Random Graphs G22.3033-007
Notes on Asymptotics

Lets start with the Taylor Series

$$\ln(1-\epsilon) = -\epsilon - \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3}\dots \tag{1}$$

valid for $|\epsilon| < 1$ though we will only be interested in ϵ small positive. This is too much information so we cut it down in a variety of ways:

$$\ln(1-\epsilon) \sim -\epsilon \text{ when } \epsilon = o(1) \tag{2}$$

and with error term

$$\ln(1-\epsilon) = -\epsilon + O(\epsilon^2) \tag{3}$$

Sometimes we need a more precise result

$$\ln(1-\epsilon) = -\epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3) \tag{4}$$

While one could continue this sequence, these will suffice for this course.

Now lets examine the asymptotics of $\binom{n}{k}$ when $n, k \to \infty$. We write:

$$\binom{n}{k} = \frac{(n)_k}{k!} \sim \frac{n^k e^k}{k^k\sqrt{2k\pi}} A \tag{5}$$

where we set

$$A := \frac{(n)_k}{n^k} = \prod_{i=0}^{k-1}(1 - \frac{i}{n}) \tag{6}$$

So if we get A we get the binomial coefficient. It is more convenient to work with

$$B := \ln A = \sum_{i=0}^{k-1} \ln(1 - \frac{i}{n}) \tag{7}$$

For $k = o(n)$ we have

$$B \sim \sum_{i=0}^{k-1} -\frac{i}{n} \sim -\frac{k^2}{2n} \tag{8}$$

and thus we can write

$$A = e^{-\frac{k^2}{2n}(1+o(1))} \tag{9}$$

This does not give the full asymptotics of A as the $1+o(1)$ is in the exponent. We go further as follows:

$$B = \sum_{i=0}^{k-1} -\frac{i}{n} + O(i^2 n^{-2}) = -\frac{k^2}{2n} + O(k^3 n^{-2}) \tag{10}$$

So *if* $k = o(n^{2/3})$, $B = -\frac{k^2}{2n} + o(1)$ and we have the asymptotic formula

$$A = e^{-\frac{k^2}{2n}}(1 + o(1)) \tag{11}$$

In particular:

$$\text{If } k = o(n^{1/2}) \text{ then } A \sim 1 \tag{12}$$

$$\text{If } k \sim cn^{1/2} \text{ then } A \sim e^{-\frac{c^2}{2}} \tag{13}$$

If $k = o(n^{3/4})$ we go to the next approximation:

$$B = \sum_{i=0}^{k-1} -\frac{i}{n} - \frac{i^2}{2n^2} + O(i^3 n^{-3} = -\frac{k^2}{2n} - \frac{k^3}{6n^2} + O(k^4 n^{-3}) \tag{14}$$

and the error term is $o(1)$ so that we have the asymptotic formula

$$A = e^{-\frac{k^2}{2n}} e^{-\frac{k^3}{6n^2}} (1 + o(1)) \tag{15}$$

In particular (this case will come up a number times)

$$\text{If } k \sim cn^{2/3} \text{ then } A \sim e^{-\frac{k^2}{2n}} e^{-\frac{c^3}{6}} \tag{16}$$

BTW, the inequality

$$\ln(1-\epsilon) < -\epsilon \text{ or, equivalently } 1 - \epsilon < e^{-\epsilon} \tag{17}$$

is valid for all $\epsilon \in (0,1)$ and can be pretty handy.