptic model problem in a bounded domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3:

$$\begin{array}{c}\text{find } u \in H_0^1(\Omega) \text{ such that}\\ -\nabla \cdot (\rho(x)\nabla u(x)) = f(x) \quad \forall x \in \Omega,\end{array} \tag{2.1}$$

where $\rho(x) \geq \rho_0 > 0$ with ρ_0 a constant. The domain Ω is subdivided into potentially many subdomains Ω_i, which may have quite irregular boundaries. Moreover, we will assume that ρ is a constant in each of these subdomains; we could easily extend our results to cases where the coefficients vary moderately in each subdomain. In the following, we will use d to denote the dimension of Ω. In our description of the algorithm, we will primarily discuss the case of $d = 3$.

An equivalent variational formulation is obtained by integrating by parts:

$$\begin{array}{l}\text{find } u \in H_0^1(\Omega) \text{ such that}\\ (\rho\nabla u, \nabla v)_{\boldsymbol{L}^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).\end{array} \tag{2.2}$$

By introducing an additional unknown, namely, $\boldsymbol{U} := \rho\nabla u$, we can recast this problem and obtain a suitable framework for our discontinuous Galerkin discretization, also known as a *two-unknown* or *saddle point* problem:

$$\begin{array}{l}\text{find } (u, \boldsymbol{U}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \text{ such that}\\ (\rho^{-1}\boldsymbol{U}, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} - (\nabla u, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \boldsymbol{V} \in \boldsymbol{L}^2(\Omega),\\ (\boldsymbol{U}, \nabla v)_{\boldsymbol{L}^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).\end{array} \tag{2.3}$$

2.2. The staggered discontinuous Galerkin discretization. Following Chung and Engquist [13, 14], we first define an initial triangulation $\mathcal{T}_u$ satisfying the standard quasi-uniform assumption in each subdomains. Thus, the domain Ω is triangulated using a set of tetrahedra in three dimensions and triangles in two dimensons. $\mathcal{F}_u$ will denote the set of all faces in this triangulation and $\mathcal{F}_u^0$ the subset of all interior faces, i.e., the set of faces in $\mathcal{F}_u$ that are not embedded in $\partial\Omega$.

For each tetrahedron, we select an interior point v and denote this tetrahedron by $\mathcal{S}(v)$. We then further subdivide each tetrahedron into four subtetrahedra by connecting the point v to the four vertices of the tetrahedron. The resulting triangulation is denoted by $\mathcal{T}$. We assume that we select points v so that this triangulation also

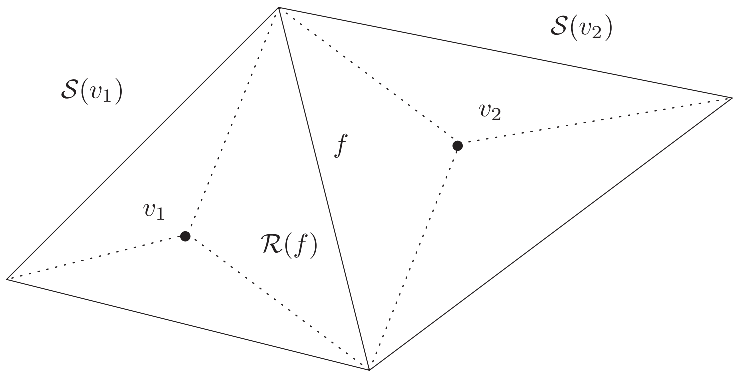

FIG. 1. *Triangulation in 2D.*

satisfies the standard shape regularity assumption. We will denote by $\mathcal{F}_p$ the set of all the new faces obtained by the second subdivision and set $\mathcal{F} := \mathcal{F}_u \cup \mathcal{F}_p$ and $\mathcal{F}^0 := \mathcal{F}_u^0 \cup \mathcal{F}_p$.

For each face $f \in \mathcal{F}_u$, we denote by $\mathcal{R}(f)$ the union of the two subtetrahedra sharing the face f. If f is a boundary face, then $\mathcal{R}(f)$ is just the one tetrahedron having this face. See Figure 1 for an illustration of this concept in two dimensions.

We define a unit normal vector $\boldsymbol{n}_f$ for each face $f \in \mathcal{F}$ as follows. If $f \in \mathcal{F}\backslash\mathcal{F}^0$, then $\boldsymbol{n}_f$ is the unit normal vector of f pointing toward the outside of Ω. If $f \in \mathcal{F}^0$, an interior face, we then fix $\boldsymbol{n}_f$ as one of the two possible unit normal vectors on f; when it is clear which face is being considered, we will simplify the notation and use $\boldsymbol{n}$ instead of $\boldsymbol{n}_f$.

We are now ready to introduce our finite element spaces. Let $k \geq 0$ be a nonnegative integer. Let $\tau \in \mathcal{T}$ and let $P^k(\tau)$ be the space of polynomials of degree less than or equal to k on τ.

We first introduce our discrete scalar field space.

Locally $H^1(\Omega)$-conforming finite element space for the scalar field.

$$\mathcal{S}_h := \{v \mid v_{|\tau} \in P^k(\tau),\ \forall \tau \in \mathcal{T};\ v \text{ continuous across } f \in \mathcal{F}_u^0;\ v_{|\partial\Omega} = 0\}. \tag{2.4}$$

We define two norms in the space $\mathcal{S}_h$, the discrete L^2-norm $\|v\|_X$ and the discrete H^1-norm $\|v\|_Z$, by

$$\|v\|_X^2 = \int_\Omega v^2\ dx + \sum_{f \in \mathcal{F}_u^0} h_f \int_f v^2\ d\sigma, \tag{2.5}$$

$$\|v\|_Z^2 = \int_\Omega |\nabla v|^2\ dx + \sum_{f \in \mathcal{F}_p} h_f^{-1} \int_f [v]^2\ d\sigma, \tag{2.6}$$

where h_f is the diameter of f and the integral of ∇v in (2.6) should be understood as defined elementwise:

$$\int_\Omega |\nabla v|^2\ dx := \sum_{\tau \in \mathcal{T}} \int_\tau |\nabla(v_{|\tau})|^2\ dx.$$

Here we recall that, by definition, $v \in \mathcal{S}_h$ is always continuous across each face of $\mathcal{F}_u^0$ but that it can be discontinuous across any face of $\mathcal{F}_p$. In the above definition, the jump $[v]$ across each $f \in \mathcal{F}_p$ is defined as

$$[v] = v_1 \boldsymbol{n}_1 + v_2 \boldsymbol{n}_2,$$

where $v_i = v_{|\tau_i}$, τ_1 and τ_2 are the two (sub-)tetrahedra sharing f, and $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the outward unit normal vectors on f for τ_1 and τ_2. We note that by using norm equivalence and a scaling argument (see also [14, Theorem 3.1]), we can show that there exists a constant $C > 0$, independent of h, such that

$$\|v\|^2_{L^2(\Omega)} \le \|v\|^2_X \le C\,\|v\|^2_{L^2(\Omega)} \quad \forall v \in \mathcal{S}_h.$$

We next introduce a discrete space of vector fields.

Locally $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming finite element space for the vector field.

$$\mathcal{V}_h = \{\boldsymbol{V} \mid \boldsymbol{V}_{|\tau} \in P^k(\tau)^d \ \forall \tau \in \mathcal{T};\ \boldsymbol{V}\cdot\boldsymbol{n} \text{ is continuous across } f \in \mathcal{F}_p\}. \tag{2.7}$$

In the space $\mathcal{V}_h$, we define two norms, the discrete $\boldsymbol{L}^2$-norm and the discrete $H(\mathrm{div};\Omega)$-norm, by

$$\|\boldsymbol{V}\|^2_{\boldsymbol{X}'} = \int_\Omega |\boldsymbol{V}|^2\,dx + \sum_{f\in\mathcal{F}_p} h_f \int_f (\boldsymbol{V}\cdot\boldsymbol{n})^2\,d\sigma, \tag{2.8}$$

$$\|\boldsymbol{V}\|^2_{\boldsymbol{Z}'} = \int_\Omega (\nabla\cdot\boldsymbol{V})^2\,dx + \sum_{f\in\mathcal{F}_u^0} h_f^{-1} \int_f [\boldsymbol{V}\cdot\boldsymbol{n}]^2\,d\sigma, \tag{2.9}$$

where the integral of $(\nabla\cdot\boldsymbol{V})^2$ in (2.9) is defined elementwise. We also recall that, by definition, $\boldsymbol{V} \in \mathcal{V}_h$ has a continuous normal component across each face $f \in \mathcal{F}_p$. In the definition above, the jump $[\boldsymbol{V}\cdot\boldsymbol{n}]$ on any $f \in \mathcal{F}_u^0$ is defined as

$$[\boldsymbol{V}\cdot\boldsymbol{n}] = \boldsymbol{V}_1\cdot\boldsymbol{n}_1 + \boldsymbol{V}_2\cdot\boldsymbol{n}_2,$$

where $\boldsymbol{V}_i = \boldsymbol{V}_{|\tau_i}$ and τ_1 and τ_2 are the two subtetrahedra with f as their common face. One can prove, by an argument used in the proof of [14, Theorem 3.2], that there exists a constant $C > 0$, independent of h, such that

$$\|\boldsymbol{V}\|^2_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{V}\|^2_{\boldsymbol{X}'} \le C\,\|\boldsymbol{V}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{V} \in \mathcal{V}_h. \tag{2.10}$$

We next define

$$\begin{aligned} b_h(\boldsymbol{U}, v) &= \int_\Omega \boldsymbol{U}\cdot\nabla v\,dx - \sum_{f\in\mathcal{F}_p}\int_f \boldsymbol{U}\cdot\boldsymbol{n}\,[v]\,d\sigma \\ &\quad - \sum_{f\in\mathcal{F}_u\setminus\mathcal{F}_u^0}\int_f v\,\boldsymbol{U}\cdot\boldsymbol{n}\,d\sigma, \quad \boldsymbol{U}\in\mathcal{V}_h, v\in\mathcal{S}_h, \end{aligned} \tag{2.11}$$

$$\begin{aligned} b_h^*(u, \boldsymbol{V}) &= -\int_\Omega u\,\nabla\cdot\boldsymbol{V}\,dx + \sum_{f\in\mathcal{F}_u^0}\int_f u\,[\boldsymbol{V}\cdot\boldsymbol{n}]\,d\sigma \\ &\quad + \sum_{f\in\mathcal{F}_u\setminus\mathcal{F}_u^0}\int_f u\boldsymbol{V}\cdot\boldsymbol{n}\,d\sigma, \quad u\in\mathcal{S}_h, \boldsymbol{V}\in\mathcal{V}_h. \end{aligned} \tag{2.12}$$

We note that when v and u in the above formulae vanish on $\partial\Omega$, the last term in both $b_h(\boldsymbol{U}, v)$ and $b_h^*(u, \boldsymbol{V})$ vanishes.

According to Lemma 2.4 of Chung and Engquist [14], we have

$$b_h(\boldsymbol{V}, v) = b_h^*(v, \boldsymbol{V}) \quad \forall (v, \boldsymbol{V}) \in \mathcal{S}_h \times \mathcal{V}_h. \tag{2.13}$$

Moreover, the following holds:

$$b_h(\boldsymbol{V}, v) \leq \|v\|_Z \, \|\boldsymbol{V}\|_{\boldsymbol{X}'} \quad \forall (v, \boldsymbol{V}) \in \mathcal{S}_h \times \mathcal{V}_h. \tag{2.14}$$

The staggered discontinuous Galerkin method reads as follows:

$$\begin{aligned} &\text{find } (u_h, \boldsymbol{U}_h) \in \mathcal{S}_h \times \mathcal{V}_h \text{ such that} \\ &(\boldsymbol{U}_h, \boldsymbol{V})_{\boldsymbol{L}^2_\rho(\Omega)} - b_h^*(u_h, \boldsymbol{V}) = 0 \quad \forall\, \boldsymbol{V} \in \mathcal{V}_h, \\ &b_h(\boldsymbol{U}_h, v) = (f, v)_{L^2(\Omega)} \quad \forall\, v \in \mathcal{S}_h. \end{aligned} \tag{2.15}$$

Here

$$(\boldsymbol{U}, \boldsymbol{V})_{\boldsymbol{L}^2_\rho(\Omega)} = \int_\Omega \frac{1}{\rho(x)} \boldsymbol{U} \cdot \boldsymbol{V} \, dx.$$

Let B_h and M_h be the matrices obtained from $b_h(\boldsymbol{V}, v)$ and $(\boldsymbol{U}, \boldsymbol{V})_{L^2_\rho(\Omega)}$ for functions in $(\boldsymbol{V}, v) \in \mathcal{V}_h \times \mathcal{S}_h$ and $(\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{V}_h \times \mathcal{V}_h$, respectively. Using that $b_h(\boldsymbol{V}, v) = b_h^*(v, \boldsymbol{V})$, the matrix B_h^T corresponds to the bilinear form $b_h^*(v, \boldsymbol{U})$ for $(v, \boldsymbol{U}) \in \mathcal{S}_h \times \mathcal{V}_h$. We can then rewrite (2.15) as an algebraic system of equations:

$$M_h \boldsymbol{U}_h - B_h^T u_h = 0, \tag{2.16}$$

$$B_h \boldsymbol{U}_h = f_h. \tag{2.17}$$

Since M_h is symmetric and positive definite and block diagonal with small blocks, we can eliminate $\boldsymbol{U}_h$ from (2.16) to obtain an equation for u_h,

$$B_h M_h^{-1} B_h^T u_h = f_h, \tag{2.18}$$

with a matrix which is symmetric and positive definite. We introduce a bilinear form for $(u, v) \in \mathcal{S}_h \times \mathcal{S}_h$,

$$a(u, v) := v^T B_h M_h^{-1} B_h^T u,$$

and use the notation A to denote the matrix $B_h M_h^{-1} B_h^T$,

$$A := B_h M_h^{-1} B_h^T. \tag{2.19}$$

We will develop two two-level overlapping Schwarz algorithms for solving the algebraic system (2.18).

In the design of the first preconditioner, we will build coarse basis functions related to a nonoverlapping subdomain partition of Ω similar to those of [18]. Let $\{\Omega_i\}$ be a nonoverlapping partition of Ω and assume that each Ω_i is connected and a union of tetrahedra/triangles in $\mathcal{T}_u$. For a given partition, we introduce local finite element spaces,

$$\mathcal{V}_{h,i} := \mathcal{V}_h|_{\Omega_i}, \; \mathcal{S}_{h,i} := \mathcal{S}_h|_{\Omega_i},$$

which are the restrictions of $\mathcal{V}_h$ and $\mathcal{S}_h$ to the subdomain Ω_i. Associated with $(\mathcal{V}_{h,i}, \mathcal{S}_{h,i})$, we introduce local bilinear forms $b_{h,i}$ and $b_{h,i}^*$ by

$$b_{h,i}(\boldsymbol{U}, v) = \int_{\Omega_i} \boldsymbol{U} \cdot \nabla v \, dx - \sum_{f \in \mathcal{F}_p \bigcap \Omega_i} \int_f \boldsymbol{U} \cdot \boldsymbol{n} \, [v] \, d\sigma, \tag{2.20}$$

$$\begin{aligned} b_{h,i}^*(u, \boldsymbol{V}) = &-\int_{\Omega_i} u \, \nabla \cdot \boldsymbol{V} \, dx + \sum_{f \in \mathcal{F}_u^0 \bigcap \Omega_i} \int_f u \, [\boldsymbol{V} \cdot \boldsymbol{n}] \, d\sigma \\ &+ \sum_{f \in \mathcal{F}_u^0 \bigcap \partial\Omega_i} \int_f u \, \boldsymbol{V} \cdot \boldsymbol{n}_i \, d\sigma, \end{aligned} \tag{2.21}$$

where $\boldsymbol{n}_i$ is the unit normal to $\partial\Omega_i$ on f. It can be seen easily that

$$b_{h,i}(\boldsymbol{V}, v) = b^*_{h,i}(v, \boldsymbol{V}) \tag{2.22}$$

and that

$$b_h(\boldsymbol{V}, v) = \sum_i b_{h,i}(\boldsymbol{V}|_{\Omega_i}, v|_{\Omega_i}), \quad b^*_h(v, \boldsymbol{V}) = \sum_i b^*_{h,i}(v|_{\Omega_i}, \boldsymbol{V}|_{\Omega_i}).$$

Let B_i and B_i^* be the matrices associated to the bilinear forms $b_{h,i}$ and $b^*_{h,i}$, respectively, i.e.,

$$\langle B_i \boldsymbol{V}|_{\Omega_i}, v|_{\Omega_i}\rangle = b_{h,i}(\boldsymbol{V}|_{\Omega_i}, v|_{\Omega_i})$$

and

$$\langle B_i^* v|_{\Omega_i}, \boldsymbol{V}|_{\Omega_i}\rangle = b^*_{h,i}(v|_{\Omega_i}, \boldsymbol{V}|_{\Omega_i}).$$

Here $\langle\cdot,\cdot\rangle$ denotes the l^2-inner product. Using (2.22), we have

$$B_i^* = B_i^T.$$

By introducing M_i, the matrix associated to the bilinear form

$$\langle M_i \boldsymbol{U}|_{\Omega_i}, \boldsymbol{V}|_{\Omega_i}\rangle = (\boldsymbol{U}|_{\Omega_i}, \boldsymbol{V}_{\Omega_i})_{L^2_\rho(\Omega_i)},$$

and R_i, the restriction from $\mathcal{S}_h$ to $\mathcal{S}_{h,i}$, we can rewrite (2.15) as

$$M_i \boldsymbol{U}_i - B_i^T R_i u_h = 0, \quad i = 1, \dots, N, \tag{2.23}$$

$$\sum_i R_i^T B_i \boldsymbol{U}_i = \sum_i R_i^T f_i, \tag{2.24}$$

where $\boldsymbol{U}_i$ is the restriction of $\boldsymbol{U}$ to Ω_i and f_i is given by

$$\langle f_i, v|_{\Omega_i}\rangle = (f, v)_{L^2(\Omega_i)}.$$

Since the M_i are invertible, by (2.23) and (2.24), we can obtain the algebraic equation (2.18) by assembling of local matrices:

$$\sum_i R_i^T B_i M_i^{-1} B_i^T R_i u_h = \sum_i R_i^T f_i. \tag{2.25}$$

Here we note that $u_h \in \mathcal{S}_h$, where functions can be discontinuous across each face $f \in \mathcal{F}_p$. We introduce the notation A_i for

$$A_i = B_i M_i^{-1} B_i^T,$$

and we introduce a bilinear form defined on $\mathcal{S}_{h,i} \times \mathcal{S}_{h,i}$ by

$$a_i(u_i, v_i) = \langle A_i u_i, v_i\rangle. \tag{2.26}$$

3. A two-level overlapping Schwarz algorithm. We consider a nonoverlapping partition of Ω, which is denoted by $\{\Omega_i\}$. The nonoverlapping partition can be obtained from the original triangulation $\mathcal{T}_u$ provided for Ω, e.g., by using a mesh partitioner; the subdomains in the resulting partition may then have quite irregular boundaries. The interface Γ is defined by $(\cup_{i\neq k}\partial\Omega_i \cap \partial\Omega_k) \setminus \partial\Omega$.

We then introduce an overlapping partition $\{\Omega'_j\}$ of Ω and for each subregion Ω'_j, and we associate two finite element spaces $\mathcal{V}_h(\Omega'_j)$ and $\mathcal{S}_h^0(\Omega'_j)$, which are the restrictions of $\mathcal{V}_h$ and $\mathcal{S}_h$ to the subregion Ω'_j. Here the superscript 0 indicates that the functions in $\mathcal{S}_h^0(\Omega'_j)$ vanish on the boundary of Ω'_j.

A bilinear form is introduced for $(u,v) \in \mathcal{S}_h^0(\Omega'_j) \times \mathcal{S}_h^0(\Omega'_j)$ by

$$a_{\Omega'_j}(u,v) := v^T B_{h,\Omega'_j} M_{\Omega'_j}^{-1} B_{h,\Omega'_j}^T u,$$

where B_{h,Ω'_j} is the matrix obtained from $b_h(\boldsymbol{U}, v)$ for $(\boldsymbol{U}, v) \in \mathcal{V}_h(\Omega'_j) \times \mathcal{S}_h^0(\Omega'_j)$ and $M_{\Omega'_j}^{-1}$ is the inverse of the weighted mass matrix obtained from $(\boldsymbol{U}, \boldsymbol{V})_{L^2_\rho(\Omega)}$, where $(\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{V}_h(\Omega'_j) \times \mathcal{V}_h(\Omega'_j)$.

To simplify the presentation, we will use the notation V'_j to denote $\mathcal{S}_h^0(\Omega'_j)$ and introduce the trivial extension by zero,

$$R_j^T \,:\, V'_j \to \mathcal{S}_h.$$

A projection P_j, related to the subregion Ω'_j, is defined by

$$P_j = R_j^T P'_j,$$

where P'_j is obtained from

$$a_{\Omega'_j}(P'_j u, v) = a(u, R_j^T v) \quad \forall v \in V'_j.$$

We now construct the coarse space V_0 based on the nonoverlapping partition $\{\Omega_i\}$. Let F_{ij} denote the common face (edge) of two subdomains Ω_i and Ω_j in three (two) dimensions. Then the union of all these F_{ij} forms a partition of $\partial\Omega_i$. For each F_{ij}, we define a face- (edge-) based function $\theta_{F_{ij}}^{(k)}(x)$ as follows: $\theta_{F_{ij}}^{(k)}$ is a piecewise constant function on $\partial\Omega_i$ with the values

$$\theta_{F_{ij}}^{(k)}(x) = \begin{cases} 1, & x \in F_{ij}, \\ 0, & x \in F_{im} \quad \forall m \neq j. \end{cases} \tag{3.1}$$

We extend these boundary values to the interior of Ω_i by a minimal energy extension with respect to the seminorm $a_i(v_i, v_i)^{1/2}$ defined in (2.26). Here we use the superscript k to stress that $\mathcal{S}_h$ is defined by piecewise polynomials of order k. For $x \in \overline{\Omega}_j$, we define $\theta_{F_{ij}}^{(k)}(x)$ similarly. We then extend it by zero to the rest of Ω as an element of $\mathcal{S}_h$. We note that $\theta_{F_{ij}}^{(k)}$ is discontinuous on the boundary of F_{ij}.

We can now define the space of coarse basis functions by

$$V_0 = \operatorname{span}\left\{\theta_{F_{ij}}^{(k)}(x)\ \forall F_{ij}\right\}.$$

The projection P_0 is then defined by

$$a(P_0 u, v) = a(u, v) \quad \forall v \in V_0,$$

and the two-level overlapping Schwarz operator is given by

$$P_{as} = \sum_{j=0}^{N} P_j.$$

4. Estimate of the condition number. We will now provide a bound of the condition number of our first two-level overlapping Schwarz algorithm. See [31, Chapter 3] for this algorithm and the theory for the standard conforming case.

For the upper bound, by the standard coloring argument and by using that the P_j are projections (see [31, Lemmas 2.6 and 2.10]), we obtain

$$a(P_{as}u, u) \le (1 + N_c)a(u, u),$$

where N_c is the number of colors required to color the overlapping subregions in $\{\Omega_j'\}$ so that no two intersecting subregions have the same color.

For the lower bound, we will prove that for some decomposition of $u \in \mathcal{S}_h$,

$$u = u_0 + \sum_{j=1}^{N} R_j^T u_j,$$

with $u_0 \in V_0$ and $u_j \in V_j'$, the following inequality holds:

$$a(u_0, u_0) + \sum_{j=1}^{N} a_{\Omega_j'}(u_j, u_j) \le C_0^2 a(u, u).$$

The condition number of P_{as} is then bounded by

$$f(P_{as}) \le (1 + N_c)C_0^2.$$

In our theory, we need an assumption on the nonoverlapping subdomain partition $\{\Omega_i\}$. A domain Ω_i is starlike if there exits a $\mathbf{x}_0 \in \Omega_i$ and a constant $c > 0$ such that

$$(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n} \ge cH_{\Omega_i} \quad \forall \mathbf{x} \in \partial\Omega_i, \tag{4.1}$$

where $\mathbf{n}$ is the unit normal to $\partial\Omega_i$ at $\mathbf{x}$ and H_{Ω_i} is the diameter of Ω_i.

Assumption 4.1. Each subdomain Ω_i satisfies the Poincaré inequalities as in [8] and the starlike property, and the number of tetrahedra along each edge of Ω_i is proportional to $(H_{\Omega_i}/h_i)^{d-2}$, where H_{Ω_i} is the diameter of Ω_i and h_i is the mesh size of the fine grid of Ω_i. The $\rho(x)$ is a positive constant ρ_i for each subdomain Ω_i.

With the above assumption on each subdomain in the nonoverlapping subdomain partition, we will prove that

$$C_0^2 \le C\left(1 + \frac{H}{\delta}\right)\left(1 + H^{2-d}\max_{F_{ij}} |\theta^c_{F_{ij}}|^2_{H^1(\Omega)}\right),$$

where $\theta^c_{F_{ij}}$ is a linear conforming face function with the boundary values

$$\theta^c_{F_{ij}}(x) = \begin{cases} 1, & x \in F^h_{ij}, \\ 0, & x \in \Gamma^h \setminus F^h_{ij}, \end{cases} \tag{4.2}$$

and which minimizes the H^1-seminorm on the space V_h. Here V_h is the space of linear conforming finite element functions on the given triangulation $\mathcal{T}$, F^h_{ij} is the set of nodes belong to F_{ij}, and Γ^h is the set of nodes belong to the boundaries of at least two substructures. We note that $\theta^c_{F_{ij}}$ is needed only for the theory.

We recall the following properties for $b_h(\boldsymbol{V}, v)$ and $b_h^*(v, \boldsymbol{V})$ (see [13, 14]):

$$|b_h^*(v, \boldsymbol{V})| \le \|\boldsymbol{V}\|_{Z'}\|v\|_X, \tag{4.3}$$

$$|b_h(\boldsymbol{V}, v)| \le \|\boldsymbol{V}\|_{X'}\|v\|_Z, \tag{4.4}$$

and

$$\inf_{\boldsymbol{V}\in\mathcal{V}_h}\sup_{v\in\mathcal{S}_h}\frac{b_h^*(v,\boldsymbol{V})}{\|v\|_X\|\boldsymbol{V}\|_{Z'}}\geq\beta, \tag{4.5}$$

$$\inf_{v\in\mathcal{S}_h}\sup_{\boldsymbol{V}\in\mathcal{V}_h}\frac{b_h(\boldsymbol{V},v)}{\|\boldsymbol{V}\|_{X'}\|v\|_Z}\geq\beta, \tag{4.6}$$

where β is a positive constant independent of h and H.

We note that using (4.4) and (4.6), we obtain for $u\in\mathcal{S}_h$

$$c(\rho)\beta^2\|u\|_Z^2\leq a(u,u)\leq C(\rho)\|u\|_Z^2,$$

where $c(\rho)$ and $C(\rho)$ are positive constants depending on $\rho(x)$. Similarly, we obtain for $u_i\in\mathcal{S}_{h,i}$

$$c\beta^2\rho_i\|u_i\|_{Z_i}^2\leq a_i(u_i,u_i)\leq C\rho_i\|u_i\|_{Z_i}^2, \tag{4.7}$$

where

$$\|u_i\|_{Z_i}^2:=\int_{\Omega_i}|\nabla u_i|^2\,dx+\sum_{f\in\mathcal{F}_p\bigcap\Omega_i}h_f^{-1}\int_f[u_i]^2\,ds$$

and c and C are positive constants, which do not depend on $\rho(x)$. We recall that we assume that $\rho(x)=\rho_i$ for all $x\in\Omega_i$, where ρ_i is a positive constant.

We list some auxiliary results which will be useful in our analysis.

- Poincaré–Friedrichs inequalities (Brenner [8]):

$$\|v\|_{L_2(\Omega)}^2\leq C\left(\sum_{\tau\in\mathcal{T}}|v|_{H_1(\tau)}^2+\sum_{f\in\mathcal{F}}h_f^{-1}\int_f[v]^2\,ds+\left(\int_\Omega v\,ds\right)^2\right), \tag{4.8}$$

$$\|v\|_{L_2(\Omega)}^2\leq C\left(\sum_{\tau\in\mathcal{T}}|v|_{H_1(\tau)}^2+\sum_{f\in\mathcal{F}}h_f^{-1}\int_f[v]^2\,ds+\left(\int_\Gamma v\,ds\right)^2\right). \tag{4.9}$$

- A trace inequality (Feng and Karakashian [24, Lemma 3.6]):

$$\|v\|_{L^2(\partial\Omega)}^2\leq C\left(H_\Omega^{-1}\|v\|_{L^2(\Omega)}^2+H_\Omega\left(\sum_{\tau\in\mathcal{T}}|v|_{H^1(\tau)}^2+\sum_{f\in\mathcal{F}}h_f^{-1}\int_f[v]^2\,ds\right)\right). \tag{4.10}$$

Let Ω_δ be the thin layer of Ω which consists of $\mathbf{x}\in\Omega$ such that $\operatorname{dist}(\mathbf{x},\partial\Omega)\leq\delta$.

- A generalized Poincaré inequality (Feng and Karakashian [24, Lemma 3.7]):

$$\|v\|_{L^2(\Omega_\delta)}\leq C\delta\left(H_\Omega^{-1}\|v\|_{L^2(\Omega)}^2+H_\Omega\left(\sum_{\tau\in\mathcal{T}}|v|_{H^1(\tau)}^2+\sum_{f\in\mathcal{F}}h_f^{-1}\int_f[v]^2\,ds\right)\right). \tag{4.11}$$

We note that these results hold for any piecewise polynomial function u given in terms of a partition $\mathcal{T}$ of $\mathcal{F}$, the set of all interior faces (edges) in $\mathcal{T}$. In our case, $\mathcal{F}$ is the union of $\mathcal{F}_p$ and $\mathcal{F}_u^0$. Γ is a measurable subset of $\partial\Omega$ with a positive $(d-1)$-dimensional measure. H_Ω and h_f denote the diameter of the domain Ω and f, respectively.

The inequalities in (4.8) and (4.9) hold for any Ω which satisfies the standard Poincaré–Friedrichs inequalities. The inequalities in (4.10) and (4.11) hold for any bounded polyhedral domain which is starlike. The constant C in (4.10) depends on the constant c appearing in (4.1), the definition of the starlike property. We note that Ω need not be convex. The result in (4.11) is a general version of Lemma 3.10 in [31]. In our theory, these results will be applied to each subdomain Ω_i.

For a given function $u \in \mathcal{S}_h$, we consider

$$u_0(x) = \sum_{ij} \overline{u}_{F_{ij}} \theta^{(k)}_{F_{ij}}(x), \tag{4.12}$$

where $\overline{u}_{F_{ij}}$ is the average of u over F_{ij}, i.e.,

$$\overline{u}_{F_{ij}} = \frac{\int_{F_{ij}} u(x(s))\, ds}{\int_{F_{ij}} 1\, ds}. \tag{4.13}$$

We note that $\theta^{(k)}_{F_{ij}}(x)$ is discontinuous on the boundary of F_{ij}, while the coarse basis function $\theta^{c}_{F_{ij}}(x)$ of the standard conforming finite elements is continuous and vanishes on the boundary of the face. In addition we note that $\theta^{(k)}_{F_{ij}}$ satisfies

$$\sum_{F_{ij}\subset \partial\Omega_i} \theta^{(k)}_{F_{ij}}(x) = 1 \ \forall x \in \overline{\Omega}_i.$$

Let I^h be an interpolant of $v \in \prod_{f\in\mathcal{F}_u} H^1(\mathcal{R}(f))$, which is the space of piecewise H^1-functions in $\mathcal{T}$ such that $[v]_f = 0$ for all f in $\mathcal{F}^0_u$ into $\mathcal{S}_h$, which satisfies

$$\begin{aligned} &\int_f (I^h v - v) q\, ds = 0 \ \forall q \in P^k(f) \quad \forall f \in \mathcal{F}_u, \\ &\int_\tau (I^h v - v) q\, dx = 0 \ \forall q \in P^{k-1}(\tau) \quad \forall \tau \in \mathcal{T}. \end{aligned} \tag{4.14}$$

From the above conditions, we can see that $I^h v$ satisfies $b_h(\boldsymbol{V}, I^h v - v) = 0$ for all $\boldsymbol{V} \in \mathcal{V}_h$ (see [14]). Consequently, by the continuity condition (4.4) and the inf-sup condition (4.6), we have

$$\|I^h v\|_Z \le \frac{1}{\beta} \sup_{\boldsymbol{V}\in\mathcal{V}_h} \frac{b_h(\boldsymbol{V}, I^h v)}{\|\boldsymbol{V}\|_{X'}} = \frac{1}{\beta} \sup_{\boldsymbol{V}\in\mathcal{V}_h} \frac{b_h(\boldsymbol{V}, v)}{\|\boldsymbol{V}\|_{X'}} \le \frac{1}{\beta}\|v\|_Z.$$

Hence, we have proved the following lemma, which will be used in our analysis.

LEMMA 4.2. *For $v \in \prod_{f\in\mathcal{F}_u} H^1(\mathcal{R}(f))$, we have*

$$\|I^h v\|_Z \le C\|v\|_Z.$$

Next, we recall the definition of $\theta^{(k)}_{F_{ij}}$ in (3.1) and obtain the following estimate.

LEMMA 4.3. *With the assumption that the number of tetrahedra along each edge of Ω_i is proportional to $(H/h)^{d-2}$, the coarse basis function satisfies*

$$\|\theta^{(k)}_{F_{ij}}\|^2_Z \le C(H^{d-2} + |\theta^c_{F_{ij}}|^2_{H^1(\Omega)})$$

for all $k \ge 0$, where $\theta^c_{F_{ij}}(x)$ is the standard linear conforming face function.

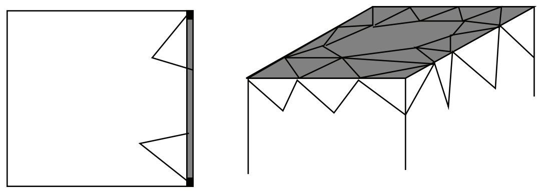

FIG. 2. *The face F_{ij} (in gray) for two dimensions (left) and three dimensions (right), the set V_{ij} with triangles/tetrahedra (in figure) which intersect ∂F_{ij} and intersect F_{ij} on an edge/a face f in F_{ij}.*

Proof. We first consider the case of $k = 1$ and later extend the result to the other cases where $k = 0$ and $k \geq 2$. Let V_{ij} be the set of all triangles/tetrahedra in the triangulation $\mathcal{T}$ that have a nonempty intersection with ∂F_{ij} and intersect F_{ij} along an edge/a face f in F_{ij}; see Figure 2.

For $v \in \mathcal{S}_h$, we denote by $H(v)$ the discrete harmonic extension into $\mathcal{S}_h$ which minimizes the discrete H^1-norm, $\|H(v)\|_Z$ for a given value v on Γ_h. By the definition of $\theta_{F_{ij}}^{(1)}$, we see that

$$H(\theta_{F_{ij}}^{(1)}) = \theta_{F_{ij}}^{(1)}.$$

In addition, for $v \in \mathcal{S}_h$, let $E(v)$ the zero extension into $\mathcal{S}_h$ for a given value v on Γ_h.

We then obtain that

$$\begin{aligned} \|\theta_{F_{ij}}^{(1)}\|_Z &= \|H(\theta_{F_{ij}}^{(1)})\|_Z \\ &\leq \|H(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})\|_Z + \|H(\theta_{F_{ij}}^{c})\|_Z \\ &\leq \|E(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})\|_Z + \|\theta_{F_{ij}}^{c}\|_Z, \end{aligned} \tag{4.15}$$

since the operator H is linear and $H(v)$ minimizes the norm $\|\cdot\|_Z$ for the given value v on Γ_h.

Using that $0 \leq E(\theta_{F_{ij}}^{(1)}(x) - \theta_{F_{ij}}^{c}(x)) \leq 1$, we obtain

$$|E(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})|_{H^1(\tau)}^2 \leq Ch^{d-2}$$

and

$$h_f^{-1}\|[E(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})]\|_{L^2(f)}^2 \leq Ch^{d-2}.$$

By the definition of $E(v)$, it suffices to consider only the tetrahedra τ with nonzero value of v and we note that $E(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})$ has nonzero values only in the tetrahedra in V_{ij}. Since the number of tetrahedra/triangles along each edge of Ω_i is proportional to $(H/h)^{d-2}$, the number of tetrahedra/triangles in V_{ij} is proportional to H/h and to a constant for $d = 3$ and $d = 2$, respectively. We then obtain

$$\begin{aligned} \|E(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})\|_Z^2 &\leq \sum_{\tau \in V_{ij}} |E(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})|_{H^1(\tau)}^2 \\ &\quad + \sum_{f \in \mathcal{F}_p \bigcap \overline{\tau}, \tau \in V_{ij}} h_f^{-1}\|[E(\theta_{F_{ij}}^{(1)} - \theta_{F_{ij}}^{c})]\|_{L^2(f)}^2 \\ &\leq Ch^{d-2}\left(\frac{H}{h}\right)^{d-2} \leq CH^{d-2}. \end{aligned} \tag{4.16}$$

We therefore obtain that

$$\|\theta_{F_{ij}}^{(1)}\|_Z^2 \le C\left(H^{d-2} + |\theta_{F_{ij}}^c|_{1,\Omega}^2\right), \tag{4.17}$$

where we have used (4.15), (4.16), and that $\|\theta_{F_{ij}}^c\|_Z = |\theta_{F_{ij}}^c|_{1,\Omega}$.

For $k = 0$ or $k \ge 2$, we consider $I^h\theta_{F_{ij}}^{(1)}$, where I^h is the interpolant into $\mathcal{S}_h$ for the given k, as defined in (4.14). Using that $I^h\theta_{F_{ij}}^{(1)}$ has the same boundary data as $\theta_{F_{ij}}^{(k)}(x)$ on Γ_h and by the stability of the interpolant given in Lemma 4.2, we obtain

$$\|\theta_{F_{ij}}^{(k)}\|_Z^2 \le \|I^h\theta_{F_{ij}}^{(1)}\|_Z^2 \le C\|\theta_{F_{ij}}^{(1)}\|_Z^2.$$

This inequality combined with the result for $k = 1$ shows that the result also holds for the general case of $k = 0$ and $k \ge 2$. ∎

LEMMA 4.4. *With the assumption that the subdomains Ω_i satisfy the Poincaré inequality and the starlike property, the u_0 of (4.12) satisfies*

$$a(u_0, u_0) \le Ca(u, u)\,(1 + H^{2-d} \max_{F_{ij}} |\theta_{F_{ij}}^c|_{H^1(\Omega)}^2).$$

Here C depends on the Poincaré and the starlike parameters of the subdomains.

Proof. We consider

$$a(u - u_0, u - u_0) = \sum_i a_i(R_i(u - u_0), R_i(u - u_0)),$$

where R_i is the restriction to Ω_i. Each term above is bounded by

$$\begin{aligned} a_i(R_i(u - u_0), R_i(u - u_0)) &\le 2a_i(R_iu, R_iu) + 2a_i(R_iu_0, R_iu_0) \\ &\le C\left(a_i(R_iu, R_iu) + \sum_{F_{ij}\subset\partial\Omega_i} \overline{u}_{F_{ij}}^2 a_i(R_i\theta_{F_{ij}}^{(k)}, R_i\theta_{F_{ij}}^{(k)})\right) \\ &\le C\left(a_i(R_iu, R_iu) + \sum_{F_{ij}\subset\partial\Omega_i} \overline{u}_{F_{ij}}^2 \rho_i\|\theta_{F_{ij}}^{(k)}\|_Z^2\right). \end{aligned} \tag{4.18}$$

Here we use the inequalities in (4.7).

For the term, $\overline{u}_{F_{ij}}^2$, we obtain by applying (4.10) to Ω_i

$$\int_{F_{ij}} u^2\,ds \le C\left(H\left(|u|_{H^1(\Omega_i)}^2 + \sum_{f\in\Omega_i\bigcap\mathcal{F}_p} h_f^{-1}\|[u]\|_{L^2(f)}^2\right) + \frac{1}{H}\|u\|_{L^2(\Omega_i)}^2\right). \tag{4.19}$$

Using the fact that $u - u_0$ is invariant to a shift by a constant and applying the Poincaré inequality (4.8) to the bound above, we obtain

$$\overline{u}_{F_{ij}}^2 \le CH^{2-d}\left(|u|_{H^1(\Omega_i)}^2 + \sum_{f\in\Omega_i\bigcap\mathcal{F}_p} h_f^{-1}\|[u]\|_{L^2(f)}^2\right). \tag{4.20}$$

Combining (4.18) with (4.20), we get

$$a_i(R_i(u-u_0), R_i(u-u_0)) \le C\left(a_i(R_iu, R_iu) + \sum_{F_{ij}\subset\partial\Omega_i} \rho_i\|R_iu\|_{Z_i}^2 H^{2-d} \max_{F_{ij}} \|\theta_{F_{ij}}^{(k)}\|_Z^2\right),$$

and by the bound $\rho_i\|R_iu\|^2_{Z_i} \leq Ca_i(R_iu, R_iu)$ (see (4.7)) and Lemma 4.3 we finally obtain the following theorem.

$$a(u_0,u_0) \leq C\left(1 + H^{2-d}\max_{F_{ij}} |\theta^c_{F_{ij}}|^2_{H^1(\Omega)}\right) a(u,u). \qquad \square \tag{4.21}$$

We now turn to the bounds for the local components. Let $\{\theta_j\}$ be a partition of unity provided for $\{\Omega'_j\}$ and where $\theta_j \in R^T_j V'_j$ with $|\nabla\theta_j| \leq C/\delta$ and let $u_j = I^h(\theta_j(u-u_0)) \in R^T_jV'_j$, where I^h interpolates into $\mathcal{S}_h$ as defined in (4.14).

We obtain the following theorem.

THEOREM 4.5. *For $u \in \mathcal{S}_h$, and with subdomains Ω_i which satisfy Assumption* 4.1, *there is a partition $u = \sum_{j=0}^N u_j$, which satisfies*

$$a(u_0,u_0) + \sum_{j=1}^N a_{\Omega'_j}(u_j,u_j) \leq C\left(1+\frac{H}{\delta}\right)\left(1 + H^{2-d}\max_{F_{ij}} |\theta^c_{F_{ij}}|^2_{H^1(\Omega)}\right) a(u,u),$$

where C depends on the Poincaré and starlike parameters of the subdomains and the number of colors N_c and where $\theta^c_{F_{ij}}(x)$ is the standard linear conforming coarse basis function defined in (4.2).

Proof. We let $w = u - u_0$ and then let

$$u_j = I^h(\theta_j w) \in \mathcal{S}_h.$$

We consider

$$\begin{aligned} a(u_j,u_j) &\leq C\sum_i \rho_i\|u_j\|^2_{Z_i} \leq C\sum_i \rho_i \|\theta_j w\|^2_{Z_i} \\ &\leq C\sum_i \rho_i\left(\sum_{\tau\in\mathcal{T}\bigcap\Omega'_j\bigcap\Omega_i} |\theta_j w|^2_{H_1(\tau)} + \sum_{f\in\mathcal{F}_p\bigcap\Omega'_j\bigcap\Omega_i} h_f^{-1}\int_f [w]^2\,ds\right) \\ &\leq C\sum_i \rho_i\left(\sum_{\tau\in\mathcal{T}\bigcap\Omega'_j\bigcap\Omega_i} \|\nabla\theta_j w\|^2_{L_2(\tau)} + \sum_{\tau\in\mathcal{T}\bigcap\Omega'_j\bigcap\Omega_i} |w|^2_{H_1(\tau)}\right. \qquad (4.22) \\ &\qquad \left. + \sum_{f\in\mathcal{F}_p\bigcap\Omega'_j\bigcap\Omega_i} h_f^{-1}\int_f [w]^2\,ds\right). \end{aligned}$$

We consider the first term in (4.22):

$$\begin{aligned} &\sum_{\tau\in\mathcal{T}\bigcap\Omega'_j\bigcap\Omega_i} \|\nabla\theta_j w\|^2_{L_2(\tau)} \\ &\quad \leq C\frac{1}{\delta^2}\sum_{\tau\in\mathcal{T}\bigcap\Omega'_{j,\delta}\bigcap\Omega_i} \|w\|^2_{L^2(\tau)} = C\frac{1}{\delta^2}\|w\|^2_{L^2(\Omega'_{j,\delta}\bigcap\Omega_i)} \\ &\quad \leq C\frac{1}{\delta^2}\delta\left(H^{-1}_{\Omega'_j}\|w\|^2_{L^2(\Omega'_j\bigcap\Omega_i)}\right. \qquad (4.23) \\ &\qquad \left. + H_{\Omega'_j}\left(\sum_{\tau\in\Omega'_j\bigcap\Omega_i} |w|^2_{H^1(\tau)} + \sum_{f\in\mathcal{F}_p\bigcap\Omega'_j\bigcap\Omega_i} h_f^{-1}\int_f [w]^2\,ds\right)\right). \end{aligned}$$

Here $\Omega'_{j,\delta}$ is the union of the elements $\tau \in \mathcal{T}$ where $\nabla\theta_j$ does not vanish, and the bound (4.11) is applied to $\Omega'_{j,\delta} \bigcap \Omega_i$.

For the term $\|w\|^2_{L_2(\Omega'_j \bigcap \Omega_i)}$, we use the Poincaré–Friedrichs inequality (4.9)

$$\|w\|^2_{L^2(\Omega_i)} \le C(\Omega_i) \left(\sum_{\tau\in\Omega_i} |w|^2_{H_1(\tau)} + \sum_{f\in\Omega_i \bigcap \mathcal{F}_p} h_f^{-1} \|[w]\|^2_{L_2(f)} + \left(\int_\Gamma w\, ds \right)^2 \right).$$

By choosing $\Gamma = F_{ij}$, we have $\int_\Gamma w\, ds = 0$ (see (4.12) and (4.13)), and from a scaling argument, we obtain

$$\|w\|^2_{L^2(\Omega_i)} \le CH^2 \left(\sum_{\tau\in\Omega_i} |w|^2_{H_1(\tau)} + \sum_{f\in\Omega_i \bigcap \mathcal{F}_p} h_f^{-1} \|[w]\|^2_{L_2(f)} \right).$$

Summing (4.23) over i, combining with the above bound, and assuming that $H_{\Omega'_j}$ is comparable to the diameter H of the Ω_i, which intersects Ω'_j, we obtain

$$\begin{aligned} &\sum_i \rho_i \sum_{\tau\in\mathcal{T} \bigcap \Omega'_j \bigcap \Omega_i} \|\nabla\theta_j w\|^2_{L_2(\tau)} \\ &\quad \le C\frac{H}{\delta} \sum_{i,\Omega_i \bigcap \Omega'_j \neq \emptyset} \rho_i \left(\sum_{\tau\in\mathcal{T}\bigcap\Omega_i} |w|^2_{H_1(\tau)} + \sum_{f\in\mathcal{F}_p \bigcap \Omega_i} h_f^{-1} \|[w]\|^2_{L_2(f)} \right). \end{aligned} \tag{4.24}$$

Here we note that the sum on the right-hand side runs over only the subdomains Ω_i which intersect the subregion Ω'_j. Summing (4.22) over j and combining with (4.24), we finally obtain

$$\sum_j a(u_j, u_j) \le C\left(1 + \frac{H}{\delta}\right) \sum_i \rho_i \|R_i w\|^2_{Z_i} \le C\left(1 + \frac{H}{\delta}\right) a(w,w), \tag{4.25}$$

where $w = u - u_0$. The bound in Lemma 4.4 then completes the proof. ∎

Remark 4.6. The above result holds for quite general subdomains Ω_i, which satisfy the standard Poincaré–Friedrichs inequalities and the starlike property, and each has a number of tetrahedra across each of its subdomain edges which is proportional to $(H/h)^{d-2}$. The resulting bound depends on the energy of the linear conforming coarse basis function, $\theta^c_{F_{ij}}(x)$. In the standard case, when Ω_i is tetrahedral ($d = 3$) and rectangular or triangular ($d = 2$), we have

$$|\theta^c_{F_{ij}}|^2_{H^1(\Omega)} \le CH^{d-2}\left(1 + \log\frac{H}{h}\right),$$

where C is a positive constant independent of any mesh parameters. We also note that for John domains Ω_i in two dimensions the above bound was proved in [27]; we refer to [7, 25, 10] for the definition of John domains. John domains satisfy Poincaré inequalities but they do not in general have the starlike property. Instead of the trace inequality in (4.10), we can apply the finite element Sobolev inequality

$$\overline{u}^2_{F_{ij}} \le \max_{x\in\Omega_i} |u(x)|^2 \le C\left(1 + \log\frac{H}{h}\right) \|R_i u\|^2_{Z_i}$$

to get the bound

$$a(u_0, u_0) + \sum_j a_{\Omega_j'}(u_j, u_j) \le \left(1 + \log \frac{H}{h}\right)^2 \left(1 + \frac{H}{\delta}\right)$$

for the two-dimensional case when Ω_i are John domains; see [18]. Here one additional logarithmic factor comes from the finite element Sobolev inequality. We refer to some recent works [32, 19, 20] for theory of domain decomposition methods for quite general subdomains.

In three dimensions, with an assumption that Ω_i are Lipschitz, we obtain the following result.

LEMMA 4.7. *For a Lipschitz Ω_i in three dimensions, there exists a function $\theta_F^c \in V_h(\Omega_i)$ with the bound*

$$|\theta_F^c(x)|^2_{H^1(\Omega_i)} \le CH\left(1 + \log \frac{H}{h}\right),$$

where $V_h(\Omega_i)$ is the space of linear conforming finite element space on the given triangulation $\mathcal{T}(\Omega_i)$, which consists of the elements τ in $\mathcal{T}$ which belong to Ω_i.

Proof. Let $V = \{x \in \Omega_i : \mathrm{dist}(x, F) \le \sin\alpha\, \mathrm{dist}(x, \partial F)\}$. Since Ω_i is a Lipschitz domain, we may select α so that $F_2 := \partial V \setminus \overline{F}$ does not touch $\partial\Omega_i$.

For $x \in V$, we define

$$d(x) = \frac{d_2(x)}{d_1(x) + d_2(x)},$$

where $d_1(x) = \mathrm{dist}(x, F)$ and $d_2(x) = \mathrm{dist}(x, F_2)$ and where we extend $d(x)$ by zero for $x \in \Omega_i \setminus V$. We note that the construction of such a function $d(x)$ was first given by Dohrmann and Widlund in [19]. Let $d_{\partial F}(x) = \mathrm{dist}(x, \partial F)$. We will show that for $x \in V$, there exists $c > 0$ such that

$$d_1(x) + d_2(x) \ge c d_{\partial F}(x).$$

For $x \in V$, let x_1 and x_2 be points on F and F_2 such that $d_1(x) = |x - x_1|$ and $d_2(x) = |x - x_2|$. Let a_{x_2} be points on ∂F such that $d_{\partial F}(x_2) = |x_2 - a_{x_2}|$. We then have

$$d_{\partial F}(x) \le |x - a_{x_2}| \le |x - x_2| + |x_2 - a_{x_2}|. \tag{4.26}$$

Since $x_2 \in F_2$, we have

$$|x_2 - a_{x_2}| = d_{\partial F}(x_2) = \frac{1}{\sin\alpha} d_1(x_2),$$

by using $d_1(x_2) \le |x_2 - x_1|$, we obtain

$$|x_2 - a_{x_2}| \le \frac{1}{\sin\alpha}\left(|x_2 - x| + |x_1 - x|\right),$$

and from the bound in (4.26) combined with the above, we prove that

$$d_{\partial F}(x) \le \left(1 + \frac{1}{\sin\alpha}\right)(d_1(x) + d_2(x)). \tag{4.27}$$

We interpolate $d(x)$ into the finite element space $V_h(\Omega_i)$ and obtain $\tilde{\theta}_F^c(x)$. We note that $\tilde{\theta}_F^c(x)$ vanishes on the boundary of F. The function $\tilde{\theta}_F^c(x)$ satisfies the required boundary condition, i.e., it has value one in the interior of F and zero at the rest of the boundary of Ω_i. We will prove that

$$|\tilde{\theta}_F^c|^2_{H^1(\Omega_i)} \le CH(1+\log(H/h)).$$

This then provides a bound for $|\theta_F^c|^2_{H^1(\Omega_i)}$ as required.

By the construction, it suffices to consider all tetrahedra covering V. For each tetrahedron τ touching the boundary of F, we have

$$|\tilde{\theta}_F^c|^2_{H^1(\tau)} \le Ch,$$

and using that the number of such tetrahedra is $O(H/h)$, we obtain

$$\sum_{\overline{\tau}\bigcap \partial F\neq\emptyset} |\tilde{\theta}_F^c|^2_{H^1(\tau)} \le CH. \tag{4.28}$$

For those tetrahedra not touching the boundary of F, by using (4.27) combined with

$$|\nabla\tilde{\theta}_F^c(x)| \le C\frac{1}{d_1(x)+d_2(x)},$$

we obtain

$$|\nabla\tilde{\theta}_F^c(x)| \le C\frac{1}{d_{\partial F}(x)},$$

and by integrating the above over all tetrahedra, which are away from ∂F by more than a mesh width, we obtain

$$\sum_{\overline{\tau}\bigcap \partial F=\emptyset} |\tilde{\theta}_F^c|^2_{H^1(\tau)} \le CH\log(H/h). \tag{4.29}$$

We complete the proof by using (4.28) and (4.29). ☐

5. Coarse problem from an additional coarse triangulation. By introducing an additional coarse triangulation and an alternative coarse space, we can obtain an alternative, often better, bound,

$$a(u_0,u_0) \le C(\rho)a(u,u),$$

which results in

$$a(u_0,u_0)+\sum_{j=1}^{N} a_{\Omega'_j}(u_j,u_j) \le C(\rho)\left(1+\frac{H}{\delta}\right)a(u,u).$$

However, $C(\rho)$ may depend on $\rho(x)$.

Let $\mathcal{T}_H$ be the additional coarse triangulation. Here the subdomains need not be a union of tetrahedra/triangles in $\mathcal{T}_H$ but we need the assumption that any subdomain diameter is comparable to the diameters of the tetrahedra/triangles which intersect it. The union of all the coarse tetrahedra/triangles in $\mathcal{T}_H$ need not be Ω. We note that [12] contains pioneering results on such methods. In that paper, Neumann boundary conditions were also considered and it was shown that that union should contain the part of $\partial\Omega$ where Neumann boundary conditions are enforced and that it always must occupy a significant part of Ω. In addition, no coarse triangle should be located entirely outside Ω.

Let V_H be the linear conforming finite element space on $\mathcal{T}_H$ and let $I_h^H u$ be the interpolant into V_H defined by

$$(I_h^H u)(x_l) = \frac{1}{|K_l \bigcap \Omega_i|} \int_{K_l \bigcap \Omega_i} u\,dx,$$

where K_l is the union of coarse tetrahedra/triangles with x_l one of their vertices and Ω_i the subdomain containing the node x_l; see [31, section 3.5] and references therein. Here we note that when x_l is on the subdomain interface, we may choose any subdomain with x_l on its boundary. We then introduce

$$u_0 = \mathcal{J}_H^h(I_h^H u) \in \mathcal{S}_h,$$

where $\mathcal{J}_H^h$ is the nodal interpolant from V_H into $\mathcal{S}_h$, i.e.,

$$(\mathcal{J}_H^h v)(x_l) = v(x_l).$$

Since $u_0 \in H_0^1(\Omega)$, we have

$$\begin{aligned} a(u_0,u_0) &\le C\sum_i \rho_i \|R_i u_0\|_{Z_i}^2 = C\sum_i \rho_i |R_i u_0|_{H^1(\Omega_i)}^2 \\ &= \sum_i \rho_i |R_i \mathcal{J}_H^h I_h^H u|_{H^1(\Omega_i)}^2 \le C\sum_i \rho_i |I_h^H u|_{H^1(\Omega_i)}^2 \\ &\le C(\rho)\sum_i \rho_i \|R_i u\|_{Z_i}^2 \qquad (5.1) \\ &\le C(\rho)\sum_i a_i(R_i u, R_i u) = C(\rho) a(u,u), \end{aligned}$$

where R_i is the restriction to the subdomain Ω_i and the inequality (5.1) can be proved in a way similar to [26, Lemma 9] and by using the Poincaré–Friedrichs inequality (4.9). Here the parameter $C(\rho)$ is determined by

$$C(\rho) \le \max_{x_l \in \mathcal{N}^H} \frac{\max_{\Omega_i \bigcap K_l \neq \emptyset} \rho_i}{\min_{\Omega_i \bigcap K_l \neq \emptyset} \rho_i},$$

where $\mathcal{N}^H$ is the set of all nodes in the coarse triangulation $\mathcal{T}_H$ and K_l is the union of the coarse tetrahedra/triangles with x_l as one of their vertices.

We note that the preconditioner is of the form

$$\mathcal{J}_H^h A_H^{-1} (\mathcal{J}_H^h)^T + \sum_j R_j^T A_j^{-1} R_j,$$

where

$$A_H = (\mathcal{J}_H^h)^T A \mathcal{J}_H^h, \quad A_j = R_j^T A R_j,$$

where R_j is the restriction to Ω_j' and A is the matrix in (2.19). When the subdomains are unions of tetrahedra/triangles in $\mathcal{T}_H$, the preconditioner is the same as the one in [6].

TABLE 1

Performance of the algorithms with the two types of coarse problems (method1 and method2) and an increasing number of subdomains N with a fixed local problem size ($H/h = 4$) and with $\delta = h$: the number of iterations is Iter, the condition number κ.

	method1		method2	
N	Iter	κ	Iter	κ
4^2	18	7.12	17	6.09
8^2	21	8.90	18	6.15
16^2	23	9.66	17	5.94
32^2	24	9.85	17	5.91

TABLE 2

Performance of the algorithms with the first type of coarse problem (method1) and the second type of coarse problem (method2) with an increasing overlapping width δ with a fixed subdomain partition ($N = 4^2$) and local problem size ($H/h = 16$): the number of iterations is Iter, the condition number κ.

	method1		method2	
H/δ	Iter	κ	Iter	κ
16	29	18.59	24	12.52
8	23	12.44	20	7.54
4	19	8.57	18	6.05
2	18	5.52	17	5.29

6. Numerical results.

In this section, we present numerical tests of our two-level Schwarz algorithms for the model elliptic problem (2.1) with Ω the unit square in two dimensions.

We partition Ω into uniform triangles of mesh size h and then divide each triangle into three subtriangles. The domain Ω is then divided into nonoverlapping subdomains so that each subdomain is a union of triangles before the subdivision. By construction, the test functions in $\mathcal{S}_h$ are continuous across each edge on the subdomain boundary. In our experiments, we take $k = 0$ in the definition of $\mathcal{S}_h$. The overlapping subregion partition for the local solver is obtained by extending each subdomain with an overlapping width δ. For the second type of the coarse problem, we consider both structured and unstructured coarse triangulations. For a structured coarse triangulation, 4^2 means that the square domain Ω is partitioned into 4×4 uniform rectangles and each rectangle is divided into two triangles. For an unstructured coarse triangulation, 4^2 means that the size of each triangle is comparable to $H_\Omega/4$, where H_Ω is the diameter of Ω. The triangles in an unstructured coarse triangulation may not be unions of triangles in $\mathcal{T}$, while those in a structured coarse triangulation are. In the CG iteration, we stop when the relative residual norm has dropped by a factor 10^{-6}.

In Table 1, we present results for the algorithms with the two cases of coarse problems with an increasing number of subdomains, a fixed local problem size, and a fixed overlapping width. We observe stable behavior of the condition number and iteration count for both cases.

In Table 2, we present the performance of our methods with varying overlapping width δ with a fixed local problem size and a fixed number of subdomains. We observe a linear increase in the condition number of the preconditioned systems with respect to H/δ for both types of coarse problems. The results agree well with our theoretical bounds.

TABLE 3

Performance of the algorithms with the first type of coarse problem (method1) and the second type of coarse problem (method2) with respect to jumps in the coefficient $\rho(x)$. The overlapping width $\delta = h$, subdomain partition $N = 8^2$, and local problem size $H/h = 4$: the number of iterations is Iter, the condition number κ.

	method1		method2 (structured $\mathcal{T}_H$)		method2 (unstructured $\mathcal{T}_H$)	
ρ_i	Iter	κ	Iter	κ	Iter	κ
10^{-6}	21	9.30	20	11.55	19	11.01
10^{-3}	21	9.29	20	11.45	19	10.91
1	20	9.51	16	6.18	15	5.52
10^3	20	8.55	33	65.64	38	76.81
10^6	21	8.68	38	78.75	45	99.60

In Table 3, we present tests of the effects on our methods of jumps in the coefficient $\rho(x)$. In our tests, $\rho(x) = \rho_i$ in the subdomains located on the diagonal in an 8×8 uniform partition and $\rho(x) = 1$ for the other subdomains. From the results, we see that the condition number of the preconditioned system arising from the first method, with the coarse problem defined by the face basis functions, is insensitive to the jumps in $\rho(x)$, while the condition number of the method with the second type of coarse problem increases very slowly with increasing jumps in $\rho(x)$. It is likely that the growth of the iteration counts would be faster for the second method for problems in three dimensions. In addition, we test the second method regarding the choice of coarse triangulations. In the structured case, each coarse triangle is a union of triangles in $\mathcal{T}$ and the coefficient $\rho(x)$ is constant in each coarse triangle. In contrast, for the unstructured one, the coarse triangles may not resolve jumps in the coefficient $\rho(x)$ and they may not be unions of triangles in $\mathcal{T}$. We observe relatively good performance in the unstructured coarse triangulation but observe that the results are a little more sensitive to jumps in the coefficient $\rho(x)$.

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