Random Graphs G22.3033-007
Assignment 5. Due Monday, March 6,2006

1. Show that the complete graph on k vertices is strictly balanced. Show that all trees are strictly balanced.

2. Let X be the number of *isolated* triangles in $G(n,p)$ with $p = \frac{c}{n}$. Last assignment you computed $E[X]$. Now find a good upper bound on $Var[X]$ (Hint: The contribution from the covariances will be negligible and you'll only need coarse upper bounds there.) Use Chebyschev's Inequality to deduce that X becomes sharply concentrated about its mean as c gets large. More precisely, letting μ denote the mean, show $\lim_{c\to\infty} \Pr[|X = \mu| > \epsilon\mu] \to 0]$.

3. Let x be chosen uniformly from $\{1,\ldots,n\}$. Set $m = n^{1/10}$. For p prime, $p \le m$ let X_p be the indicator random variable for p dividing x. Let $X = \sum X_p$, the sum over all primes $p \le m$. Set $\mu_p = E[X_p]$ and $\mu = E[X]$.

   (a) Show $E[(X-\mu)^4] \sim 3(\ln\ln n)^2$. [This is one of the steps toward the Erdős-Kac theorem. Use as a fact that $\sum_{p\le m} p^{-1} \sim \ln\ln m \sim \ln\ln n$. You'll want, similarly to a previous assignment, to expand the fourth power and consider the different cross terms. The main term shall come from $\sum_{p\ne q} E[(X_p-\mu_p)^2(X_q-\mu_q)^2]$ – for the other terms one can get away with rougher upper bounds, as long as you show the contribution is $o((\ln\ln n)^2)$.]

   (b) The Turan Theorem can be written in the following way: For any positive λ the number of x with $1 \le x \le n$ with $|v(x) - \ln\ln n| > \lambda(\ln\ln n)^{1/2}$ is at most $\lambda^{-2}n$. This comes from Chebyschev but lets see it directly: Each such x would contribute $\frac{1}{n}\lambda^2 \ln\ln n$ to $E[(X - E[X])^2]$ (the $\frac{1}{n}$ being the probability of selecting x) so since, as we showed, $E[(X-E[X])^2] \sim \ln\ln n$ this can only happen at most $\lambda^{-2}n + o(n)$ times. Now for the problem: Use the result of the first part to find a new bound on the number of x with $1 \le x \le n$ with $|v(x) - \ln\ln n| > \lambda(\ln\ln n)^{1/2}$. Your bound should be better than the Turan bound for λ appropriately large.

4. Consider a drawing (in the intuitive sense) of a graph G on the plane with v vertices, e edges and κ crossings. [When P,Q,R,S are distinct vertices and the edges PQ, RS cross that counts as one crossing.] In

this exercise we find a lower bound for κ as a function of v, e. If you don't get part i please assume it and go on to part $i + 1$.

(a) Show that if $\kappa = 0$ then $e \leq 3v - 6$. (A classic result, involving no probability.)

(b) Show that $\kappa \geq e - (3v - 6)$. (Easy once you see it.)

(c) Take a random subset U of the vertices, where for each vertex P $\Pr[P \in U] = p$ (for the moment, a parameter) and these events are mutually independent. Let X, Y, Z be the expected number of vertices, edges and crossings respectively in the restriction of the drawing to U. Show $E[X] = vp$, $E[Y] = ep^2$ and $E[Z] = \kappa p^4$.

(d) From parts two and three give a condition on κ of the form: For all $p \in [0, 1]$ yadda yadda yadda.

(e) For technical convenience replace part two by $\kappa \geq e - 3v$ and make the appropriate replacement in part four. Now use your calculus skills to derive a theorem in the form

If e is at least blip blip blip then κ is at least blah blah blah

The blip blip blip condition will simply reflect that p must lie in $[0, 1]$.

Dear Sir,

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 25 years of age. I have no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics... I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as "startling"
– Ramanujan