# Random Graphs G22.3033-007
# Assignment 4. Due Monday, February 27,2006

`Note: NO CLASS President's Day, Monday, February 20!!`

1. Let $P, Q, R, S$ be uniformly and independently selected from the unit square. Let $f(\epsilon)$ be the probability that triangles $PQR$ and $QRS$ both have area less than $\epsilon$. Find the asymptotics of $f(\epsilon)$ (neglecting constant factors) as $\epsilon$ approaches zero. [Idea: Look first at the length $QR$. There is a tricky part - if you get an integral that looks infinite you are on the right track but you'll have to fix it.]

2. Let $X$ be the number of triangles in $G(n, p)$ with $p = c/n$. Find the asymptotic [$c$ fixed, $n \to \infty$, in terms of $c$] values for the expectation and variance of $X$.

3. Let $X$ be the number of isolated triangles in $G(n, p)$ with $p = c/n$. Find the asymptotic [$c$ fixed, $n \to \infty$, in terms of $c$] value for the expectation of $X$. (A triangle $v, w, u$ is isolated if there are no other edges including any one of those three vertices.)

4. For $1 \le i \le n$ let $X_i$ be independent random variables with $\Pr[X_i = 1] = \frac{1}{i}$, $\Pr[X_i = 0] = 1 - \frac{1}{i}$. Set $Y_n = \sum_{i=1}^{n} X_i$. Let $\mu_i, \sigma_i^2$ equal the mean and variance of $X_i$ and let $\mu, \sigma^2$ denote the mean and variance of $Y_n$.

   (a) Find asymptotic formulae of $\mu, \sigma^2$.

   (b) Use Chebyschev's Inequality to show that for any $\epsilon > 0$

   $$\lim_{n\to\infty} \Pr[|Y_n - E[Y_n]| > \epsilon E[Y_n]] = 0$$

   (c) Show that *if* $\lambda = o(1)$ then

   $$E[e^{\lambda(X_i - \mu_i)}] = 1 + \frac{\lambda^2}{2}\sigma_i^2(1 + o(1))$$

   (Hint: For $t \ge 3$ bound $E[(X_i - \mu_i)^t] \le E[(X_i - \mu_i)^2]$ as $|X - i - \mu_i| \le 1$ always.)

   (d) Deduce that $\ln[E[e^{\lambda(Y_n - \mu)}]] \sim \frac{1}{2}\lambda^2\sigma^2$.

(e) Use Chernoff bounds to show that *if* $a = o(\sigma)$ then

$$\Pr[Y_n - E[Y_n] > a\sigma] < e^{-\frac{a^2}{2}(1+o(1))}$$

5. Let $X_i$, $1 \leq i \leq n$, be uniform on $\{1, \ldots, 6\}$ (throws of a fair die), $Y_i = X_i - \frac{7}{2}$ (to move to zero mean) and $Y = \sum_{i=1}^{n} Y_i$. Use Chernoff Bounds to give $A$ as small as possible (include the constant factor!) so that

   (a) $\Pr[Y > A] < n^{-1}$

   (b) $\Pr[Y > A] < n^{-10}$

   (c) $\Pr[Y > A] < e^{-\sqrt{n}}$

I was interviewed in the Israeli Radio for five minutes and I said that more than 2000 years ago, Euclid proved that there are infinitely many primes. Immediately the host interrupted me and asked "Are there still infinitely many primes?"
Noga Alon