Some Remarks on the Theory of Graphs

BAMS 1947

Theorem 1. Let $k \geq 3$. Then

$$2^{k/2} < f(k,k) \leq C_{2k-2,k-1} < 4^{k-1}$$

[. . .] Let $N \leq 2^{k/2}$. Clearly the number of different graphs of N vertices equals $2^{N(N-1)/2}$. [. . .] The number of different graphs containing a given complete graph of order k is clearly $2^{N(N-1)/2}/2^{k(k-1)/2}$. Thus the number of graphs of $N \leq 2^{k/2}$ vertices containing a complete graph of order k is less than

$$C_{N.k}\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{N^k}{k!}\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{2^{N(N-1)/2}}{2} \tag{1}$$

since by a simple calculation for $N \leq 2^{k/2}$ and $k \geq 3$

$$2N^k < k!2^{k(k-1)/2}$$

But it follows immediately from (1) that there exists a graph such that neither it nor its complementary graph contains a complete subgraph of order k, which completes the proof of Theorem 1.

Graph Theory and Probability. II,

Canad J Math 1961

Lemma 1. Almost all G_α^n have the property that for every $G^{(x)}$ there is an edge $e_{\alpha,x}$ contained in both G_α^n and $G^{(x)}$, which is not contained in any triangle whose edges are in G_α^n and whose third vertex is not in $G^{(x)}$.

Lemma 5. Almost all G_α^n have the property that for every $G^{(x)}$ there are more than $\frac{1}{2}\binom{x}{2}$ edges of $G^{(x)}$ which do not occur in any triangle, the other two sides of which are in G_α^n and whose third vertex is not in $G^{(x)}$.

Paul Erdős and Alfred Rényi

Magyar Tud Akad Mat Kut Int Közl 1960

ON THE *EVOLUTION* OF RANDOM GRAPHS

The study of the evolution of graphs leads to rather surprising results. For a number of fundamental structural properties A there exists a function $A(n)$ tending monotonically to $+\infty$ for $n \to +\infty$ such that

$$\lim_{n\to\infty} P_{n,N(n)}(A) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} \frac{N(n)}{A(n)} = 0 \\ 1 & \text{if } \lim_{n\to\infty} \frac{N(n)}{A(n)} = \infty \end{cases} \tag{1}$$

If such a function $A(n)$ exists we shall call it a "*threshold function*" of the property A.

If a graph G has n vertices and N edges we call the number $\frac{2N}{n}$ the "*degree*" of the graph. [. . .] If a graph G has the property that G has no subgraph having a larger degree than G itself, we call G a *balanced* graph.

THE DOUBLE JUMP

There is however a surprisingly abrupt change in the structure of $\Gamma_{n,N}$ with $N \sim cn$ when c surpasses the value $\frac{1}{2}$.

...

This double "jump" of the size of the largest component when $\frac{N(n)}{n}$ passes the value 1/2 is one of the most striking facts concerning random graphs.

On a Combinatorial problem, I.

Nordisk Mat Tidskr 1963

Hajnal and I [2] recently published a paper on the property B and its generalizations. One of the unsolved problems we state asks: What is the smallest integer $m(p)$ for which there exists a family F of finite sets $A_1, \ldots, A_{m(p)}$, each having p elements, which does not possess property B?

Theorem 1. Let $\{A_i\}$, $1 \le i \le k$ be a family F of finite sets, $|A_i| = \alpha_i \ge 2$. If

$$\sum_{i=1}^{k} \frac{1}{2^{\alpha_i}} \le \frac{1}{2} \tag{3}$$

[...] holds, then F has property B.

Put $\cup_{i=1} A_i = T$, $|T| = n$. [...] Denote by F_T the family of sets S for which

$$S \subset T, A_i \cap S \neq \emptyset, A_i \not\subset S, 1 \le i \le k \tag{6}$$

We have to show that if (3) holds then $|F_T| > 0$ (since this implies that the family of sets $\{A_i\}$ satisfying (3) has property B.)

Denote by F_i the family of sets S satisfying

$$S \subset T, A_i \subset S \text{ or } A_i \cap S = \emptyset \qquad (7)$$

Clearly an $S \subset T$ is in the family F_T if it is in none of the families F_i, $1 \leq i \leq k$ (that is, it satisfies (6) if it does not satisfy (7) for any i, $1 \leq i \leq k$. By a simple sieve process we thus have

$$|F_T| \geq 2^n - \sum_{i=1}^{k} |F_i| + 1 \qquad (8)$$

[. . .] We evidently have

$$|F_i| = 2^{n-\alpha_i+1} \tag{9}$$

since clearly there are $2^{n-\alpha_i}$ sets $S \subset T$ satisfying $A_i \subset S$ and $2^{n-\alpha_i}$ sets satsfying $A_i \cap S = \emptyset$. From (8)and (9) we have $|F_T| \geq 1$ if (3) is satisfied.

Now one can ask the following problem which I cannot answer: Let $\{A_i\}$ be a finite or infinite family of finite sets which does not have property B and for which $|A_i| \geq p \geq 2$ for all i. What is the upper bound $C^{(p)}$ of $\prod_i (1-2^{-\alpha_i})$ and the lower bound C_p of $\sum_i 2^{-\alpha_i}$. [...] Probably

$$\lim_{p \to \infty} C^{(p)} = 0, \lim_{p \to \infty} C_p = \infty$$

On a Combinatorial problem, II.

Acta math Acad Sci Hungar 1964

Theorem 1. $m(n) < n^2 2^{n+1}$

Theorem 1 thus implies $\lim_{n\to\infty} m(n)^{1/n} = 2$. [...] It would be interesting to improve the bounds for $m(n)$. A reasonable guess seems to be that $m(n)$ is of the order $n2^n$.

Paul Erdős and John Moon

On Sets of Consistent Arcs in a Tournament

Canad Math Bull 1965

$$f(n) \leq \frac{1+\epsilon}{2}\binom{n}{2} \qquad (2)$$

In a tournament T_n there are $n!$ ways of re-labelling the nodes and $N = \binom{n}{2}$ pairs of distinct nodes. Hence, there are at most $n!\binom{N}{k}$ such tournament whose largest set of consistent arcs contains k arcs. So, an upper bound for the number of tournaments T_n which contain a set of more than $(1+\epsilon)N/2$ consistent arcs is given by

$$n! \sum_{k>(1+\epsilon)N/2} \binom{N}{k} < \cdots < n! 2^N e^{-\epsilon^2 N/4} \qquad (3)$$

[. . .] But for all sufficiently large n the last quantity in (3) is easily seen to be less than 2^N, the total number of tournaments with n nodes. Hence, there must be at least one tournament T_n which does not contain any set of more than $(1+\epsilon)N/2$ consistent arcs.

The argument employed in the preceding paragraph illustrates the usefulness of probabilistic methods in extremal problems in graph theory, for while we can easily infer the existence of a tournament with a certain required property we are unable to give an explicit construction actually exhibiting such a tournament in general.

With a more careful analysis of inequality (3) this argument actually implies that

$$f(n) < \frac{1}{2}\binom{n}{2} + (\frac{1}{2} + o(1))(n^3 \log n)^{1/2} \qquad (4)$$

It would be desirable to obtain a better estimate for $f(n)$.