

# Breaking $e^n$ barrier for deterministic poly-time approximation of the permanent and settling Friedland's conjecture on the Monomer-Dimer Entropy

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Two important breakthroughs on the permanent had been accomplished in 1998: A. Schrijver proved Schrijver-Valiant Conjecture on the minimal exponential growth of the number of perfect matchings in  $k$ -regular bipartite graphs with multiple edges; N. Linial, A. Samorodnitsky and A. Wigderson introduced strongly poly-time deterministic algorithm to approximate the permanent of **general** non-negative matrices within the multiplicative factor  $e^n$ . Many things happened since them, notably the prize-winning Jerrum, Vigoda, Sinclair **FPRAS** for the permanent. Schrijver's lower bound was vastly generalized and improved; moreover the new proofs, based on hyperbolic polynomials, are transparent, easy, non-computational. Yet, until now there were no deterministic poly-time algorithms to approximate the permanent of **general** non-negative matrices within the multiplicative factor  $C^n, C < e$ . We prove the following double-sided inequality for doubly-stochastic matrices  $A$ :

$$\prod_{1 \leq i, j \leq n} (1 - A(i, j))^{1 - A(i, j)} \leq \text{per}(A) \leq C^n \prod_{1 \leq i, j \leq n} (1 - A(i, j))^{1 - A(i, j)}, \quad (1)$$

where  $C \approx 1.9022$ . A fairly direct corollary is  $C^n$  deterministic poly-time algorithm for the permanent. A slightly more involved argument proves S. Friedland's conjecture on the Monomer-Dimer Entropy, which is a generalization of Schrijver-Valiant Conjecture to partial matchings (aka dimers) that in the limit cover a given fraction  $t \in [0, 1]$  of vertices. The talk is for general mathematical/computer science/physics audience. I will gently go through the very rich and surprising history of the topic: amazingly, the popular statistical physics (and lately machine learning) heuristic, called Bethe Approximation, is one of the (completely rigorous) keys in our approach. The Bethe Approximation was already applied by physicists to the Monomer-Dimer Problem in late 1930s and was mysteriously hidden in Schrijver's 1998 paper.

Time permitting, a conjecture, connecting the Bethe Approximation, Correlation Inequalities and hyperbolic polynomials, will be stated and discussed.

(Joint work with Alex Samorodnitsky (Hebrew University).)