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Short Descriptions of Random Structures

Joel Spencer

Courant Institute

Succint Definitions

General First Order Structure

Def: $D(G) =$ smallest *quantifier depth*

of A that defines G

What is $D(G)$ for random n-element model?

Kim/Pikhurko/Verbitsky/JS

$G(n, \frac{1}{2})$: $\Theta(\ln n)$

StJohn/JS:

$G_<(n, \frac{1}{2})$: $\Theta(\ln^* n)$

BitString $U(n, \frac{1}{2})$: $\Theta(\ln \ln n)$

$$G(n, \tfrac{1}{2})$$

Lower Bound

k-extension: All witnesses on all $\leq k$ vertices

Let $k = (1-\epsilon)\log_2 n$

$\Pr[k - \text{extension}] \to 1$

k-extension determines $\equiv_{k+1}$

Therefore most $D(G) > k+1$

Upper Bound

Let $k = (2+\epsilon)\log_2 n$

Random k vertices X

All other vertices have distinct profiles

$$\Pr[\text{FAIL}] \leq \binom{n}{2} 2^{-k} \to 0$$

Therefore most $D(G) \leq k+2$

Tightening Upper Bound

Let $k = (1 + \epsilon) \log_2 n$

Random k vertices X

$Y :=$ those y with unique profile

$$\Pr[y \notin Y] \leq n2^{-k} \to 0$$

$|Y| \sim n$

All z have distinct profile to Y

Therefore $D(G) \leq k + 5$

Tenacity

$T_\epsilon(n) :=$ maximal k so that $n_1, n_2 \geq n$, G_1, G_2 random n_1, n_2 models

$$\Pr[\text{Duplicator wins EHR}[G_1, G_2; k]] \geq 1 - \epsilon$$

ϵ fixed, $n \to \infty$

Zero-One Law implies $T_\epsilon(n) \to \infty$

If random G has $D(G) \leq k$ then $T_\epsilon \leq k$

$G \sim G(n, \frac{1}{2})$, $T_\epsilon(n) \sim \log_2 n$

What about $G \sim G(n, n^{-\alpha})$

with $\alpha \in (0, 1)$, irrational?

Should depend on approximations

of α by rationals

Random Bit String $U(n,p)$, $p = \frac{1}{2}$

Lower Bound

$1^m \equiv_k 1^{m+1}$ for $k = \Omega(\ln m)$

$p1^m s \equiv_k p1^{m+1} s$ for $k = \Omega(\ln m)$

Random $\tau = p1^m s$ for $m = \Omega(\ln n)$

Therefore $D(\tau) = \Omega(\ln \ln n)$

Random Bit String $U(n,p)$, $p = \frac{1}{2}$

Upper Bound

Every m-string τ has $D(\tau) = O(\ln m)$

$m = 10 \ln n$ All $\frac{m}{2}$ strings unique

- Describe all m-strings
- Describe first and last m-string

Now n-string determined

$D(U) = O(\ln m) = O(\ln \ln n)$

Random Ordered $G_<(n,p)$, $p = \frac{1}{2}$

No Convergence via Dance Marathon

n points flip fair coins. Drop out if tails

$f(n) :=$ Pr[unique winner]

$$f(n) = \sum_k n 2^{-k-1}(1-2^{-k})^{n-1}$$

$n = 2^u\theta$, $\theta \in (0,1)$, $k = u + s$

$$f(n) \sim g(\theta) := \sum_{s=-\infty}^{+\infty} 2^{-s-1}\theta e^{-\theta 2^{-s}}$$

$g(\theta)$ not constant. $\lim_n f(n)$ does not exist

$$A : \exists_k \exists!_x (x > k) \wedge ((y \leq k) \rightarrow (x \sim y))$$

$\lim_n \Pr[A]$ does not exist

NonSeparability

Interval $I = [a, b]$ given by a, b

BINARY ADJ$[x, y]$ on $[a, b]$ given by c, d

$x \neq y$ and there exists $c \leq y \leq d$ adjacent to precisely x, y in I

If $|I| \leq \ln^{0.4} n$ get all ADJ

Replace A on graphs with

$$A^* : \exists_{a,b,c,d} A^{**}$$

Traktenbrot-Vought: No Decision Procedure for existence of finite graph models

$\Rightarrow$ Nonseparability of $\Pr[A] \to 1$ and $\Pr[A] = 0$

Big and Small Functions

$\mathrm{TOWER}(1) = 2$

$\mathrm{TOWER}(k+1) := 2^{\mathrm{TOWER}(k)}$

$\log^*(n) :=$ least k, $\mathrm{TOWER}(k) \geq n$

Very Robust

Any First Order System bound $\equiv_k$-classes

$x_{i,k} :=$ number $(x_1, \ldots, x_i)$ "types"

$(x_1, \ldots, x_k)$ types $\exp[k^{O(1)}]$

$x_{i-1,k}$ determined by *set* of reachable

$(x_1, \ldots, x_i)$ types $x_{i-1,k} \leq 2^{x_{i,k}}$

Number of $\equiv_k$-classes

$= x_{0,k} \leq \mathrm{TOWER}(k + O(1))$

$D(U)$ for Random BitString

General Lower Bound

The number of U with $D(U) \leq k$ is

at most number of $\equiv_k$-classes

which is $\leq \mathsf{TOWER}(k + O(1)) \ll n$

for $k = \Omega(\ln^* n)$

Therefore most U have $D(U) = \Omega(\ln^* n)$

$D(U)$ for Random BitString

Upper Bound

$\ln^* n = x_1 < x_2 < \ldots < x_s = n$

x_{i+1} least so that all $y \in (x_i, x_{i+1}]$ have unique

profie to $[1, x_i]$

$D(U) \leq x_1 + O(s)$

Usually $x_{i+1} > 2^{x_i/2}$

Robustness: $s = O(\ln^* n)$

$D(U) = O(\ln^*(n))$

A Limit in Theory

Following equivalent for $x \in E_k$:

- $\forall_y \exists_z x + y + z = x$
- $\forall_y \exists_z z + y + x = x$
- $\exists_p \exists_s \forall_y p + y + s = x$
- x persistent in Markov Chain

x called k-persistent.

There exist (many) persistent x

Persistency not dependent on edge effects

x persistent implies $p + x + s$ persistent

$\lim_n \Pr[k - \text{persistent}] = 1$

But how long must persistent x be?

Very long!

Counting Ehrenfeucht Classes

$f(s, k) :=$ number $\equiv_k$-classes

over s-element alphabet Σ

$f(s + 1, k + 2) \geq 2^{f(s,k)}$

$\Sigma^+ = \Sigma \cup \{\alpha\}$

Set of $\equiv_k$-classes between consecutive α

$f(k, k) \geq \text{TOWER}(\Omega(k))$

Encode $\Sigma = \{1, \ldots, m\}$ to $\Sigma = \{0, 1, 2\}$

$352701 \to 011\underline{2}101\underline{2}010\underline{2}111\underline{2}000\underline{2}001$

$f(3, k) \geq$ TOWER$(\Omega(k))$

$k + 2$-persistent over $\Sigma = \{0, 1, 2, \beta\}$

Need *every* $\equiv_k$-class between consecutive β

Length $\geq f(3, k) \geq$ TOWER$(\Omega(k))$

Technical: Reduce to $\Sigma = \{0, 1\}$

A_k: σ is k-persistent

Pr$[A_k] \to 1$ but equals zero for $n <$ TOWER$(\Omega(k))$